Lecture Notes in

Structural Analysis

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University of Colorado, Boulder

2022

Notice

Intentionally, this document can not be printed.

It is best read on a computer to easily follow the multiple hyperlinks and bookmarks.

Structural Analysis

Role of Technology

Victor E. Saouma
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Spring 2019

- Teaching methods must evolve with time and account for societal changes.
- By now all students have used computers since middle-high schools (if not earlier for games).
- Most computer program no longer have manuals, or steep learning curves. For a software to gain public trust and support it has to be simple, elegant, intuitive, scalable.
- This has affected
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  - Computing
  - Drafting
  - Structural Analysis
  - Learning

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    - Get a feel for numbers $3200 \times 400 = 1,280,000$
    - Perform simple trigonometric identities, calculus operations $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ or $\int \frac{1}{x} dx$
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Loads

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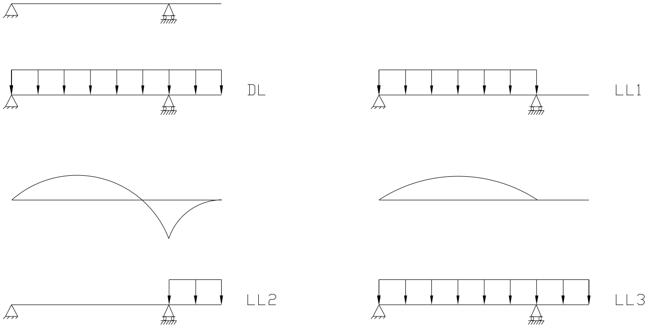
Table of Contents I

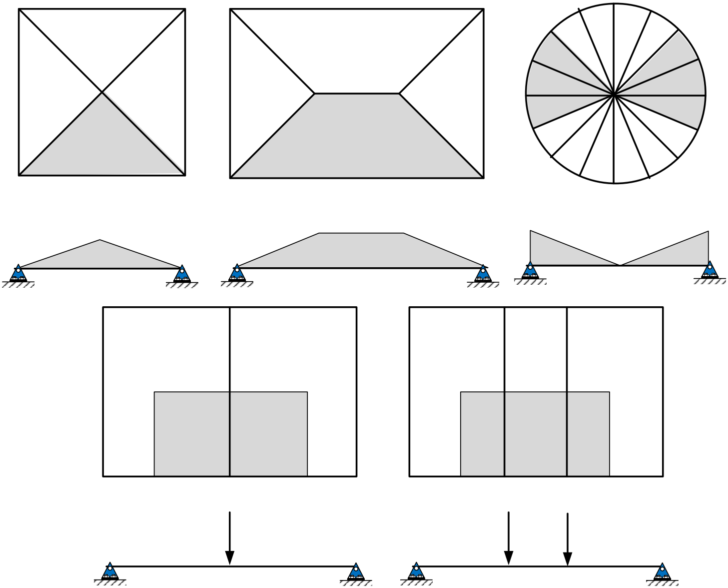
Table of Contents II

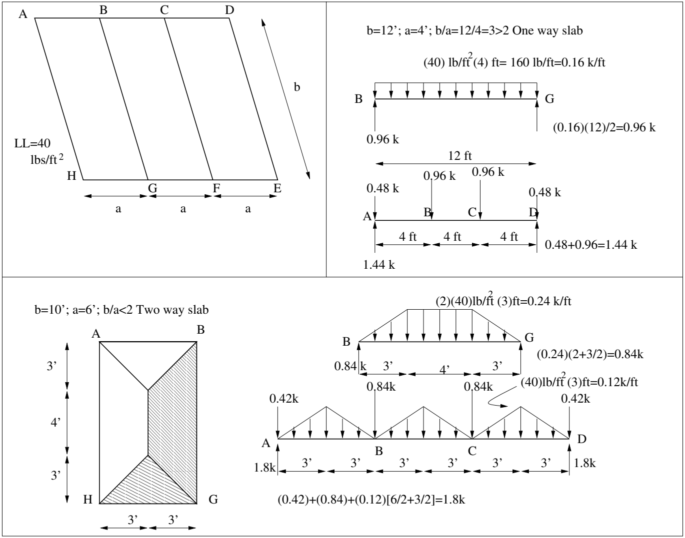
- The main purpose of a structure is to transfer load from one point to another: bridge deck to pier; slab to beam; beam to girder; girder to column; column to foundation; foundation to soil, etc.
- There can also be secondary loads such as thermal (in restrained structures), shrinkage (concrete), differential settlement of foundations, P-Delta effects (additional moment caused by the product of the vertical force and the lateral displacement caused by lateral load in a high rise building), misfit between structural elements. Often those loads are ignored, yet they may potentially cause substantial damage.
- Loads are generally subdivided into two categories: vertical and horizontal loads. In linear elastic analysis, it is common to consider each load type separately.
- Vertical loads are the predominant ones and include dead and live loads.
- Horizontal loads act horizontally on the structure and caused by Wind and earthquakes
- Other loads include, hydrostatic, active/passive soil pressures, and thermal.

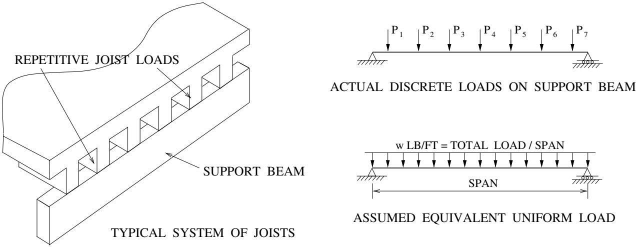
- Live loads specified by codes represent the maximum possible loads and the likelihood of all these loads occurring simultaneously is remote. Hence, building codes allow some reduction when certain loads are combined together.
- Only the dead load is static. The live load on the other hand may or may not be applied on a given component of a structure. Hence, the load placement arrangement resulting in the highest internal forces (moment +ve or -ve, shear) at different locations must be considered.



- For preliminary analysis, the tributary area of a structural component can determine the total applied load. The uniform load per unit area over the shaded area is transferred as a linear load over the adjacent structural element.




A B C D
b
LL=40 lbs/ft2
H G F E
a a a
b=12'; a=4'; b/a=12/4=3>2 One way slab
(40) lb/ft2 (4) ft= 160 lb/ft=0.16 k/ft
B G
(0.16)(12)/2=0.96 k
0.96 k
12 ft
0.96 k 0.96 k
0.48 k 0.48 k
A B C D
4 ft 4 ft 4 ft
0.48+0.96=1.44 k
1.44 k
b=10'; a=6'; b/a<2 Two way slab
A B
3' 4' 3'
H G
3' 3'
(2)(40)lb/ft2 (3)ft=0.24 k/ft
B G
(0.24)(2+3/2)=0.84k
0.84 k 3' 4' 3'
0.84k 0.84k
(40)lb/ft2 (3)ft=0.12k/ft
0.42k 0.42k
A B C D
1.8k 3' 3' 3' 3' 3' 3' 1.8k
(0.42)+(0.84)+(0.12)[6/2+3/2]=1.8k


REPETITIVE JOIST LOADS
SUPPORT BEAM
TYPICAL SYSTEM OF JOISTS
P1 P2 P3 P4 P5 P6 P7
ACTUAL DISCRETE LOADS ON SUPPORT BEAM
w LB/FT = TOTAL LOAD / SPAN
SPAN
ASSUMED EQUIVALENT UNIFORM LOAD

- Dead loads (DL) consist of the weight of the structure itself, and other permanent fixtures (such as walls, slabs, machinery).
- DL can easily be determined from the structure's dimensions and density

Material	lb/ft^3	kN/m^3
Aluminum	173	27.2
Brick	120	18.9
Concrete	145	33.8
Steel	490	77.0
Wood (pine)	40	6.3

- For design purposes, dead loads must be estimated and verified at the end of the design cycle. This makes the design process iterative.

- Live loads (LL) are movable or moving and may be horizontal.
- In analysis load placement should be such that their effect (shear/moment) are maximized.

Use or Occupancy	lb/ft^2
Assembly areas	50
Cornices, marquees, residential balconies	60
Corridors, stairs	100
Garage	50
Office buildings	50
Residential	40
Storage	125-250

- For small areas (30 to 50 sq ft) the effect of **concentrated load** should be considered separately.

- Since there is a small probability that large tributary areas are fully loaded, a reduction of the live load L_0 when the influence area $K_{LL}A_T$ is larger than 400 ft^2, however the reduced load must not exceed 50% of L_0 for members supporting one floor or a section of a single floor, nor less than 40% of L_0 for members supporting two or more floors:

$$L = L_0 \left(0.25 + \frac{15}{\sqrt{K_{LL}A_T}} \right) \tag{1}$$

  where K_{LL} is equal to 4 for interior columns and exterior columns without cantiliver slab, and 2 for interior beams and edge beams without cantiliver slabs.

- The reduced live load for flat roofs is

$$L = L_0 R_1 R_2 \tag{2}$$

  where $R_1 = 1.2 - 0.001A_T$ for 200 ft$^2 < A_T <$ 600 ft^2, $R_1 = 1.0$ for $A_T \leq$ 200 ft^2, and $R_1 = 0.6$ for $A_T \geq$ 600 ft^2. $R_2 = 1.0$ for $F \leq 4$, $R_2 = 1.2 - 0.05F$ for $4 < F < 12$, and $R_2 = 0.6$ for $F \geq 12$ where F is the number of inches of rise of the roof per foot of span.

- For columns or beams supporting more than one floor, A_T is the sum of the tributary area from all the floors.

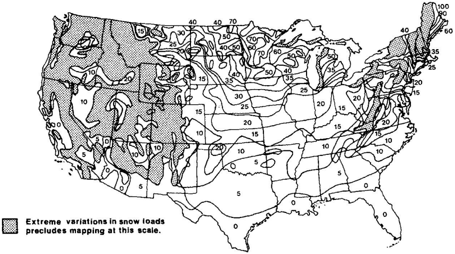
A four storey office building has interior columns spaced 30 ft apart in the two directions. If the flat roof loading is 50 lb/ft^2, determine the reduced live load supported by a typical interior column located on the ground level

$$
\begin{aligned}
L_0 &= 50 \text{ psf} \\
A_T &= (30)(30) = 900 \text{ ft}^2 (> 400 \text{ ft}^2 \surd) \\
L_{floor} &= L_0 \left(0.25 + \frac{15}{\sqrt{K_{LL} A_T}} \right) = 50 \left(0.25 + \frac{15}{\sqrt{4(900)}} \right) = 25 \text{ psf} \\
\% \text{ Reduction} &= \frac{25}{50} = 50\% > 40\% \surd \\
L_{roof} &= L_0 R_1 R_2 = (50)(0.6)(1) = 30 \text{ psf} \\
F_1 &= \left[\underbrace{(25 \text{ psf})(3 \times 900 \text{ ft}^2)}_{1^{st} \text{ 3 columns}} + \underbrace{(30 \text{ psf})(900 \text{ ft}^2)}_{\text{Roof column}} \right] \frac{1}{1,000} \\
&= 67.5 + 27.0 = 94.5 \text{ k}
\end{aligned}
$$

Note that without reduction the total load would have been

$$F_2 = 4(50\text{ psf})(900\text{ ft}^2)\frac{1}{1,000} = 180.0\text{ k}$$

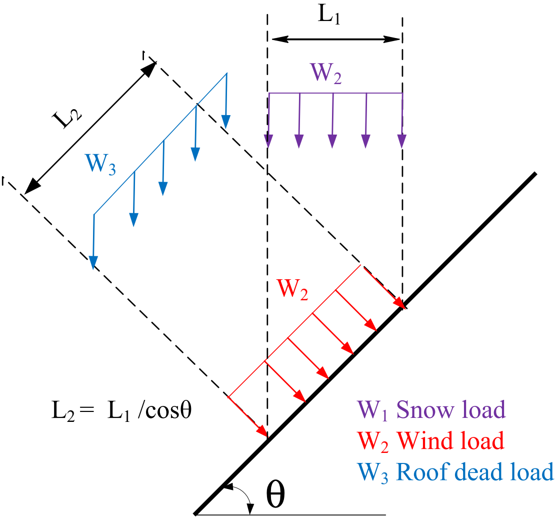
- Must be determined from local codes and depend on geographical locations.

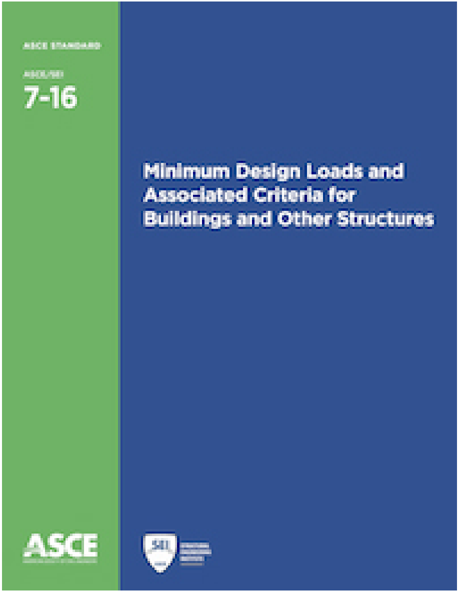


- Snow loads are always given on the projected length or area on a slope.
- The steeper the roof, the lower the snow retention. For snow loads greater than 20 psf and roof pitches α more than 20° the snow load p may be reduced by

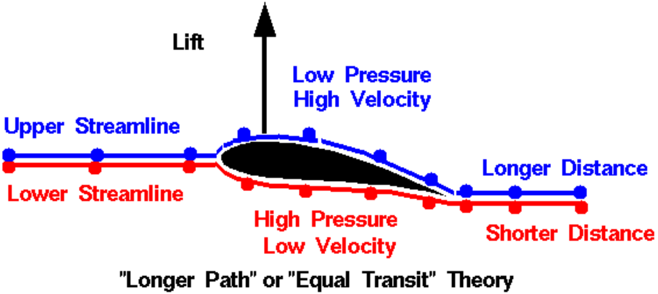
$$R = (\alpha - 20)\left(\frac{p}{40} - 0.5\right) \qquad \text{(psf)}$$

- Extremely important figure.




ASCE STANDARD
ASCE/SEI
7-16
Minimum Design Loads and
Associated Criteria for
Buildings and Other Structures
ASCE
SEI

- Let us pull back a step from the textbook, and tie together fluid and structures.
- Bernouilli (1700-1782) Principle: $P + \frac{1}{2}\rho v^2 = cst \Rightarrow$ velocity increases, the pressure decreases. This explains airfoil and negative pressures (suction) on roofs.



- When a steady streamline airflow of velocity V is completely stopped by a rigid body ($P + \frac{1}{2}\rho v^2 = 0$), the stagnation pressure (or velocity pressure) q_s becomes

$$q_s = \frac{1}{2}\rho V^2 \tag{3}$$

where ρ is the air mass density of air.

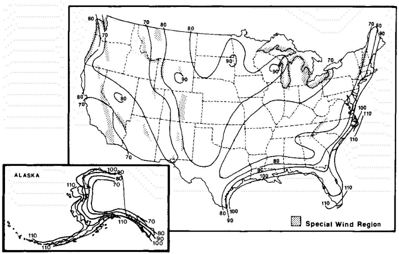
- At sea level and a temperature of 15^{o}C (59^{o}F), the air specific weight γ is 0.0765 lb/ft^3, thus the air mass density will be

$$\rho = \frac{\gamma}{g} = \frac{0.0765}{32.2} \tag{4}$$

this would yield a pressure of

$$\begin{aligned} q_s &= \frac{1}{2}\frac{(0.0765)\text{lb/ft}^3}{(32.2)\text{ft/sec}^2}\left(\frac{(5280)\text{ft/mile}}{(3600)\text{sec/hr}}V\right)^2 \\ &= 0.00256V^2 \end{aligned} \tag{5}$$

where V is the maximum wind velocity (in miles per hour) and q_s is in psf and can be obtained from wind maps (in the United States $70 \leq V \leq 110$)



- The previous equation can now be generalized through an emperical equation to account for the shape and surroundings of the building. Thus, the design pressure q_z (psf) is given by

$$q_z = \underbrace{0.00256 V^2}_{q_s} K_z K_{zt} K_d K_e \tag{6}$$

where

- q_z Velocity wind pressure at height z above ground.
- V Velocity, mph
- K_z Velocity pressure exposure coefficient accounts for height and exposure $K_x = [B|C|D]$, *Exposure B is for urban and suburban, or wooded areas with low structures; C for open terrain with scattered obstructions generally less than 30 ft; D for unobstructed areas exposed to wind*.
- K_{zt} Topological factor accounts for hills (usually 1.0)
- K_d Directionality factor reflects the fact that the climatologically and aerodynamically or dynamically most unfavorable wind directions typically do not coincide.
- K_e Ground elevation factor accounts for variability of air density in terms of elevation above sea-level

- Last step:

$$p = q_z G C_p \tag{7}$$

where

- G Gust factor = 0.85
- C_p External pressure coefficient (usually $\pm$ 0.8) fraction of the wind acting on

we need K_z, K_d, K_e, G, C_p.

K_z

K_z			
z (ft)	Exposure		
	B	C	D
0-25	0.57	0.85	1.03
20	0.62	0.90	1.08
25	0.66	0.94	1.12
30	0.70	0.98	1.16
40	0.76	1.04	1.22
50	0.81	1.09	1.27
100	0.99	1.26	1.43
160	1.13	1.39	1.55

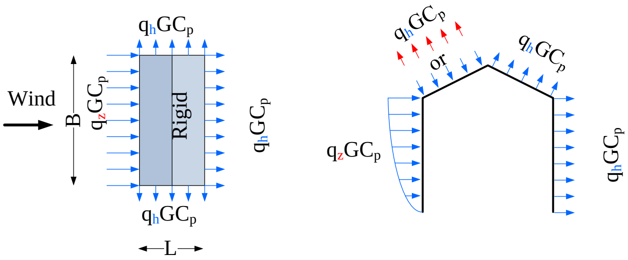
K_d

Buildings	
Main Wind Force Resisting System	0.85
Components and Cladding	0.85
Arched roofs	0.85
Chimneys, Tanks. and Similar Structures	
Square	0.90
Hexagonal	0.95
Round	0.95
Open Signs and Lattice Frameworks	0.85
Trussed Towers	
Triangular. square. rectangular	0.85
All other cross sections	0.95

K_e

Altitude (ft)	Air Density (pcf)	K_e Factor
0	0.0765	1.00
1,000	0.0742	0.96
2,000	0.072	0.93
3,000	0.0699	0.90
4,000	0.0678	0.86
5,000	0.0659	0.83
6,000	0.0639	0.80

C_p Wall pressure coefficient

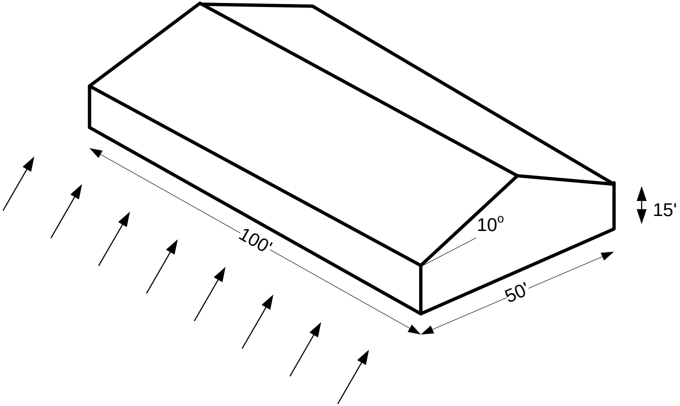


Surface	L/B	C_p	Use with
Windward wall	All Values	0.8	q_z
	0-1	-0.5	
Leeward wall	2	-0.3	q_h
	≥4	-0.2	
Side Walls	All values	-0.7	q_h

	Windward								Leeward		
θ	10	15	20	25	30	35	45	≥60	10	15	≥20
C_p	-0.9	-0.7	-0.4	-0.3	-0.2	-0.2	0.0	0.01θ	-0.5	-0.5	-0.6
			0.0	0.2	0.2	0.3	0.4				

Note: Two values of C_p: must design for both

Wind blows on the side of the fully enclosed agricultural building located on open flat terrain in Oklahoma. Determine the external pressure acting on the roof. Use linear interpolation to determine q_h.



$$
\begin{aligned}
q_z &= 0.00256 V^2 K_z K_{Zt} K_d K_e \\
C &= 0.85 \;\; \text{Exposure: Open Terrain} \\
K_z^{0-15} &= 0.85 \\
K_z^{20} &= 0.90 \\
K_{zt} &= 1 \text{ on level ground} \\
K_d &= 0.85 \;\; \text{Main building} \\
K_e &= = 1.0 \;\; \text{Elevation less than 1,000 ft} \\
q_{15} &= 0.00256(90)^2(0.85)(1)(1)(0.85) = 14.9 \;\; \text{psf} \\
q_{20} &= 0.00256(90)^2(0.90)(1)(1)(0.85) = 15.9 \;\; \text{psf} \\
h &= 15 + \frac{1}{2}(25 \tan 10^\circ) = 17.20 \;\; \text{ft} \;\; \text{Mean elevation} \\
\frac{q_h - 14.9}{17.20 - 15} &= \frac{15.9 - 14.9}{20 - 15} \\
\Rightarrow q_h &= 15.34 \;\; \text{psf}
\end{aligned}
$$

External pressure on winward side of roof

$$\begin{aligned} p &= q_h G C_p \\ C_p &= -0.9 \\ p &= q_h G C_p = 15.34(0.85)(-0.9) = -11.7 \text{ psf} \end{aligned}$$

External pressure on Leeeward side of roof

$$\begin{aligned} p &= q_h G C_p \\ C_p &= -0.5 \\ p &= q_h G C_p = 15.34(0.85)(-0.5) \\ &= -6.5 \text{ psf} \end{aligned}$$

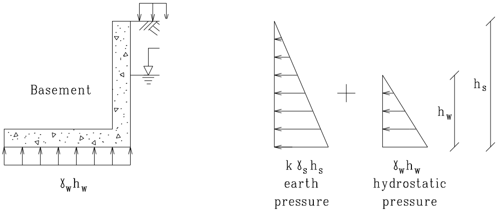
- Structures below ground must resist lateral earth pressure.

$$q = K\gamma h$$

where γ is the soil density, $K = \frac{1-\sin\Phi}{1+\sin\Phi}$ is the pressure coefficient, h is the height.

- For sand and gravel $\gamma = 120$ lb/ ft^3, and $\Phi \approx 30^\circ$.

- If the structure is partially submerged, it must also resist hydrostatic pressure of water



$$q = \gamma_w h$$

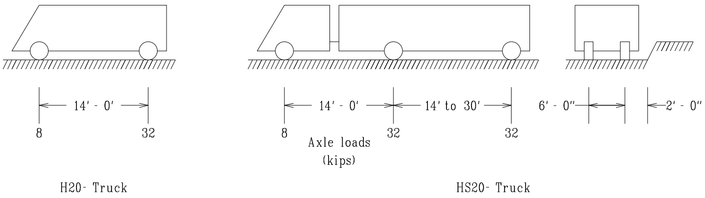
where $\gamma_w = 62.4$ lbs/ft^3.

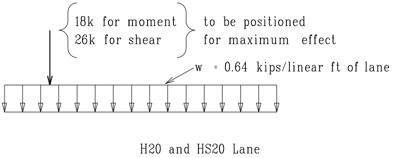
Example The basement of a building is 12 ft below grade. Ground water is located 9 ft below grade, what thickness concrete slab is required to exactly balance the hydrostatic uplift?

The hydrostatic pressure must be countered by the pressure caused by the weight of concrete. Since $p = \gamma h$ we equate the two pressures and solve for h the height of the concrete slab

$$\underbrace{(62.4)\ \text{lbs}/\text{ft}^3 \times (12-9)\ \text{ft}}_{\text{water}} = \underbrace{(150)\ \text{lbs}/\text{ft}^3 \times h}_{\text{concrete}}$$

$$\Rightarrow h = \frac{(62.4)\ \text{lbs}/\text{ft}^3}{(150)\ \text{lbs}/\text{ft}^3}(3)\ \text{ft}(12)\ \text{in/ft} = 14.976\ \text{in} \simeq \boxed{15.0\ \text{inch}}$$


14' - 0'
8
32
H20- Truck
14' - 0'
14' to 30'
6' - 0''
2' - 0''
8
32
32
Axle loads
(kips)
HS20- Truck


18k for moment
26k for shear
to be positioned
for maximum effect
w = 0.64 kips/linear ft of lane
H20 and HS20 Lane

Structural Analysis

Equilibrium & Reactions

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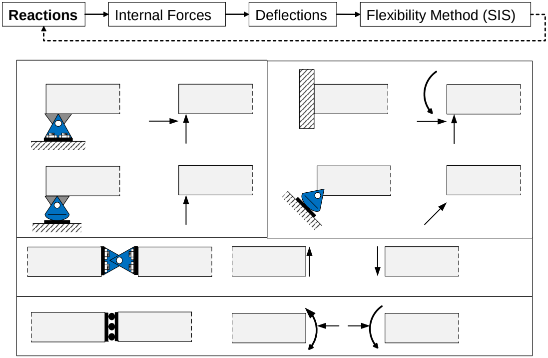
Newton's Third Law

To every action there is an equal and opposite reaction.


Reactions
Internal Forces
Deflections
Flexibility Method (SIS)

- Summation of forces and moments, in a static system must be equal to zero.
- In a 3D cartesian coordinate system there are a total of 6 independent equations of equilibrium:

$$\begin{aligned} \Sigma F_x &= \Sigma F_y = \Sigma F_z = 0 \\ \Sigma M_x &= \Sigma M_y = \Sigma M_z = 0 \end{aligned}$$

- In a 2D cartesian coordinate system there are a total of 3 independent equations of equilibrium:

$$\Sigma F_x = \Sigma F_y = \Sigma M_z = 0$$

- For reaction calculations, the externally applied load may be reduced to an equivalent force; For internal forces (shear and moment) we must use the actual load distribution.
- Summation of the moments can be taken with respect to any arbitrary point.
- Whereas forces are represented by a vector, moments are also vectorial quantities and are represented by a curved arrow or a double arrow vector.
- Not all equations are applicable to all structures

Structure Type	Equations					
Beam, no axial forces		ΣF_y				ΣM_z
2D Truss, Frame, Beam	ΣF_x	ΣF_y				ΣM_z
Grid			ΣF_z	ΣM_x	ΣM_y	
3D Truss, Frame	ΣF_x	ΣF_y	ΣF_z	ΣM_x	ΣM_y	ΣM_z
Alternate Set						
Beams, no axial Force	ΣM_z^A	ΣM_z^B				
2 D Truss, Frame, Beam	ΣF_x	ΣM_z^A	ΣM_z^B			
	ΣM_z^A	ΣM_z^B	ΣM_z^C			

- The three conventional equations of equilibrium in 2D: $\Sigma F_x, \Sigma F_y$ and ΣM_z can be replaced by the independent moment equations ΣM_z^A, ΣM_z^B, ΣM_z^C provided that A, B, and C are not colinear.
- It is always preferable to check calculations by another equation of equilibrium.
- Before you write an equation of equilibrium,
  1. Arbitrarily decide which is the **+ve** direction
  2. Assume a direction for the unknown quantities

3. The right hand side of the equation should be zero

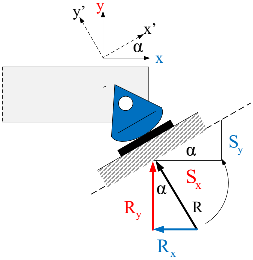
If your reaction is negative, then it will be in a direction opposite from the one assumed.

- Summation of all external forces (including reactions) is not necessarily zero: dynamic problem.
- Summation of external forces is equal and opposite to the internal ones. Thus the net force/moment is equal to zero.
- The external forces give rise to the (non-zero) shear and moment diagram.

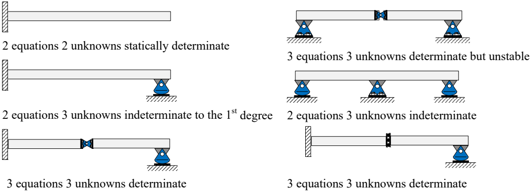
- If a structure has an internal hinge (which may connect two or more substructures), then this will provide an additional equation ($\Sigma M = 0$ at the hinge) which can be exploited to determine the reactions.
- Those equations are often exploited in trusses (where each connection is a hinge) to determine reactions.

- In an inclined roller support with S_x and S_y horizontal and vertical projection, then the reaction R would have

$$\frac{R_x}{R_y} = \frac{S_y}{S_x}$$

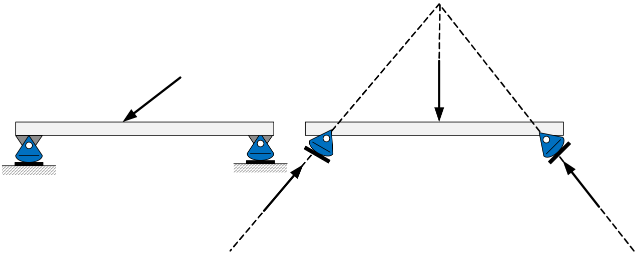


- In statically determinate structures, reactions depend only on the geometry, boundary conditions and loads.
- If the reactions can not be determined simply from the equations of static equilibrium (and equations of conditions if present), then the reactions of the structure are said to be statically indeterminate.
- the degree of static indeterminacy is equal to the difference between the number of reactions and the number of equations of equilibrium



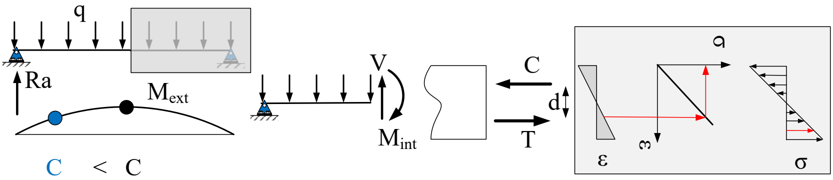
- Failure of one support in a statically determinate system results in the collapse of the structures. Thus a statically indeterminate structure is safer than a statically determinate one.

- Geometric instability will occur if:
  1. All reactions are parallel and a non-parallel load is applied to the structure.
  2. All reactions are concurrent

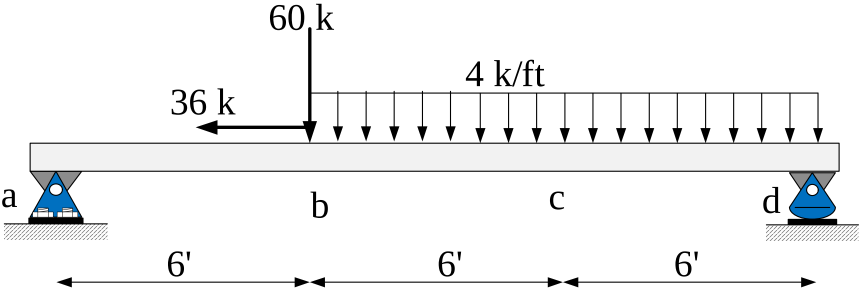


  3. The number of reactions is smaller than the number of equations of equilibrium, that is a mechanism is present in the structure.

- Free-body diagrams are diagrams used to show the relative magnitude and direction of all forces/moments acting upon an object in a given situation. It is not a scaled drawing, it is a diagram
- Free body diagrams consist of:
    - A simplified version of the body
    - Forces shown as straight arrows pointing in the direction they act on the body, moments are shown as curves with an arrow head or a vector with two arrow heads pointing in the direction they act on the body
    - One or more reference coordinate systems
    - By convention (though not always followed), reactions to applied forces are shown with hash marks through the stem of the vector
- All forces and moments must balance to zero.
- Free body diagrams do not necessarily represent an entire physical body. Portions of a body can be selected for analysis. This would allows calculation of internal forces, making them appear external, allowing analysis. This can be used multiple times to calculate internal forces at different locations within a physical body.


q
Ra
Mext
C < C
V
Mint
C
T
d
ε
σ

Determine the reactions of the simply supported beam shown below.



The beam has 3 reactions, we have 3 equations of static equilibrium, hence it is statically determinate.

$$
\begin{aligned}
(+\text{rgt})\ \Sigma F_x = 0; &\quad \Rightarrow \quad R_{ax} - 36\text{ k} = 0 \\
(+\uparrow)\ \Sigma F_y = 0; &\quad \Rightarrow \quad R_{ay} + R_{dy} - 60\text{ k} - (4)\text{ k/ft}(12)\text{ ft} = 0 \\
(+\circlearrowleft)\ \Sigma M_z^c = 0; &\quad \Rightarrow \quad 12R_{ay} - 6R_{dy} - (60)(6) = 0
\end{aligned}
$$

or

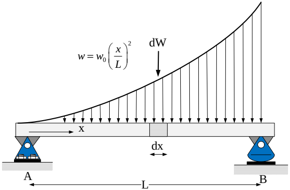
$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 12 & -6 \end{bmatrix} \begin{Bmatrix} R_{ax} \\ R_{ay} \\ R_{dy} \end{Bmatrix} = \begin{Bmatrix} 36 \\ 108 \\ 360 \end{Bmatrix} \Rightarrow \begin{Bmatrix} R_{ax} \\ R_{ay} \\ R_{dy} \end{Bmatrix} = \begin{Bmatrix} 36\text{ k} \\ 56\text{ k} \\ 52\text{ k} \end{Bmatrix}
$$

Alternatively we could have used another set of equations:

$$
\begin{aligned}
(+\circlearrowleft)\ \Sigma M_z^a = 0; &\quad (60)(6) + (48)(12) - (R_{dy})(18) = 0 \quad \Rightarrow \quad R_{dy} = 52\text{ k}\uparrow \\
(+\circlearrowleft)\ \Sigma M_z^d = 0; &\quad (R_{ay})(18) - (60)(12) - (48)(6) = 0 \quad \Rightarrow \quad R_{ay} = 56\text{ k}\uparrow
\end{aligned}
$$

Check:

$$
(+\uparrow)\ \Sigma F_y = 0; 56 + 52 - 60 - 48 = 0\surd
$$



There are two unknowns and two equations of equilibrium (ΣF_y and ΣM), we judiciously start with the second one, as it would directly give us the reaction at B Considering an infinitesimal element of length dx, weight dW, and moment dM:

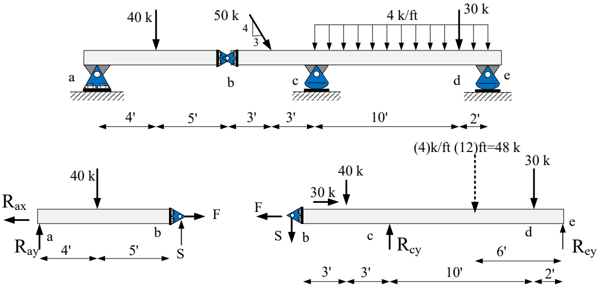
$$\left(+\circlearrowleft\right)\ \Sigma M_z^A = 0; \quad \underbrace{\int_{x=0}^{x=L} \underbrace{\underbrace{\underbrace{w_0 \left(\frac{x}{L}\right)^2}_{w} dx}_{dW} \times x}_{dM}}_{M} - (R_B)(L) \quad = 0$$

$$\Rightarrow R_B = \frac{1}{L} w_0 \left(\frac{L^4}{4L^2}\right) \qquad = \frac{1}{4} w_0 L$$

With R_B determined, we solve for R_A from

$$\left(+\uparrow\right)\ \Sigma F_y = 0; \quad R_A + \underbrace{\frac{1}{4} w_0 L}_{R_b} - \int_{x=0}^{x=L} w_0 \left(\frac{x}{L}\right)^2 dx \quad = 0$$

$$\Rightarrow R_A = \frac{w_0}{L^2}\frac{L^3}{3} - \frac{1}{4} w_0 L \qquad = \frac{1}{12} w_0 L$$



- 4 unknowns (R_{ax}, R_{ay}, R_{cy} and R_{ey}), three equations of equilibrium and one equation of condition ($\Sigma M_b = 0$), thus structure is statically determinate.
- Though there are many approaches to solve for those four unknowns (all of them correct), some are simpler.
- In this case, it is easiest to break the structure into two substructures, and examine the free body diagram of each one of them separately.

1. Isolating ab:

$$\begin{aligned}
(+\circlearrowleft)\ \Sigma M_z^b = 0; &\quad (9)(R_{ay}) - (40)(5) = 0 &\quad \Rightarrow R_{Ay} = 22.2 \text{ k} \uparrow \\
(+\circlearrowleft)\ \Sigma M_z^a = 0; &\quad (40)(4) - (S)(9) = 0 &\quad \Rightarrow S = 17.7 \text{ k} \uparrow \\
\Sigma F_x = 0; & &\quad \Rightarrow R_{ax} = 30 \text{ k}_{\text{lft}}
\end{aligned}$$

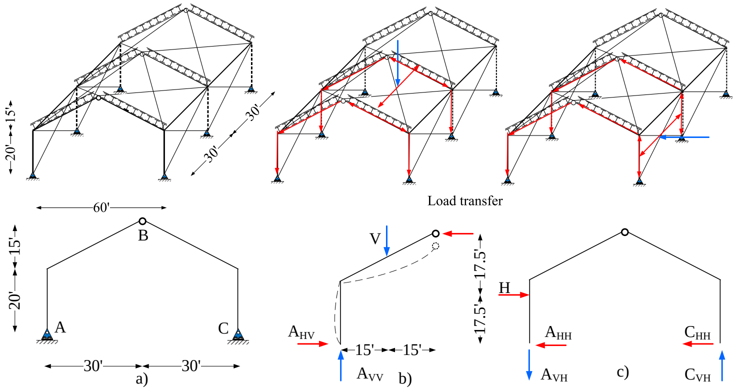
2. Isolating be:

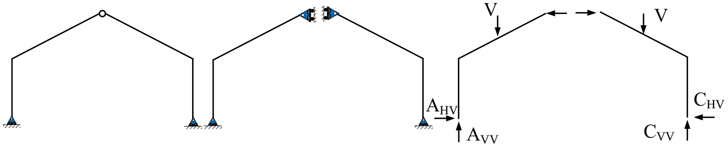
$$\begin{aligned}
(+\circlearrowleft)\ \Sigma M_z^e = 0; &\quad -(17.7)(18) - (40)(15) - (4)(12)(6) - (30)(2) + R_{cy}(12) = 0 \\
&\quad \Rightarrow R_{cy} = \tfrac{1{,}266.6}{12} = 105.6 \text{ k} \uparrow \\
(+\circlearrowleft)\ \Sigma M_z^c = 0; &\quad -(17.7)(6) - (40)(3) + (4)(12)(6) + (30)(10) - R_{ey}(12) = 0 \\
&\quad \Rightarrow R_{ey} = \tfrac{361.8}{12} = 30.15 \text{ k} \uparrow
\end{aligned}$$

3. Check

$$\Sigma F_y = 0;\ \uparrow;\ 22.2 - 40 - 40 + 105.6 - 4(12) - 30 + 30.15 = -0.050 \simeq 0.\surd$$

The three-hinged gable frames spaced at 30 ft. on center. Determine the reactions components on the frame due to: 1) Roof dead load, of 20 psf of roof area; 2) Snow load, of 30 psf of horizontal projection; 3) Wind load of 15 psf of vertical projection. Determine the critical design values for the vertical and horizontal reactions.





1. Due to symmetry, there is no vertical force transmitted by the hinge for snow and dead load, and thus we can consider only the left (or right) side of the frame.
2. Point equivalent loads:

   2.1 Roof dead load per one side of frame is

$$DL = (20)\text{ psf}(30)\text{ ft}\left(\sqrt{30^2+15^2}\right)\text{ ft}\frac{1}{1,000}\text{ lbs/k} = 20.2\text{ k}\downarrow$$

   2.2 Snow load per one side of frame is

$$SL = (30)\text{ psf}(30)\text{ ft}(30)\text{ ft}\frac{1}{1,000}\text{ lbs/k} = 27.\text{ k}\downarrow$$

2.3 Wind load per frame (ignoring the suction) is

$$WL = (15)\text{ psf}(30)\text{ ft}(20+15)\text{ ft}\frac{1}{1,000}\text{ lbs/k} = 15.75\text{ k}\text{rgt}$$

3. There are 4 reactions, 3 equations of equilibrium and one equation of condition $\Rightarrow$ statically determinate.

   Alternatively, by symmetry there is no shear at the hinge C, and we would have for the substructure two reactions at the support and one (horizontal) at the hinge.

   Relationship between the horizontal and vertical reactions at A due to a centered vertical load, A_{HV} and A_{VV} respectively is determined by taking the moment with respect to the hinge (b):

$$\left.\begin{array}{lcrrcl} (+\circlearrowleft)\ \Sigma M_z^B & = & 0 & 15V - 30A_{VV} + 35A_{HV} & = & 0 \\ (+\uparrow)\ \Sigma F_y & = & 0 & A_{VV} - V & = & 0 \end{array}\right\} A_{HV} = \frac{15A_{VV}}{35} = .429A_{VV}$$

   Substituting for roof dead and snow load we obtain

$$\begin{array}{lclclcl} A_{VV}^{DL} & = & C_{VV}^{DL} & = & & & 20.2\text{ k}\uparrow \\ A_{HV}^{DL} & = & C_{HV}^{DL} & = & (.429)(20.2) & = & 8.66\text{ k}\text{rgt} \\ A_{VV}^{SL} & = & C_{VV}^{SL} & = & & & 27.\text{ k}\uparrow \\ A_{HV}^{SL} & = & C_{HV}^{SL} & = & (.429)(27.) & = & 11.58\text{ k}\text{rgt} \end{array}$$

4. The reactions due to wind load (blowing from the left), are determined as follows:

   4.1 Vertical reaction at A is determined by considering the entire structure and taking the moment with respect to C, (c)

$$(+\circlearrowleft)\ \Sigma M_z^C = 0; \quad (15.75)(\tfrac{20+15}{2}) - 60A_{VH} = 0 \quad \Rightarrow A_{VH} = 4.60 \text{ k} \uparrow$$

   A_{VH} is the Vertical reaction at A due to the Horizontal load (The double subscript notation is extensively used in structural analysis. X_{yz} typically implies quantity X at y due to some action z.) and from equilibrium of forces in the y direction, we have $B_{VH} = -A_{VH} = -4.60 \downarrow$ (note that wind load does not have any vertical component).

   4.2 The horizontal reaction at B is determined by considering the right substructure and taking moment with respect to the internal hinge at B

$$(+\circlearrowleft)\ \Sigma M_z^B = 0; \quad 35C_{HH} - (4.6)(30) = 0 \quad \Rightarrow C_{HH} = 3.95 \text{ klft}$$

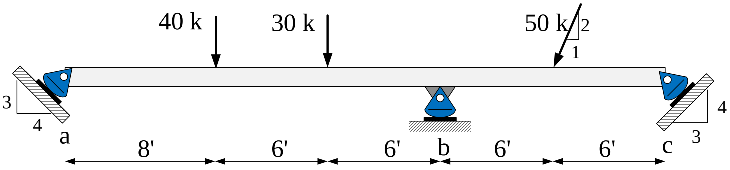
4.3 Horizontal reaction at A is taken by considering the entire structure and summing forces in the x direction:

$$(+\text{rgt})\ \Sigma F_x = 0; \quad 15.75 - 3.95 - A_{HH} = 0 \quad \Rightarrow A_{HH} = 11.80 \text{ k}\text{lft}$$

and note that A carries most of the horizontal load.

5. Finally, the supports should be designed for themost critical (plausible) combination of reactions:

$$\begin{aligned} H = &\quad 8.66 \text{ k} + 11.58 \text{ k} + 11.8 \text{ k} &= \quad 32.04 \text{ k} \\ V = &\quad 20.2 \text{ k} + 27.0 \text{ k} + 4.60 \text{ k} &= \quad 51.8 \text{ k} \end{aligned}$$



A priori we would identify 5 reactions, however we do have 2 equations of conditions (one at each inclined support), thus with three equations of equilibrium, we have a statically determinate system.

$$\left(+\circlearrowleft\right)\ \Sigma M_z^b = 0;\quad (R_{ay})(20) - (40)(12) - (30)(6) + (44.72)(6) - (R_{cy})(12) = 0$$
$$\Rightarrow 20R_{ay} = 12R_{cy} + 391.68$$
$$(+\text{rgt})\ \Sigma F_x = 0;\quad \frac{3}{4}R_{ay} - 22.36 - \frac{4}{3}R_{cy} = 0$$
$$\Rightarrow R_{cy} = 0.5625R_{ay} - 16.77$$

Solving for those two equations:

$$\begin{bmatrix} 20 & -12 \\ 0.5625 & -1 \end{bmatrix} \left\{ \begin{array}{c} R_{ay} \\ R_{cy} \end{array} \right\} = \left\{ \begin{array}{c} 391.68 \\ 16.77 \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} R_{ay} \\ R_{cy} \end{array} \right\} = \left\{ \begin{array}{c} 14.37 \text{ k} \uparrow \\ -8.69 \text{ k} \downarrow \end{array} \right\}$$

The horizontal components of the reactions at a and c are

$$\begin{aligned} R_{ax} &= \tfrac{3}{4}14.37 &= &\ 10.78 \text{ krgt} \\ R_{cx} &= \tfrac{4}{3}8.69 &= &\ -11.59 \text{ krgt} \end{aligned}$$

Finally we solve for R_{by}

$$\left(+ \circlearrowleft\right) \Sigma M_z^a = 0; \quad (40)(8) + (30)(14) - (R_{by})(20) + (44.72)(26) + (8.69)(32) = 0$$

$$\Rightarrow R_{by} = 109.04 \text{ k} \uparrow$$

We check our results

$$\begin{aligned} \left(+ \uparrow\right) \Sigma F_y = 0; &\quad 14.37 - 40 - 30 + 109.04 - 44.72 - 8.69 &= 0\surd \\ (+\text{rgt})\ \Sigma F_x = 0; &\quad 10.78 - 22.36 + 11.59 &= 0\surd \end{aligned}$$

Structural Analysis

Trusses

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Table of Contents I

- Cables and trusses are 2D or 3D structures composed of an assemblage of simple one dimensional components which transfer only axial forces along their axis.
- Trusses are extensively used for bridges, long span roofs, electric tower, space structures.
- For trusses, it is assumed that
  - Bars are pin-connected (equation of conditions)
  - Joints are frictionless hinges [1].
  - Loads are applied at the joints only.
- A truss would typically be composed of triangular elements with the bars on the upper chord under compression and those along the lower chord under tension. Depending on the orientation of the diagonals, they can be under either tension or compression.

[1] In practice the bars are riveted, bolted, or welded directly to each other or to gusset plates, thus the bars are not free to rotate and so-called secondary bending moments are developed at the bars. Another source of secondary moments is the dead weight of the element.

Sign Convention: Tension positive, compression negative. On a truss the axial forces are indicated as forces acting on the joints.

Stress-Force: $\sigma = \frac{P}{A}$

Stress-Strain: $\sigma = E\varepsilon$

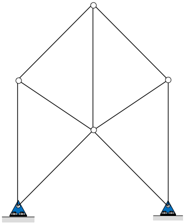
Force-Displacement: $\varepsilon = \frac{\Delta L}{L}$

Equilibrium: $\Sigma F = 0$

- Trusses are statically determinate when all the bar forces can be determined from the equations of statics alone. Otherwise the truss is **statically indeterminate**.
- A truss may be externally or internally determinate or indeterminate.
- If we refer to j as the number of joints, R the number of reactions and m the number of members, then we would have a total of $m + R$ unknowns and $2j$ (or $3j$) equations of statics (2D or 3D at each joint). If we do not have enough equations of statics then the problem is indeterminate, if we have too many equations then the truss is unstable.

	2D	3D
Static Indeterminacy		
External	$R > 3$	$R > 6$
Internal	$m + R > 2j$	$m + R > 3j$
Unstable	$m + R < 2j$	$m + R < 3j$

- If $m < 2j - 3$ (in 2D) the truss is unstable, and it will not remain a rigid body when it is detached from its supports. However, when attached to the supports, the truss will be rigid.
- The external equations of equilibrium which can be used to determine the reactions are
  - 2D $\Sigma F_X = 0$, $\Sigma F_Y = 0$ and $\Sigma M_Z = 0$.
  - For 3D trusses the available equations are $\Sigma F_X = 0$, $\Sigma F_Y = 0$, $\Sigma F_Z = 0$ and $\Sigma M_X = 0$, $\Sigma M_Y = 0$, $\Sigma M_Z = 0$.
- At each joint
  - For a 2D truss: $\Sigma F_X = 0$ and $\Sigma F_Y = 0$.
  - For 3D trusses: $\Sigma F_X = 0$, $\Sigma F_Y = 0$ and $\Sigma F_Z = 0$.



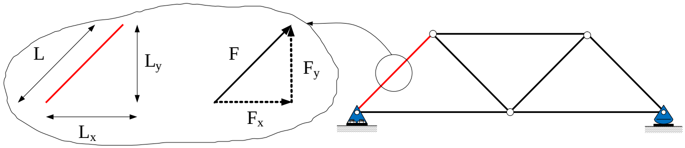
- 4 reactions, thus it is externally indeterminate.
- 6 joints ($j = 6$), 4 reactions ($R = 4$) and 9 members ($m = 9$).
- A total of $m + R = 9 + 4 = 13$ unknowns and $2 \times j = 2 \times 6 = 12$ equations of equilibrium, thus the truss is internally statically indeterminate.

- There are two methods of analysis for statically determinate trusses
  1. The Method of joints
  2. The Method of sections

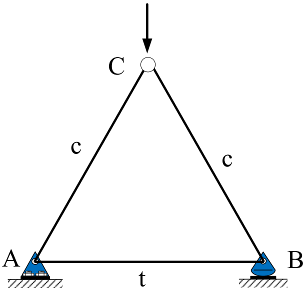
- The method of joints can be summarized as follows
  - Determine if the structure is statically determinate
  - Compute all reactions
  - Sketch a free body diagram showing all joint loads (including reactions)
  - For each joint, and starting with the loaded ones, apply the appropriate equations of equilibrium (ΣF_x and ΣF_y in 2D; ΣF_x, ΣF_y and ΣF_z in 3D).
  - Because truss elements can only carry axial forces, the resultant force ($\vec{F} = \vec{F_x} + \vec{F_y}$) must be along the member.

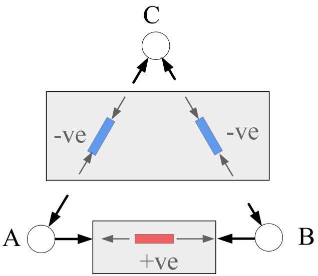
$$\frac{F}{L} = \frac{F_x}{L_x} = \frac{F_y}{L_y}$$

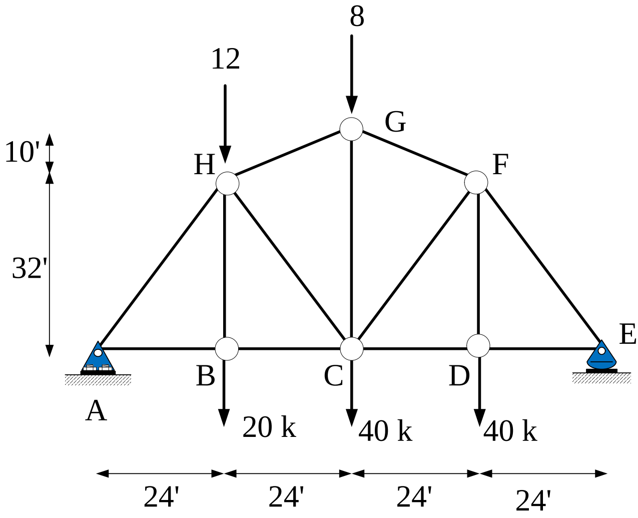
- Always keep track of the x and y components of a member force (F_x, F_y), as those might be needed later on when considering the force equilibrium at another joint to which the member is connected.



- This method should be used when all member forces must be determined.
- In truss analysis, there is no sign convention. A member is assumed to be under tension (or compression). If after analysis, the force is found to be negative, then this would imply that the wrong assumption was made, and that the member should have been under compression (or tension).
- On a free body diagram, the internal forces are represented by arrow acting on the joints and not as end forces on the element itself. That is for tension, the arrow is pointing away from the joint, and for compression toward the joint.


C
c
c
A
B
t


C
-ve
-ve
A
+ve
B

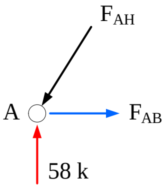

8
12
G
H
F
10'
32'
E
A
B
C
D
20 k
40 k
40 k
24'
24'
24'
24'

1. $R = 3$, $m = 13$, $2j = 16$, and $m + R = 2j\surd$
2. We compute the reactions

$$
\begin{aligned}
(+\circlearrowleft)\ \Sigma M_z^E = 0; \quad &\Rightarrow \quad (20+12)(3)(24) + (40+8)(2)(24) + (40)(24) \\
& \quad -R_{A_y}(4)(24) = 0 \Rightarrow R_{A_y} = \boxed{58 \text{ k} \uparrow} \\
(+\downarrow)\ \Sigma F_y = 0; \quad &\Rightarrow \quad 20+12+40+8+40-58-R_{E_y} = 0 \\
&\Rightarrow \quad R_{E_y} = \boxed{62 \text{ k} \uparrow}
\end{aligned}
\tag{1}
$$

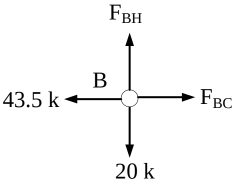
3. Consider each joint separately:

Node A: Clearly *AH* is under compression, and *AB* under tension.



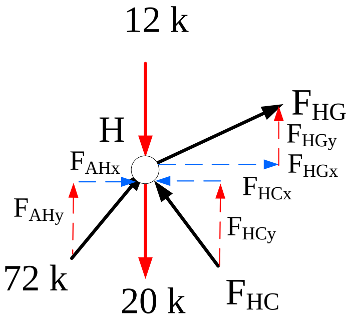
$$
\begin{aligned}
(+\uparrow)\ \Sigma F_y = 0; \quad &\Rightarrow \quad -F_{AH_y} + 58 = 0 \\
&\quad\ \ F_{AH} = \frac{l}{l_y}(F_{AH_y}) \\
&\quad\ \ l_y = 32; l = \sqrt{32^2+24^2} = 40 \\
&\Rightarrow \quad F_{AH} = \frac{40}{32}(58) = \boxed{72.5 \text{ k Compression}} \\
(+\text{rgt})\ \Sigma F_x = 0; \quad &\Rightarrow \quad -F_{AH_x} + F_{AB} = 0 \\
&\quad\ \ F_{AB} = \frac{L_x}{L_y}(F_{AH_y}) = \frac{24}{32}(58) = \boxed{43.5 \text{ k Tension}}
\end{aligned} \tag{2}
$$

Node B:



$$
\begin{aligned}
(+\text{rgt})\ \Sigma F_x = 0; \quad &\Rightarrow \quad F_{BC} = \boxed{43.5 \text{ k Tension}} \\
(+\uparrow)\ \Sigma F_y = 0; \quad &\Rightarrow \quad F_{BH} = \boxed{20 \text{ k Tension}}
\end{aligned} \tag{3}
$$

Node H:


12 k
H
F_HG
F_HGy
F_HGx
F_AHx
F_HCx
F_AHy
F_HCy
72 k
20 k
F_HC

$$
\begin{array}{lll}
(+\text{rgt})\,\Sigma F_x = 0; & \Rightarrow & F_{AH_x} - F_{HC_x} + F_{HG_x} = 0 \\
 & & 43.5 - \frac{24}{\sqrt{24^2+32^2}}(F_{HC}) + \frac{24}{\sqrt{24^2+10^2}}(F_{HG}) = 0 \\
(+\uparrow)\,\Sigma F_y = 0; & \Rightarrow & F_{AH_y} + F_{HC_y} - 12 + F_{HG_y} - 20 = 0 \\
 & & 58 + \frac{32}{\sqrt{24^2+32^2}}(F_{HC}) - 12 + \frac{10}{\sqrt{24^2+10^2}}(F_{HG}) - 20 = 0
\end{array}
\tag{4}
$$

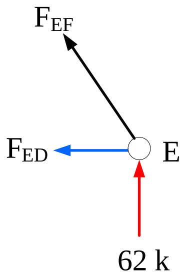
This can be most conveniently written as

$$
\square \left\{ \begin{array}{c} F_{HC} \\ F_{HG} \end{array} \right\} = \left\{ \begin{array}{c} 43.5 \\ 26.0 \end{array} \right\}
$$

Solving we obtain $F_{HC} = -7.5$ and $F_{HG} = -52$, thus we made an erroneous assumption in the free body diagram of node H, and the final answer is

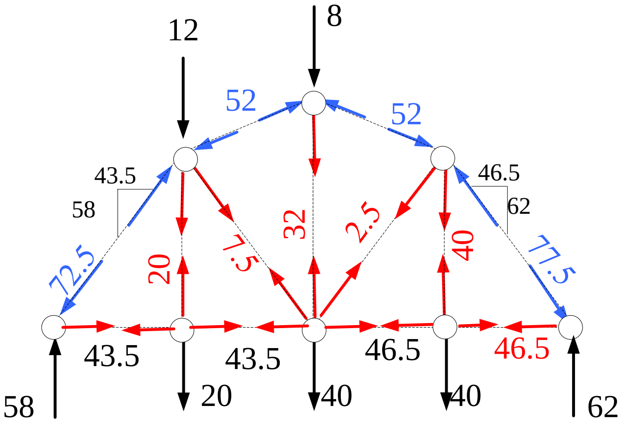
$$
\begin{array}{lll}
F_{HC} & = & \boxed{\text{7.5 k Tension}} \\
F_{HG} & = & \boxed{\text{52 k Compression}}
\end{array}
\tag{5}
$$

Node E:

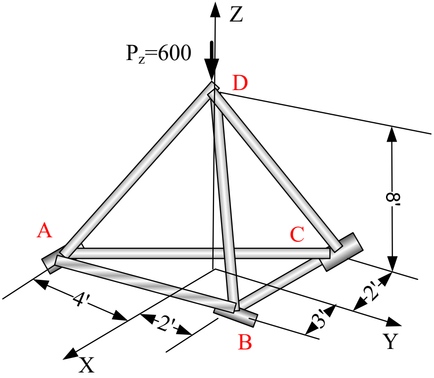


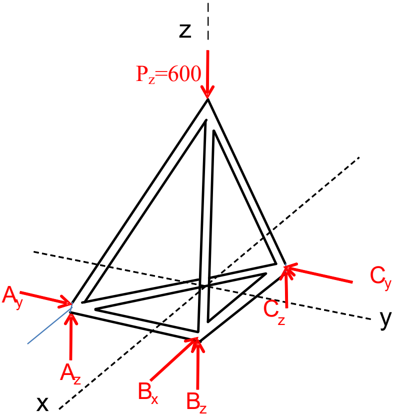
$$
\begin{array}{lllll}
\Sigma F_y = 0; & \Rightarrow & F_{EF_y} = 62 & \Rightarrow F_{EF} = \frac{\sqrt{24^2+32^2}}{32}(62) & = \boxed{77.5 \text{ k C}} \\
\Sigma F_x = 0; & \Rightarrow & F_{ED} = F_{EF_x} & \Rightarrow F_{ED} = \frac{24}{32}(F_{EF_y}) = \frac{24}{32}(62) & = \boxed{46.5 \text{ k T}}
\end{array}
\tag{6}
$$

The results of this analysis are summarized below



4. We could check our calculations by verifying equilibrium of forces at a node not previously used, such as *D*





$A(3,-4,0)$, $B(3,2,0)$, $C(-2,2,0)$, and $D(0,0,8)$.
No reactions in the x or y direction (structure is on ice, and there is no lateral load).
Steps:

1. Reactions
   1. $\Sigma M_{AB} = 0 \Rightarrow C_z \surd$
   2. $\Sigma M_{CB} = 0; \Rightarrow A_z \surd$
   3. $\Sigma F_z = 0; \Rightarrow B_z = 40.0 \surd$
   4. $\Sigma F_x = 0; \Rightarrow B_x \surd$
   5. $\Sigma F_y = 0; \Rightarrow A_y \surd; C_y \surd$
2. Joint B
   1. L_{BD}
   2. $\Sigma F_z = 0; \Rightarrow F_{BD} \surd$
   3. F_{BD}^x
   4. F_{BD}^y
   5. $\Sigma F_x = 0; \Rightarrow F_{BA} \surd$
3. joint A
   1. $\alpha \surd$
   2. $L_{AD} \surd$
   3. $\Sigma F_z = 0; F_{AD} \surd$
   4. $\Sigma F_x = 0; F_{AC} \surd$

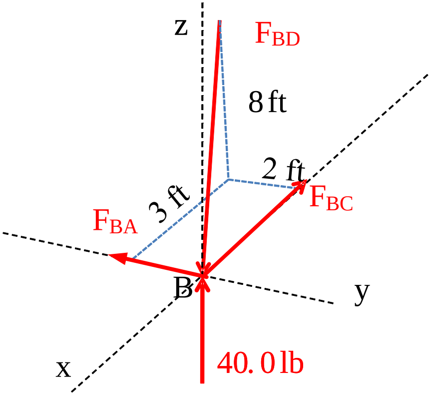
4. Joint C
   1. $F_{CD}^{Z}\surd$
   2. $\Sigma F_Z = 0 \Rightarrow F_{CD}\surd$

Solution

1. Considering the free body diagram of the entire truss

$$
\begin{array}{rll}
\Sigma M_{AB} = 0; & C_z(5) - 600(3) = 0 & \Rightarrow C_z = 360 \\
\Sigma M_{CB} = 0; & 600(2) - A_z(6) = 0 & \Rightarrow A_z = 200 \\
\Sigma F_z = 0; & B_z + 200 + 360 - 600 = 0 & \Rightarrow B_z = 40.0 \\
\Sigma F_x = 0; & & B_x = 0 \\
\Sigma F_y = 0; & A_y - C_y = 0 & \\
\Sigma M_z = 0; A_y(3) + C_y(2) = 0; & & \Rightarrow A_y = C_y = 0
\end{array}
$$

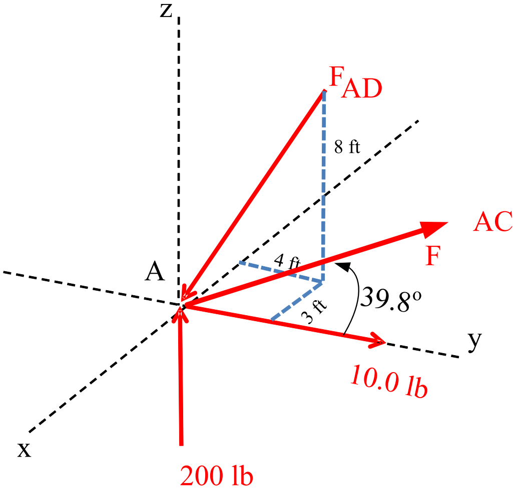
2. Considering the free body diagram of joint B



Joint B

$$\begin{aligned}
L_{BD} &= \sqrt{L_x^2 + L_y^2 + L_z^2} && = \sqrt{2^2+3^2+8^2} = \sqrt{77}\\
\Sigma F_z = 0; &\quad \frac{-8}{\sqrt{77}} F_{BD} + 40 = 0; && \Rightarrow F_{BD} = 43.87 lbf(C)\\
\Sigma F_x = 0; &\quad F_{BD}^x - F_{BC} = 0\\
F_{BD}^x = \frac{L_x}{L} F_{BD} &= \frac{3}{\sqrt{77}}(43.87)\\
F_{BC} &= 15.0 lbf(T)\\
\Sigma F_y = 0; &\quad F_{BD}^y - F_{BA} = 0\\
F_{BD}^y &= \frac{L_x}{L} F_{BD} = \frac{2}{\sqrt{77}}(43.87)\\
F_{BA} &= 10.0 lbf(T)
\end{aligned}$$

3. FBD of joint A

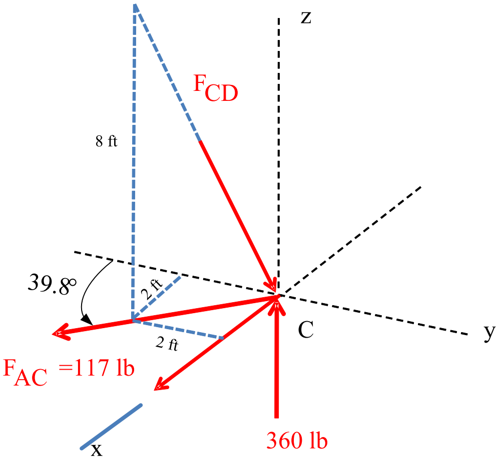

z
F_{AD}
8 ft
AC
F
A
4 ft
39.8°
3 ft
y
10.0 lb
x
200 lb

$$\begin{aligned}
\tan(\alpha) &= \frac{L_x}{L_y} = \frac{5}{6} \Rightarrow \alpha = 39.8\,\text{deg} \\
L_{AD} &= \sqrt{L_x^2 + L_y^2 + L_z^2} = \sqrt{8^2 + 3^2 + 4^2} = \sqrt{89} \\
\Sigma F_z = 0; \quad & \frac{-8}{\sqrt{89}} F_{AD} + 200 = 0 \Rightarrow F_{AD} = 236 lbf(C) \\
\Sigma F_x = 0; \quad & F_{AD}^X - F_{AC}^X = 0 \Rightarrow \frac{3}{\sqrt{89}}(235.9) - F_{AC}\sin(39.8^o) = 0 \\
& \Rightarrow F_{AC} = 117 lbf(T)
\end{aligned}$$

4. Check

$$\begin{aligned}
\Sigma F_y = 0 F_{AC}^Y - F_{AD}^Y + 10 &= 0 \\
117.2\cos(39.81^o) - \frac{4}{\sqrt{89}}(235.9) + 10.0 &= 0\surd
\end{aligned}$$

5. Joint C

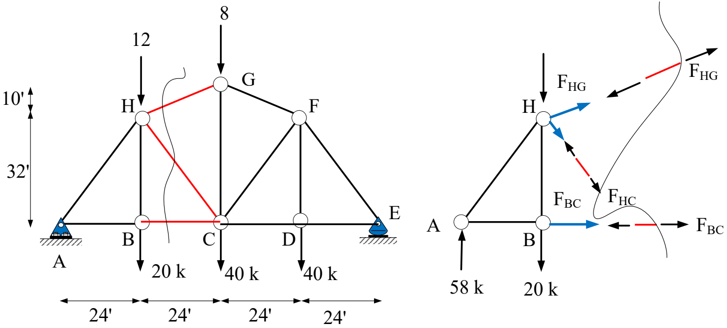


$$
\begin{aligned}
\Sigma F_Z = 0; \quad F^Z_{CD} + 360 &= 0 \\
\frac{-8}{\sqrt{72}} F_{CD} + 360 &= 0 \\
F_{CD} &= 382 lbf(C)
\end{aligned}
$$

- When only forces in selected members (away from loaded joints) is to be determined, this method should be used.
- This method can be summarized as follows
    - "Cut" the truss into two substructures by an imaginary line (not necessarily straight) such that it will at least intersect the member for which force is to be determined.
    - Consider either one of the two substructures as the free body
    - Each substructure must remain in equilibrium. Apply the equations of equilibrium
        - Summation of moments about a particular point (usually the intersection of 2 cut members) would permit the determination of other member forces
        - Summation of forces is usually used to determine forces in inclined members

Determine F_{BC} and F_{HG} in the previous example.

- Cutting through members *HG*, *HC* and *BC*, we first take the summation of forces with respect to H:



Cut through the members, but in drawing FBD, remove them and just show nodal forces

$$
\begin{array}{lll}
(+\circlearrowleft)\ \Sigma M_z^H = 0 & \Rightarrow & R_{A_y}(24) - F_{BC}(32) = 0 \\
 & & F_{BC} = \frac{24}{32}(58) = \boxed{43.5 \text{ k Tension}} \\
(+\circlearrowleft)\ \Sigma M_z^C = 0; & \Rightarrow & (58)(24)(2) - (20+12)(24) - F_{HG_x}(32) - F_{HG_y}(24) = 0 \\
 & & 2784 - 768 - (32)(F_{HG})\frac{24}{\sqrt{24^2+10^2}} - \\
 & & (24)(F_{HG})\frac{10}{\sqrt{24^2+10^2}} = 0 \\
 & & 2,000 - (29.5)F_{HG} - (9.2)F_{HG} = 0 \\
 & \Rightarrow & F_{HG} = \boxed{52 \text{ k Compression}}
\end{array} \tag{7}
$$

Structural Analysis

Internal Forces

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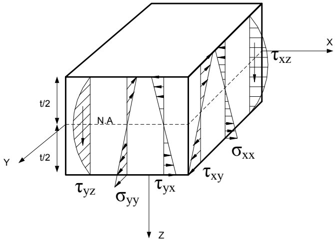
Spring 2022

Table of Contents I

Reactions → **Internal Forces** → Deflections → Flexibility Method (SIS)

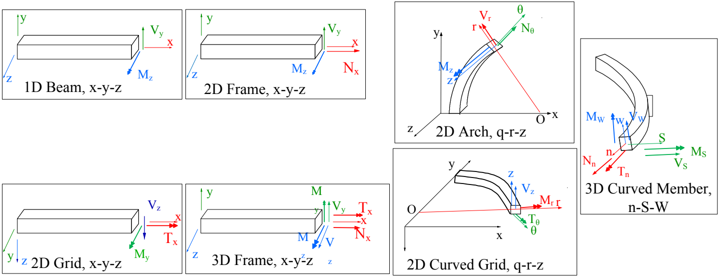
- Ultimately, we are interested in the internal stresses in a three dimensional structure.
- Problem is too complex, we thus take advantage of shape, and categorize structures as shells, plates or beams.
- In those problems, instead of solving for the stress components throughout the body, we solve for certain stress resultants (normal, shear forces, and Moments and torsions) resulting from an integration over the body.
- Alternatively, if a continuum solution is desired, and engineering theories prove to be either too restrictive or inapplicable, we can use numerical techniques (such as Finite Element Method) to solve the problem.

Internal forces are integrals of stresses in a plate/beam.

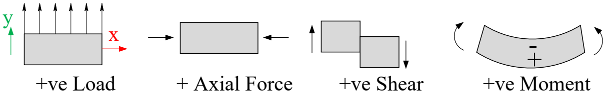


and the resultants *per unit width* are given by

Membrane (Axial) Forces	$\mathrm{N} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma dz$	$\rightarrow N_{xx}$	$= \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx} dz$ etc.
Bending Moments	$\mathrm{M} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma z dz$	$\rightarrow M_{yy}$	$= \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx} z dz$ etc.
Transverse Shear Forces	$\mathrm{V} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau dz$	$\rightarrow V_{yy}$	$= \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz} dz$ etc.



Cartesian						
	Forces			Moments		
	x	*y*	*z*	*x*	*y*	*z*
Beam		V_y				M_z
2D Frame	N_x	V_y				M_z
Grid			V_z	T_x	M_y	
Polar						
	Forces			Moments		
	r	θ	*z*	*r*	θ	*z*
Arch	V_r	N_θ				M_z
Curved Grid			V_z	M_r	T_θ	
Curved						
	Forces			Moments		
	n	*s*	*w*	*n*	*s*	*w*
Curved	N_n	V_s	V_w	T_n	M_s	M_w



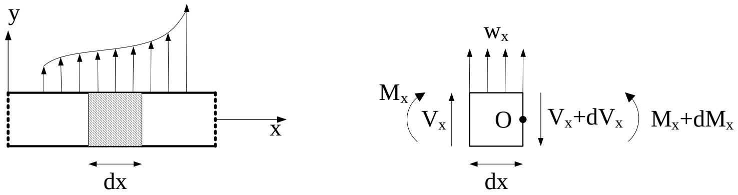
Load Positive along the beam's local y axis (assuming a right hand side convention), that is positive upward.

Axial: tension positive.

Flexure A positive moment is one which causes tension in the lower fibers, and compression in the upper ones. Moments are drawn on the compression side (useful to keep in mind for frames).

Shear A positive shear force is one which is "up" on a negative face. Alternatively, a pair of positive shear forces will cause clockwise rotation.

Torsion Counterclockwise positive



- The infinitesimal section must also be in equilibrium.
- There are no axial forces, thus we only have two equations of equilibrium to satisfy $\Sigma F_y = 0$ and $\Sigma M_z = 0$.
- Since dx is infinitesimally small, the small variation in load along it can be neglected, therefore we assume $w(x)$ to be constant along dx.
- To denote that a small change in shear and moment occurs over the length dx of the element, we add the differential quantities dV_x and dM_x to V_x and M_x on the right face.

Considering the first equation of equilibrium
$(+\uparrow)\,\Sigma F_y = 0 \Rightarrow V_x + w_x dx - (V_x + dV_x) = 0 \Rightarrow \frac{dV}{dx} = w(x)$

The slope of the shear curve at any point along the axis of a member is given by the load curve at that point.

Similarly $(+\circlearrowleft)\,\Sigma M_O = 0 \Rightarrow M_x + V_x dx - w_x dx \frac{dx}{2} - (M_x + dM_x) = 0$
Neglecting the dx^2 term, this simplifies to $\frac{dM}{dx} = V(x)$

The slope of the moment curve at any point along the axis of a member is given by the shear at that point.

$$
\begin{aligned}
V &= \int w(x)dx \\
\Delta V_{21} &= V_{x_2} - V_{x_1} = \int_{x_1}^{x_2} w(x)dx
\end{aligned}
$$

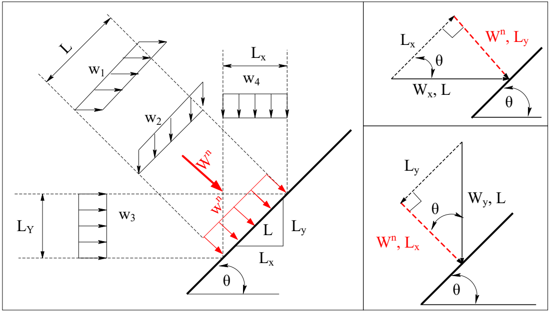
The change in shear between 1 and 2, ΔV_{1-2}, is equal to the area under the load between x_1 and x_2.

$$
\begin{aligned}
M &= \int V(x)dx \\
\Delta M_{21} &= M_{x2} - M_{x1} = \int_{x_1}^{x_2} V(x)dx
\end{aligned}
$$

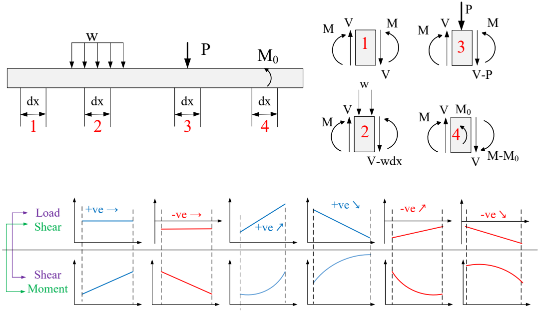
The change in moment between 1 and 2, ΔM_{21}, is equal to the area under the shear curve between x_1 and x_2.

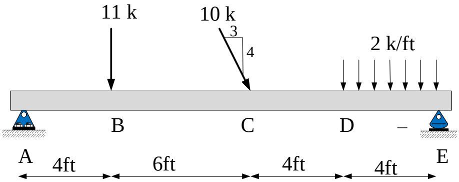
Note that we still need to have V_1 and M_1 in order to obtain V_2 and M_2.

Inclined Members/Loads



Lateral inertial force x	$W_1^x = w_1^x L;$	$W^n = W_1^x \frac{L_y}{L};$	$w^n = \frac{W^n}{L}$	$= w_1^x \frac{L_y}{L}$
Self weight y	$W_2^y = w_2^y L;$	$W^n = W_2^y \frac{L_x}{L};$	$w^n = \frac{W^n}{L}$	$= w_2^y \frac{L_x}{L}$
Wind Load	$W_3^x = w_3^x L_y;$	$W^n = W_3^x \frac{L_y}{L};$	$w^n = \frac{W^n}{L}$	$= w_3^x \frac{L_y^2}{L^2}$
Snow Load	$W_4^y = w_4^y L_x;$	$W^n = W_4^y \frac{L_x}{L};$	$w^n = \frac{W^n}{L}$	$= w_4^y \frac{L_x^2}{L^2}$


w
P
M_0
dx
1
2
3
4
M
V
V-P
V-wdx
$M-M_0$
Load
Shear
Moment
+ve →
-ve →
+ve ↗
+ve ↘
-ve ↗
-ve ↘



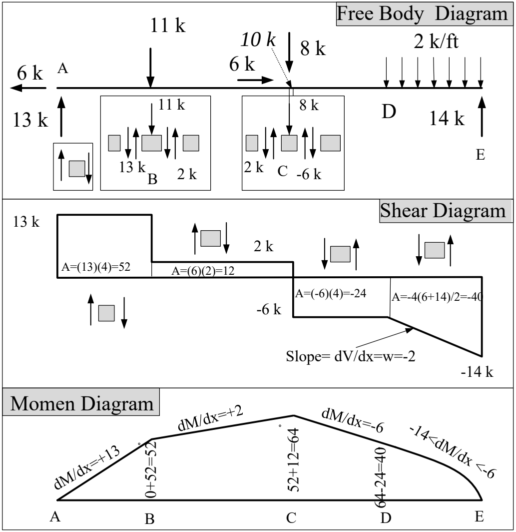
- Reactions are determined from the equilibrium equations

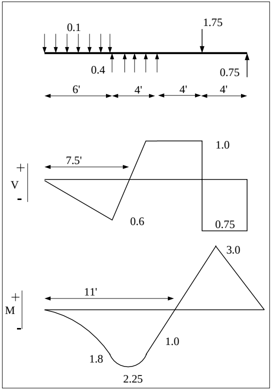
$$
\begin{aligned}
(+\text{lft})\ \Sigma F_x = 0; \quad &\Rightarrow \quad -A_x + 6 = 0 \Rightarrow A_x = 6 \text{ k} && (1)\\
(+\circlearrowright)\ \Sigma M_A = 0; \quad &\Rightarrow \quad (11)(4) + (8)(10) + (4)(2)(14+2) - E_y(18) = 0 && (2)\\
\Rightarrow R_{E_y} \quad &= \quad 14 \text{ k} && (3)\\
(+\uparrow)\ \Sigma F_y = 0; \quad &\Rightarrow \quad A_y - 11 - 8 - (4)(2) + 14 = 0 \Rightarrow A_y = 13 \text{ k} && (4)
\end{aligned}
$$

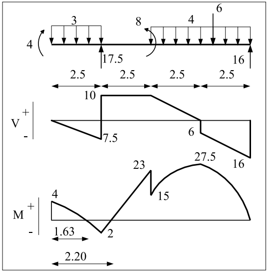
- Shear are determined next.
  1. At A the shear is equal to the reaction and is positive.
  2. At B the shear drops (negative load) by 11 k to 2 k.
  3. At C it drops again by 8 k to -6 k.

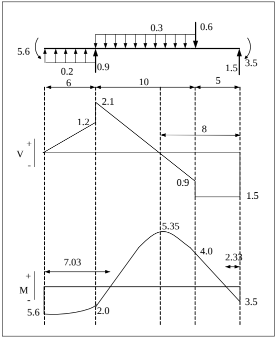
4. It stays constant up to D and then it decreases (constant negative slope since the load is uniform and negative) by 2 k per linear foot up to -14 k.
5. As a check, -14 k is also the reaction previously determined at F.

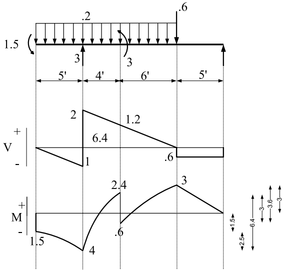
- Moment is determined last:
  1. The moment at A is zero (hinge support).
  2. The change in moment between A and B is equal to the area under the corresponding shear diagram, or $\Delta M_{B-A} = (13)(4) = 52$.
  3. etc...

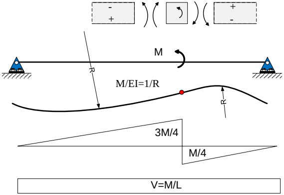

Free Body Diagram
11 k
10 k
8 k
2 k/ft
6 k
A
6 k
13 k
11 k
13 k
B
2 k
8 k
2 k
C
-6 k
D
14 k
E
Shear Diagram
13 k
A=(13)(4)=52
A=(6)(2)=12
2 k
A=(-6)(4)=-24
A=-4(6+14)/2=-40
-6 k
Slope= dV/dx=w=-2
-14 k
Momen Diagram
dM/dx=+13
dM/dx=+2
dM/dx=-6
-14<dM/dx <-6
0+52=52
52+12=64
64-24=40
A
B
C
D
E

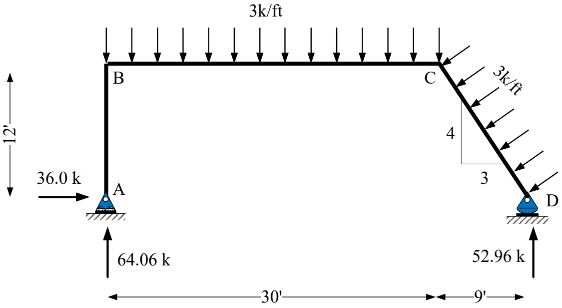

0.1
1.75
0.4
0.75
6'
4'
4'
4'
1.0
7.5'
V
0.6
0.75
3.0
11'
M
1.0
1.8
2.25

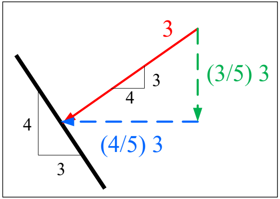

3
8
4
6
4
17.5
16
2.5
2.5
2.5
2.5
10
V
7.5
6
16
23
27.5
4
15
M
1.63
2
2.20

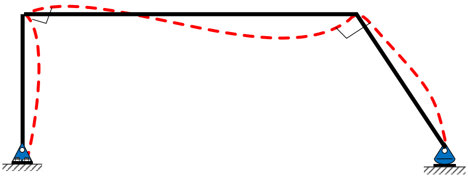

0.3
0.6
5.6
0.2
0.9
1.5
3.5
6
10
5
2.1
1.2
8
V
0.9
1.5
5.35
4.0
7.03
2.33
M
5.6
2.0
3.5


.6
.2
1.5
3
3
5'
4'
6'
5'
2
1.2
6.4
V
1
.6
2.4
3
M
1.5
4
.6


M
M/EI=1/R
R
R
3M/4
M/4
V=M/L


3k/ft
B
C
3k/ft
4
3
12'
36.0 k
A
D
64.06 k
52.96 k
30'
9'


3
3
4
(3/5) 3
4
3
(4/5) 3



Reactions

$$(+_{\text{lft}})\ \Sigma F_x = 0; \quad \Rightarrow \quad R_{A_x} - \underbrace{\frac{4}{5}(3)(15)}_{\text{load}} = 0 \Rightarrow R_{A_x} = 36 \text{ k}$$

$$\left(+\circlearrowleft\right) \Sigma M_A = 0; \quad \Rightarrow \quad (3)(30)(\frac{30}{2}) + \underbrace{\frac{3}{5}(3)(15)\left(30+\frac{9}{2}\right)}_{CD_Y} - \underbrace{\frac{4}{5}(3)(15)\frac{12}{2}}_{CD_X} - 39R_{D_y} = 0$$

$$\Rightarrow \quad R_{D_y} = 52.96 \text{ k}$$

$$\left(+\uparrow\right) \Sigma F_y = 0; \quad \Rightarrow \quad R_{A_y} - (3)(30) - \frac{3}{5}(3)(15) + 52.96 = 0$$

$$\Rightarrow \quad R_{A_y} = 64.06 \text{ k}$$

Shear Diagram:

1. For $A - B$, the shear is constant, equal to the horizontal reaction at A and negative according to our previously defined sign convention, $V_A = -36$ k
2. For member $B - C$ at B, the shear must be equal to the vertical force which was transmitted along $A - B$, and which is equal to the vertical reaction at A, $V_B = 64.06$.

3. Since $B-C$ is subjected to a uniform negative load, the shear along $B-C$ will have a slope equal to -3 and in terms of x (measured from B to C) is equal to

$$V_{B-C}(x) = 64.06 - 3x$$

4. The shear along $C-D$ is obtained by decomposing the vertical reaction at D into axial and shear components. $V = \frac{3}{5}52.96 = 31.78$ k and is negative. Slope of the shear must be equal to -3 along $C-D$. Shear at C is such that $V_c - \frac{5}{3}9(3) = -31.78$ or $V_c = 13.22$.

$$V = 13.22 - 3x$$

5. We check our calculations by verifying equilibrium of node C

$$(+_{\text{lft}})\ \Sigma F_x = 0 \quad \Rightarrow \quad \frac{3}{5}(42.37) + \frac{4}{5}(13.22) = 25.42 + 10.58 = 36\surd$$
$$(+\uparrow)\ \Sigma F_y = 0 \quad \Rightarrow \quad \frac{4}{5}(42.37) - \frac{3}{5}(13.22) = 33.90 - 7.93 = 25.97\surd$$

Moment:

1. Along $A - B$, moment is zero at A, and its slope is equal to the shear, thus at B the moment is equal to $(-36)(12) = -432$ k.ft
2. Along $B - C$, the moment is equal to

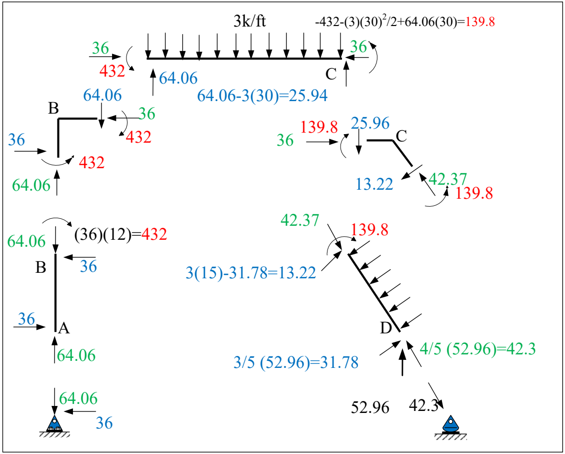
$$M_{B-C} = M_B + \int_0^x V_{B-C}(x)dx = -432 + \int_0^x (64.06 - 3x)dx = -432 + 64.06x - 3\frac{x^2}{2}$$

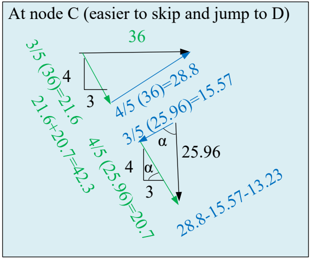
   which is a parabola. Substituting for $x = 30$, we obtain at C:
   $M_C = -432 + 64.06(30) - 3\frac{30^2}{2} = 139.8$ k.ft
3. Alternatively: $(+ \circlearrowleft)\ \Sigma M_C = 0 \Rightarrow -432 - 3\frac{(30)^2}{2} + 64.06 - M_c = 0$ Solving gives $M_c = 139.8$ positive.
4. For the maximum moment along $B - C$, we know that $\frac{dM_{B-C}}{dx} = 0$ at the point where $V_{B-C} = 0$, that is $V_{B-C}(x) = 64.06 - 3x = 0 \Rightarrow x = \frac{64.06}{3} = 21.35$ ft. i.e., maximum moment occurs where the shear is zero. Thus
   $M_{B-C}^{max} = -432 + 64.06(21.35) - 3\frac{(21.35)^2}{2} = -432 + 1,367.7 - 683.7 = -251.98$ k.ft

5. Along $C-D$, the moment varies quadratically (linear shear), the moment first increases (positive shear), and then decreases (negative shear). The moment along $C-D$ is given by

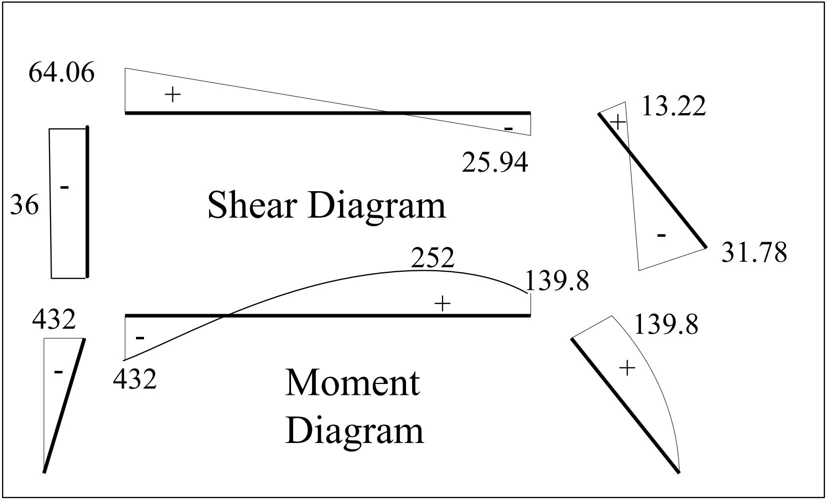
$$M_{C-D} = M_C + \int_0^x V_{C-D}(x)dx = 139.8 + \int_0^x (13.22 - 3x)dx = 139.8 + 13.22x - 3\frac{x^2}{2}$$

which is a parabola. Substituting for $x = 15$, we obtain at node C
$M_C = 139.8 + 13.22(15) - 3\frac{15^2}{2} = 139.8 + 198.3 - 337.5 = 0\surd$



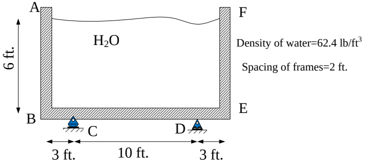


cos α=3/5=N/25.96
N=(25.96)(3)/(5)=15.57


64.06
+
-
25.94
13.22
+
36
-
Shear Diagram
-
31.78
252
139.8
432
+
-
139.8
-
432
+
Moment
Diagram

The frame shown below is the structural support of a flume. Assuming that the frames are spaced 2 ft apart along the length of the flume,

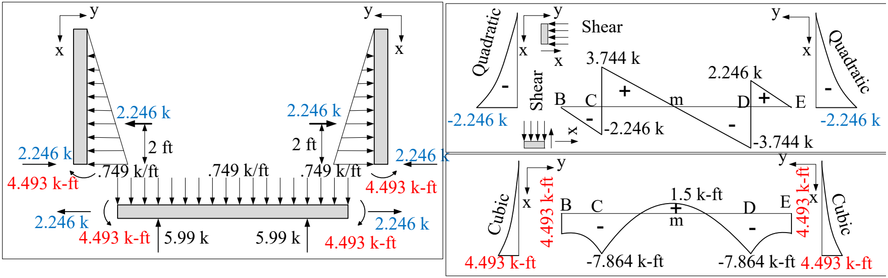
1. Determine all internal member end actions
2. Draw the shear and moment diagrams
3. Locate and compute maximum internal bending moments
4. If this is a reinforced concrete frame, show the location of the reinforcement.



The hydrostatic pressure causes lateral forces on the vertical members which can be treated as cantilevers fixed at the lower end. The pressure is linear and is given by $p = \gamma h$. Since each frame supports a 2 ft wide slice of the flume, the equation for w (pounds/foot) is

$$w = (2)(62.4)(h) = 124.8h \text{ lbs/ft}$$

At the base $w = (124.8)(6) = 749$ lbs/ft $= .749$ k/ft Note that this is both the lateral pressure on the end walls as well as the uniform load on the horizontal members.



End Actions

1. Base force at B is $F_{Bx} = (.749)\frac{6}{2} = 2.246$ k
2. Base moment at B is $M_B = (2.246)\frac{6}{3} = 4.493$ k.ft
3. End force at B for member $B-E$ are equal and opposite.
4. Reaction at C is $R_{Cy} = (.749)\frac{16}{2} = 5.99$ k

Shear forces

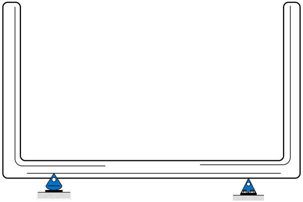
1. Base at B the shear force was determined earlier and was equal to 2.246 k. Based on the orientation of the $x-y$ axis, this is a negative shear.
2. The vertical shear at B is zero (neglecting the weight of $A-B$)

3. The shear to the left of C is $V = 0 + (-.749)(3) = -2.246$ k.
4. The shear to the right of C is $V = -2.246 + 5.99 = 3.744$ k

Moment diagrams

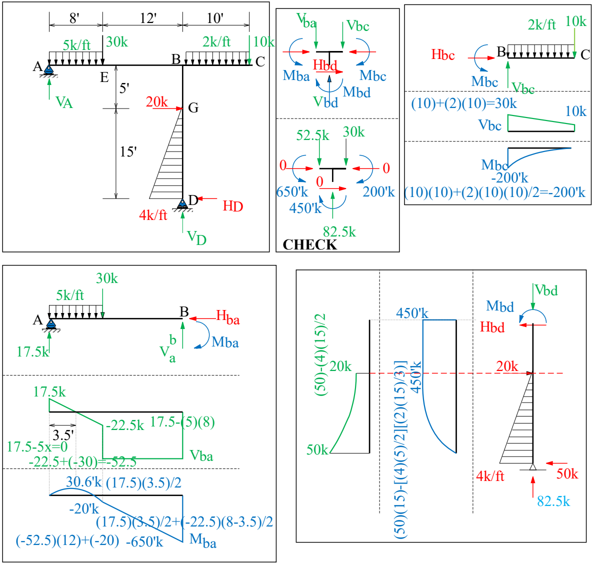
1. At the base: $B\ M = 4.493$ k.ft as determined above.
2. At the support C, $M_c = -4.493 + (-.749)(3)(\frac{3}{2}) = -7.864$ k.ft
3. The maximum moment is equal to
   $M_{max} = -7.864 + (.749)(5)(\frac{5}{2}) = 1.50$ k.ft

Design: Reinforcement should be placed along the fibers which are under tension, that is on the side of the negative moment[1]. The figure below schematically illustrates the location of the flexural[2] reinforcement.

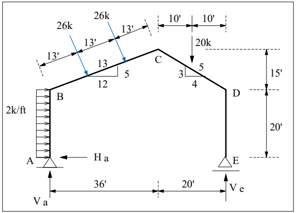


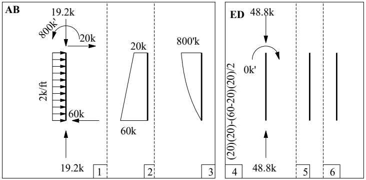
[1] That is why in most European countries, the sign convention for design moments is the opposite of the one commonly used in the U.S.A.; Reinforcement should be placed where the moment is “postive”.

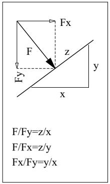
[2] Shear reinforcement is made of a series of vertical stirrups.

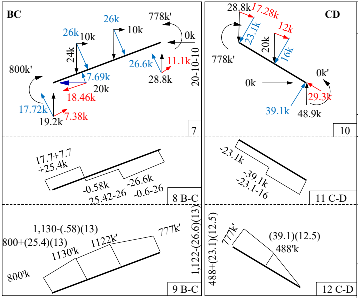


8'
12'
10'
5k/ft
30k
2k/ft
10k
A
E
B
C
5'
20k
G
15'
D
HD
4k/ft
VA
VD
Vba
Vbc
Hbd
Mba
Mbc
Mbd
Vbd
52.5k
30k
0
0
0
650'k
200'k
450'k
82.5k
CHECK
2k/ft
10k
Hbc
B
C
Mbc
Vbc
(10)+(2)(10)=30k
10k
Vbc
Mbc
-200'k
(10)(10)+(2)(10)(10)/2=-200'k
30k
5k/ft
A
B
Hba
Mba
17.5k
Vab
17.5k
-22.5k
17.5-(5)(8)
3.5'
17.5-5x=0
-22.5+(-30)=-52.5
Vba
30.6'k
(17.5)(3.5)/2
-20'k
(17.5)(3.5)/2+(-22.5)(8-3.5)/2
(-52.5)(12)+(-20)
-650'k
Mba
(50)-(4)(15)/2
20k
50k
450'k
(50)(15)-[(4)(5)/2][(2)(15)/3)]
450'k
Vbd
Mbd
Hbd
20k
4k/ft
50k
82.5k

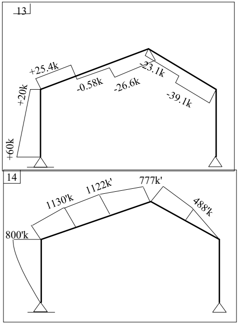


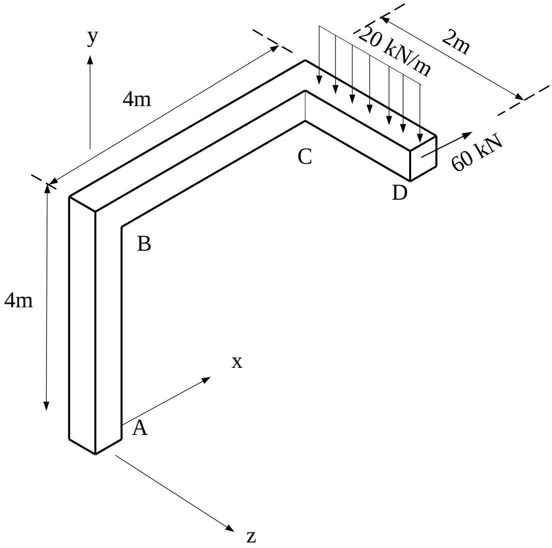






(60)(20)-(2)(20)(20)/2=800	AB
60-(2)(20)=20	
(20)(5)/13=7.7	B
(20)(12)/(13)=18.46	
(19.2)(5)/(13)=7.38	
(19.2)(12)/(13)=17.72	
(26)(12)/(13)=24	BC
(26.6)(13)/(12)=28.8	C
(26.6)(5)/(12)=11.1	
(28.8)(4)/(5)=23.1	
(28.8)(3)/(5)=17.28	CD
(20)(4)/(5)=16	
(20)(3)/(5)=12	
(39.1)(5)/(4)=48.9	D
(39.1)(3)/(4)=29.3	




y
4m
20 kN/m
2m
60 kN
C
D
B
4m
x
A
z

1. The frame has a total of 6 reactions (3 forces and 3 moments) at the support, and we have a total of 6 equations of equilibrium, thus it is statically determinate.
2. Each member has the following internal forces (defined in terms of the *local coordinate system* of each member $x' - y' - z'$ such that x is along the member)

Member	Internal Forces					
Member	Axial	Shear		Moment		Torsion
	$N_{x'}$	$V_{y'}$	$V_{z'}$	$M_{y'}$	$M_{z'}$	$T_{x'}$
$C-D$		$\surd$	$\surd$	$\surd$	$\surd$	
$B-C$	$\surd$		$\surd$	$\surd$	$\surd$	$\surd$
$A-B$	$\surd$	$\surd$		$\surd$	$\surd$	$\surd$

3. The numerical calculations for the analysis of this three dimensional frame are quite simple, however the main complexity stems from the difficulty in visualizing the inter-relationships between internal forces of adjacent members.
4. In this particular problem, rather than starting by determining the reactions, it is easier to determine the internal forces at the end of each member starting with member $C-D$. Note that temporarily we adopt a sign convention which is compatible with the local coordinate systems.

C-D

$$
\begin{array}{llllll}
\Sigma F_{y'} = 0 & \Rightarrow & V_{y'}^{C} = (20)(2) & = +40\text{kN} & \\
\Sigma F_{z'} = 0 & \Rightarrow & V_{z'}^{C} & = +60\text{kN} & \\
\Sigma M_{y'} = 0 & \Rightarrow & M_{y'}^{C} & = -(60)(2) & = -120\text{kN.m} \\
\Sigma M_{z'} = 0 & \Rightarrow & M_{z'}^{C} & = (20)(2)\frac{2}{2} & = +40\text{kN.m}
\end{array}
$$

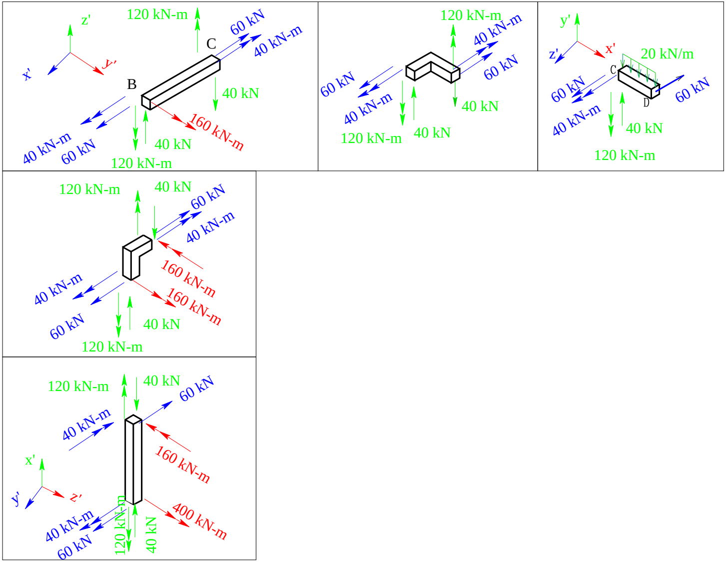
B-C

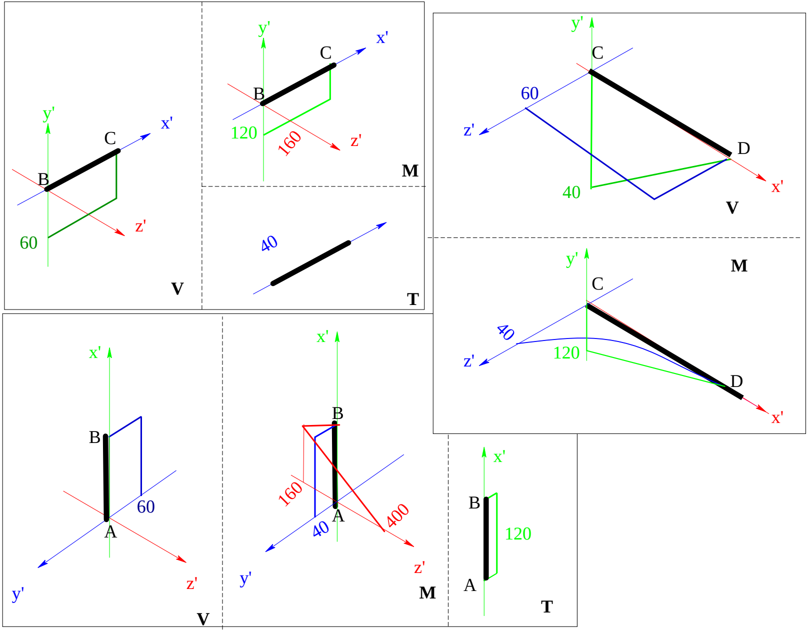
$$
\begin{array}{llllll}
\Sigma F_{x'} = 0 & \Rightarrow & N_{x'}^{B} & = V_{z'}^{C} & & = -60\text{kN} \\
\Sigma F_{y'} = 0 & \Rightarrow & V_{y'}^{B} & = V_{y'}^{C} & & = +40\text{kN} \\
\Sigma M_{y'} = 0 & \Rightarrow & M_{y'}^{B} & = M_{y'}^{C} & & = -120\text{kN.m} \\
\Sigma M_{z'} = 0 & \Rightarrow & M_{z'}^{B} & = V_{y}^{\prime C}(4) & = (40)(4) & = +160\text{kN.m} \\
\Sigma T_{x'} = 0 & \Rightarrow & T_{x'}^{B} & = -M_{z'}^{C} & & = -40\text{kN.m}
\end{array}
$$

A-B

$$
\begin{array}{lllll}
\Sigma F_{x'} = 0 & \Rightarrow & N_{x'}^{A} & = V_{y'}^{B} & = +40\text{kN} \\
\Sigma F_{y'} = 0 & \Rightarrow & V_{y'}^{A} & = N_{x'}^{B} & = +60\text{kN} \\
\Sigma M_{y'} = 0 & \Rightarrow & M_{y'}^{A} & = T_{x'}^{B} & = +40\text{kN.m} \\
\Sigma M_{z'} = 0 & \Rightarrow & M_{z'}^{A} & = M_{z'}^{B} + N_{x'}^{B}(4) = 160 + (60)(4) & = +400\text{kN.m} \\
\Sigma T_{x'} = 0 & \Rightarrow & T_{x'}^{A} & = M_{y'}^{B} & = -120\text{kN.m}
\end{array}
$$

The interaction between axial forces N and shear V as well as between moments M and torsion T is clearly highlighted by this example.


z'
x'
y'
120 kN-m
C
60 kN
40 kN-m
B
40 kN
160 kN-m
40 kN-m
60 kN
40 kN
120 kN-m
20 kN/m
D
400 kN-m



y'
x'
C
B
z'
60
V
y'
x'
C
B
120
160
z'
M
40
T
y'
C
60
z'
D
40
x'
V
y'
M
C
40
z'
120
D
x'
x'
B
60
A
y'
z'
V
x'
B
160
A
400
40
y'
z'
M
x'
B
120
A
T

Structural Analysis

Cables & Arches

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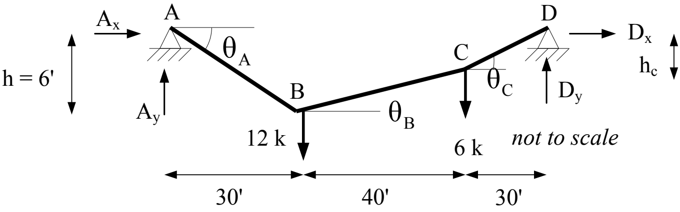
Table of Contents I

- A cable is a slender flexible member with zero or negligible flexural stiffness, thus it can only transmit tensile forces.
- The tensile force at any point acts in the direction of the tangent to the cable (as any other component will cause bending).
- Its strength stems from its ability to undergo extensive changes in geometry (slopes at points of load application) to accommodate load distribution.
- Cables resist vertical forces by undergoing sag (h) and thus developing tensile forces. The horizontal component of this force (H) is called thrust.
- The distance between the cable supports is called the chord (span).
- The sag to span ratio is denoted by

$$r = \frac{h}{l}$$

- When a set of concentrated loads is applied to a cable of negligible weight, then the cable deflects into a series of linear segments and the resulting shape is called the funicular polygon.
- If a cable supports vertical forces only, then the horizontal component H of the cable tension T remains constant (hence both horizontal reactions are equal and opposite).

- A unique characteristic of cable structures (and of flexible structures for that matter) is that not only are the internal forces unknown, but also the geometry. In other words, since geometry varies with the load (equations of equilibrium are based on the free body diagram of the deformed configuration), it also must be determined. This is also referred to as geometric nonlinearity.
- Analysis of cable structures entails not only reactions and internal forces (axial), but also geometry.



- 8 unknowns ($A_x, A_y, D_x, D_y, \theta_A, \theta_B, \theta_C$ and h_C)
- Can be analyzed by applying 2 equations of equilibrium expressed at each of the four points of interest like a truss. However, simpler to use a better tactical approach.
- Horizontal reactions are equal.
- Solution:
  - D_y

$$\left(+\circlearrowleft\right) \Sigma M_z^A = 0; \Rightarrow 12(30) + 6(70) - D_y(100) = 0 \Rightarrow D_y = 7.8 \text{ k} \tag{1}$$

  - A_y

$$\left(+\uparrow\right) \Sigma F_y = 0; \Rightarrow A_y - 12 - 6 + 7.8 = 0 \Rightarrow A_y = 10.2 \text{ k} \tag{2}$$

- Horizontal force by isolating the free body diagram AB

$$\left(+\circlearrowleft\right) \Sigma M_z^B = 0; \Rightarrow A_y(30) - H(6) = 0 \Rightarrow H = 51 \text{ k} \tag{3}$$

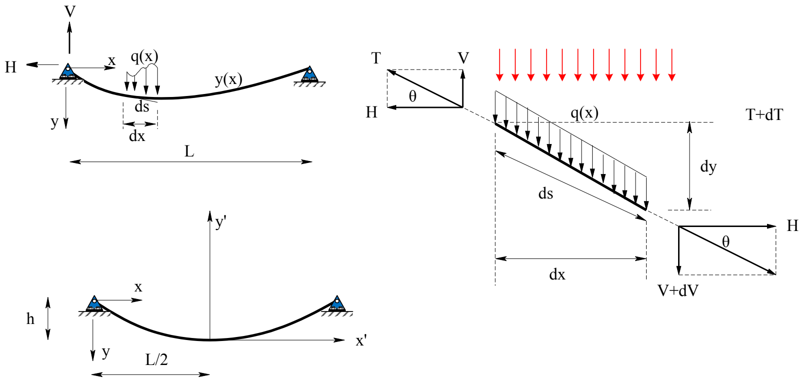
- Sag at C by isolating the free body diagram CD

$$\left(+\circlearrowleft\right) \Sigma M_z^C = 0 \Rightarrow -D_Y(30) + H(h_c) = 0 \Rightarrow h_c = \frac{30D_y}{H} = \frac{30(7.8)}{51} = 4.6 \text{ ft} \tag{4}$$

- Cable internal forces or tractions

$$\begin{aligned}
\tan\theta_A &= \frac{6}{30} = 0.200 \Rightarrow \theta_A = 11.31\,\text{deg} \\
T_{AB} &= \frac{H}{\cos\theta_A} = \frac{51}{0.981} = 51.98 \text{ k} \\
\tan\theta_B &= \frac{6-4.6}{40} = 0.035 \Rightarrow \theta_B = 2\,\text{deg} \\
T_{BC} &= \frac{H}{\cos\theta_B} = \frac{51}{0.999} = 51.03 \text{ k} \\
\tan\theta_C &= \frac{4.6}{30} = 0.153 \Rightarrow \theta_C = 8.7\,\text{deg} \\
T_{CD} &= \frac{H}{\cos\theta_C} = \frac{51}{0.988} = 51.62 \text{ k}
\end{aligned}$$

1. Governing differential equation for a cable with distributed load $q(x)$ per unit horizontal projection of the cable length.



- In the absence of any horizontal load we have $H =$constant. Summation of the vertical forces yields

$$\left(+\downarrow\right)\Sigma F_y = 0 \Rightarrow -V + q\mathrm{d}x + (V + \mathrm{d}V) = 0 \quad (5)$$

$$dV + q\mathrm{d}x = 0 \quad (6)$$

  where V is the vertical component of the cable tension at x.
- Note that if the cable was subjected to its own weight then we would have qds instead of $q\mathrm{d}x$.
- Because the cable must be tangent to T, we have

$$\tan\theta = \frac{V}{H} \quad (7)$$

- Eliminate V and rewrite in terms of H which is constant along the cable by substituting into Eq. 6

$$\mathrm{d}(H\tan\theta) + q\mathrm{d}x = 0 \quad (8)$$

  or

$$-\frac{\mathrm{d}}{\mathrm{d}x}(H\tan\theta) = q \quad (9)$$

  Since H is constant (no horizontal load is applied), this last equation can be rewritten as

$$-H\frac{d}{\mathrm{d}x}(\tan\theta) = q \quad (10)$$

- Written in terms of the vertical displacement y, $\tan\theta = \frac{dy}{dx}$ which when substituted in Eq. 10 yields the governing equation for cables

$$\frac{d^2y}{dx^2} = -\frac{q}{H} \tag{11}$$

Note analogy with flexure

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{12}$$

2. Determination of shape

- Double integration.

$$-Hy' = qx + C_1 \tag{13}$$

$$-Hy = \frac{qx^2}{2} + C_1x + C_2 \tag{14}$$

- C_1 and C_2 are obtained form the boundary conditions: $y = 0$ at $x = 0$ and at $x = L \Rightarrow$ $C_2 = 0$ and $C_1 = -\frac{qL}{2}$. Thus

$$Hy = \frac{q}{2}x(L - x) \tag{15}$$

- This equation gives the shape $y(x)$ in terms of the horizontal force H yet to be determined.

3. Horizontal force H.

- Rewrite Eq. 15 in terms of the maximum sag h which occurs at midspan, hence at $x = \frac{L}{2}$ we would have. In the current case, the moment is simply Hh.

$$Hh = \frac{qL^2}{8} \tag{16}$$

  Note the analogy between this equation and the maximum moment in a simply supported uniformly loaded beam $M = \frac{qL^2}{8}$
- The constant horizontal force H is thus

$$H = \frac{ql^2}{8h} \tag{17}$$

- This relation clearly shows that the horizontal force is inversely proportional to the sag h, as $h \searrow H \nearrow$.

4. Final Shape

- Combining Eq. 15 and 16 we obtain

$$y = \frac{4hx}{L^2}(L - x)$$

- If we shift the origin to midspan, and reverse y, then

$$y' = \frac{4h}{L^2}\left(\frac{L}{2} + x'\right)\left(\frac{L}{2} - x'\right) \tag{18}$$

- Cable has a parabolic shape (as the moment diagram of the applied load).
- Contrarily to the funicular arrangement (where geometry changes with load), the shape of the cable does not change with an increase in the magnitude of the uniform load it is supporting.

5. Maximum Tension

- The maximum tension occurs at the support where the vertical component is equal to $V = \frac{qL}{2}$ (just like in a simply supported beam with a uniform load) and the horizontal one to H, thus

$$T_{max} = \sqrt{V^2 + H^2} = \sqrt{\left(\frac{qL}{2}\right)^2 + H^2} = H\sqrt{1 + \left(\frac{qL/2}{H}\right)^2} \tag{19}$$

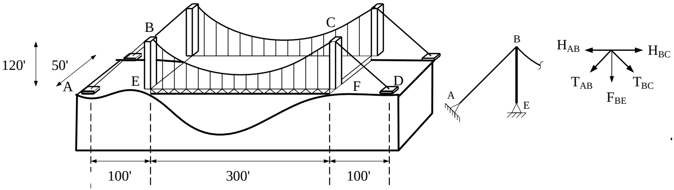
- Recall that $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdot$ or $(1+b)^n = 1 + nb + \frac{n(n-1)b^2}{2!} + \frac{n(n-1)(n-2)b^3}{3!} + \cdots$; Thus for $b^2 << 1$, $\sqrt{1+b} = (1+b)^{\frac{1}{2}} \approx 1 + \frac{b}{2}$.
- Eq. 16 can be rewritten as

$$\frac{qL}{H} = \frac{8h}{L} = 8r \tag{20}$$

where $r = h/L$
- Combining Eqs. 19 and 20 we obtain

$$T_{max} = H\sqrt{1 + 16r^2} \approx H(1 + 8r^2) \tag{21}$$

Design the following 4 lanes suspension bridge by selecting the cable diameters assuming an allowable cable strength σ_{all} of 190 ksi. The bases of the tower are hinged in order to avoid large bending moments.



The total dead load is estimated at 200 psf. Assume a sag to span ratio of $\frac{1}{5}$

1. The dead load carried by each cable will be one half the total dead load or $p_1 = \frac{1}{2}(200)\text{ psf}(50)\text{ ft}\frac{1}{1{,}000} = 5.0\text{ k/ft}$
2. Using the HS 20 truck (or its distributed equivalent load of 0.64 k/ft per lane), the uniform additional load per cable is $p_2 = (2)\text{lanes/cable}(.64)\text{k/ft/lane} = 1.28\text{ k/ft/cable}$. Thus, the total design load is $p_1 + p_2 = 5 + 1.28 = 6.28\text{ k/ft}$

3. The thrust H is determined from Eq. 16

$$H = \frac{pl^2}{8h} = \frac{(6.28)\text{ k/ft}(300)^2\text{ ft}^2}{(8)\,(60)\text{ ft}} = 1,177\text{ k}$$

Note that h is given from $r = h/L = 1/5$.

4. From Eq. 21 the maximum tension is

$$\begin{aligned} T_{max} &= H\sqrt{1+16r^2} \\ &= (1,177)\text{ k}\sqrt{1+(16)\left(\frac{1}{5}\right)^2} \\ &= 1,507\text{ k} \end{aligned}$$

5. Note that if we used the approximate formula in Eq. 21 we would have obtained

$$\begin{aligned} T_{max} &= H(1+8r^2) \\ &= 1,177\left(1+8\left(\frac{1}{5}\right)^2\right) \\ &= 1,554\text{ k} \end{aligned}$$

or 3% difference!

6. The required cross sectional area of the cable along the main span should be equal to $A = \frac{T_{max}}{\sigma_{all}} = \frac{1{,}507 \text{ k}}{190 \text{ ksi}} = 7.93 \text{ in}^2$ which corresponds to a diameter $d = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{(4)(7.93)}{\pi}} = 3.18 \text{ in}$
7. blah
8. We seek next to determine the cable force in AB. Since the pylon can not take any horizontal force, we should have the horizontal component of T_{max} equal and opposite to the horizontal component of T_{AB} or $\frac{T_{AB}}{H} = \frac{\sqrt{(100)^2+(120)^2}}{100}$ thus

$$T_{AB} = H\frac{\sqrt{(100)^2+(120)^2}}{100} = (1,177)(1.562) = 1,838 \text{ k}$$

the cable area should be $A = \frac{1{,}838 \text{ k}}{190 \text{ ksi}} = 9.68 \text{ in}^2$ which corresponds to a diameter $d = \sqrt{\frac{(4)(9.68)}{\pi}} = 3.51 \text{ in}$

9. To determine the vertical load acting on the pylon, we must add the vertical components of T_{max} and of T_{AB} (V_{BC} and V_{AB} respectively). We can determine V_{BC} from H and T_{max}, thus

$$P = \underbrace{\frac{120}{100}(1,177)}_{V_{AB}} + \underbrace{\sqrt{(1,507)^2 - (1,177)^2}}_{V_{BC}} = 1,412 + 941 = 2,353 \text{ k}$$

Using A36 steel with an allowable stress of 21 ksi, the cross sectional area of the tower should be $A = \frac{2{,}353}{21} = 112 \text{ in}^2$. Note that buckling of such a high tower might govern the final dimensions.

10. If the cables were to be anchored to a concrete block, the volume of the block should be at least equal to $V = \frac{(1{,}412) \text{ k}(1{,}000)}{150 \text{ lbs}/\text{ft}^3} = 9,413 \text{ ft}^3$ or a cube of approximately 21 ft

- Let us consider now the case where the cable is subjected to its own weight (plus ice and wind if any). We would have to replace $q\mathrm{d}x$ by $q\mathrm{d}s$ in Eq. 6

$$dV + q\mathrm{d}s = 0 \tag{22}$$

- The differential equation for this new case will be derived exactly as before, but we substitute $q\mathrm{d}x$ by $q\mathrm{d}s$, thus Eq. 11 becomes

$$\frac{d^2y}{\mathrm{d}x^2} = -\frac{q}{H}\frac{\mathrm{d}s}{\mathrm{d}x} \tag{23}$$

- But $\mathrm{d}s^2 = \mathrm{d}x^2 + dy^2$, hence:

$$\frac{d^2y}{\mathrm{d}x^2} = -\frac{q}{H}\sqrt{1 + \left(\frac{dy}{\mathrm{d}x}\right)^2} \tag{24}$$

- solution of this differential equation is considerably more complicated than Eq. 11.
- We let $dy/\mathrm{d}x = p$, then

$$\frac{dp}{\mathrm{d}x} = -\frac{q}{H}\sqrt{1 + p^2} \tag{25}$$

- Rearranging

$$\int \frac{dp}{\sqrt{1 + p^2}} = -\int \frac{q}{H}\mathrm{d}x \tag{26}$$

- From Mathematica (or handbooks), the left hand side is equal to

$$\int \frac{dp}{\sqrt{1+p^2}} = \log_e(p+\sqrt{1+p^2}) \tag{27}$$

- Substituting, we obtain

$$\log_e(p+\sqrt{1+p^2}) = \underbrace{-\frac{qx}{H}+C_1}_{A} \tag{28}$$

$$p+\sqrt{1+p^2} = e^A \tag{29}$$

$$\sqrt{1+p^2} = -p+e^A \tag{30}$$

$$1+p^2 = p^2-2pe^A+e^{2A} \tag{31}$$

$$p = \frac{e^{2A}-1}{2e^A} = \frac{e^A-e^{-A}}{2} = \sinh A \tag{32}$$

$$= \frac{dy}{\mathrm{d}x} = \sinh\left(-\frac{qx}{H}+C_1\right) \tag{33}$$

$$y = \int \sinh\left(-\frac{qx}{H}+C_1\right)\mathrm{d}x = -\frac{H}{q}\cosh\left(-\frac{qx}{H}+C_1\right)+C_2 \tag{34}$$

- To determine the two constants, we set

$$\frac{dy}{dx} = 0 \quad \text{at } x = \frac{L}{2} \tag{35}$$

$$\frac{dy}{dx} = -\frac{q}{H}\frac{H}{q}\sinh\left(-\frac{qx}{H} + C_1\right) \tag{36}$$

$$\Rightarrow 0 = \sinh\left(-\frac{q}{H}\frac{L}{2} + C_1\right) \Rightarrow C_1 = \frac{q}{H}\frac{L}{2} \tag{37}$$

$$\Rightarrow y = -\frac{H}{q}\cosh\left[\frac{q}{H}\left(\frac{L}{2} - x\right)\right] + C_2 \tag{38}$$

- At midspan, the sag is equal to h, thus

$$h = -\frac{H}{q}\cosh\left[\frac{q}{H}\left(\frac{L}{2} - \frac{L}{2}\right)\right] + C_2 \tag{39}$$

$$C_2 = h + \frac{H}{q} \tag{40}$$

- If we move the origin at the lowest point along the cable at $x' = x - L/2$ and $y' = h - y$, we obtain

$$\frac{q}{H}y = \cosh\left(\frac{q}{H}x\right) - 1 \tag{41}$$

- This equation is to be contrasted with 18, we can rewrite those two equations as:

$$\frac{q}{H}y = \frac{1}{2}\left(\frac{q}{H}x\right)^2 \quad \text{Parabola} \tag{42}$$

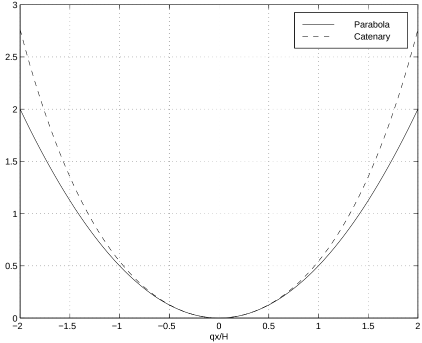
$$\frac{q}{H}y = \cosh\left(\frac{q}{H}x\right) - 1 \quad \text{Catenary} \tag{43}$$

- The hyperbolic cosine of the catenary can be expanded into a Taylor power series as

$$\frac{qy}{H} = \frac{1}{2}\left(\frac{qx}{H}\right)^2 + \frac{1}{24}\left(\frac{qx}{H}\right)^4 + \frac{1}{720}\left(\frac{qx}{H}\right)^6 + ... \tag{44}$$

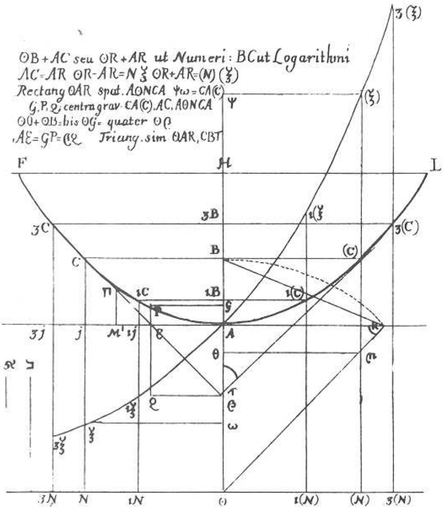
  The first term of this development is identical as the formula for the parabola, and the other terms constitute the difference between the two.

- The difference becomes significant only for large qx/H, that is for large sags in comparison with the span.

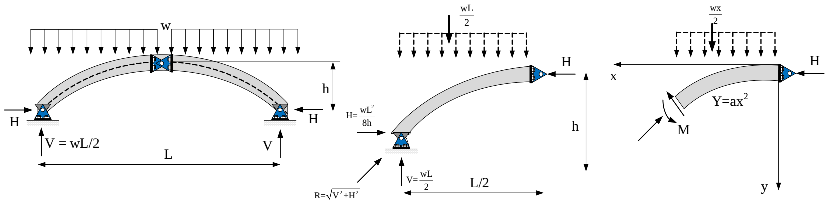


Parabola
Catenary
3
2.5
2
1.5
1
0.5
0
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
qx/H

- Solution of the catenary problem constitued one of the major mathematical/Mechanics challenges of the early 18th century.
- Around 1684, differential and integral calculus took their first effective forms, and those powerful new techniques allowed scientists to tackle complex problems for the first time.
- One of these problems was the solution to the catenary problem as presented by Jakob Bernouilli. Immediately thereafter, Leibniz presented a solution based on infinitesimal calculus, another one was presented by Huygens.
- Finally, the brother of the challenger, Johann Bernoulli did also present a solution.
- Huygens solution was complex and relied on geometrical arguments. The one of Leibniz was elegant and correct ($y/a = (b^{x/a} + b^{-x/a})/2$ (we recognize Eq. 41 albeit written in slightly different form



- Finally, Bernoulli presented two correct solution, and in his solution he did for the first time express equations of equilibrium in differential form.



- Due to symmetry, the vertical reaction is simply $V = \frac{wL}{2}$, and there is no shear across the midspan of the arch (nor a moment). Taking moment about the crown,

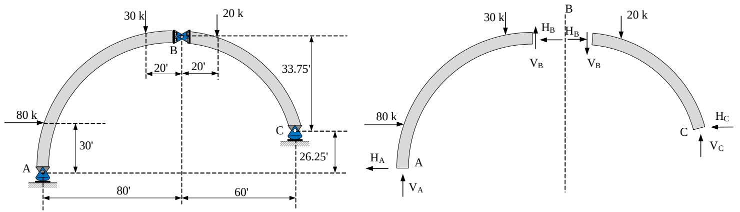
$$M = Hh - \frac{wL}{2}\left(\frac{L}{2} - \frac{L}{4}\right) = 0$$

Solving for H

$$H = \frac{wL^2}{8h} \tag{45}$$

- Note analogy with $M = wL^2/8$ for beams and $H = wL^2/8h$ for cables.
- In general an arch will carry the vertical load across the span through a combination of axial forces and flexural ones. A well dimensioned arch will have a small to negligible moment, and relatively high normal compressive stresses.

- The “perfect” parabolic shape of a simply supported three-hinged arch
  - There should not be any shear or moment along any section.
  - Moment at x: $M = Hy - wx^2/2 = Hax^2 - wx^2/2 = 0$,
  - Solving for a: $a = w/(2H)$
  - Substitute for H and conclude that a must be equal to $4h/L^2$.
  - For any span L, there is only one height h which would yield a parabola with zero moment and shear.
- Three-hinged arches are statically determinate structures which shape can accommodate support settlements and thermal expansion without secondary internal stresses. They are also easy to analyze through statics.
- An arch is far more efficient than a beam, and possibly more economical and aesthetic than a truss in carrying loads over long spans.



- Four unknowns, three equations of equilibrium, one equation of condition $\Rightarrow$ statically determinate.

$$
\begin{array}{rcll}
(+\circlearrowleft)\ \Sigma M_z^C & = & 0; & (R_{Ay})(140)+(80)(3.75)-(30)(80)-(20)(40)+R_{Ax}(26.25) = 0 \\
 & & & \Rightarrow 140R_{Ay}+26.25R_{Ax} = 2.900 \\
(+\text{rgt})\ \Sigma F_x & = & 0; & 80-R_{Ax}-R_{Cx} = 0 \\
(+\uparrow)\ \Sigma F_y & = & 0; & R_{Ay}+R_{Cy}-30-20 = 0 \\
(+\circlearrowleft)\ \Sigma M_z^B & = & 0; & (R_{ax})(60)-(80)(30)-(30)(20)+(R_{Ay})(80) = 0 \\
 & & & \Rightarrow 80R_{Ay}+60R_{Ax} = 3,000
\end{array}
$$

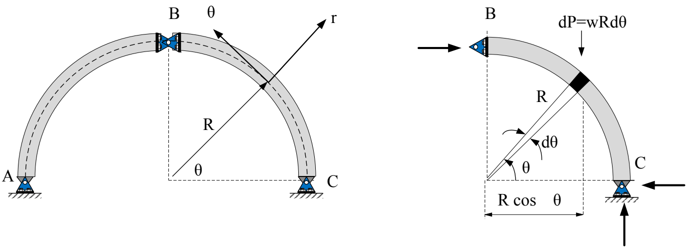
- Solving those four equations simultaneously we have:

$$
\begin{bmatrix} 140 & 26.25 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 80 & 60 & 0 & 0 \end{bmatrix}
\begin{Bmatrix} R_{Ay} \\ R_{Ax} \\ R_{Cy} \\ R_{Cx} \end{Bmatrix}
=
\begin{Bmatrix} 2,900 \\ 80 \\ 50 \\ 3,000 \end{Bmatrix}
\Rightarrow
\begin{Bmatrix} R_{Ay} \\ R_{Ax} \\ R_{Cy} \\ R_{Cx} \end{Bmatrix}
=
\begin{Bmatrix} 15.1\text{ k} \\ 29.8\text{ k} \\ 34.9\text{ k} \\ 50.2\text{ k} \end{Bmatrix}
$$

- We can check our results by considering the summation with respect to B from the right:

$$xxx$$

Determine the reactions of the three hinged statically determined semi-circular arch under its own dead weight w (per unit arc length s, where $ds = rd\theta$).



- The reactions can be determined by integrating the load over the entire structure

- Vertical Reaction

$$\left(+\circlearrowleft\right)\Sigma M_A = 0; (C_y)(2R) - \int_{\theta=0}^{\theta=\pi} \underbrace{wRd\theta}_{dP}\underbrace{R(1+\cos\theta)}_{\text{moment arm}} = 0$$

$$\Rightarrow C_y = \frac{wR}{2}\int_{\theta=0}^{\theta=\pi}(1+\cos\theta)d\theta = \frac{wR}{2}[\theta+\sin\theta]\,|_{\theta=0}^{\theta=\pi}$$

$$= \frac{wR}{2}[(\pi+\sin\pi)-(0+\sin 0)]$$

$$= \frac{\pi}{2}wR$$

- Horizontal Reactions

$$\left(+\circlearrowleft\right)\Sigma M_B = 0; -(C_x)(R)+(C_y)(R) - \int_{\theta=0}^{\theta=\frac{\pi}{2}} \underbrace{wRd\theta}_{dP}\;\underbrace{R\cos\theta}_{\text{moment arm}} = 0$$

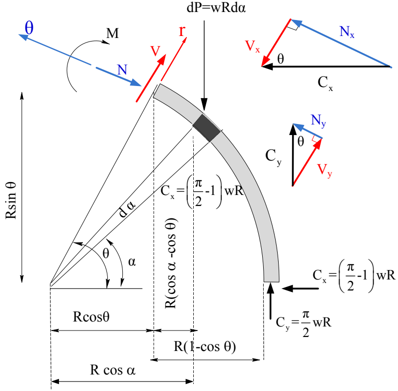
$$\Rightarrow C_x = \frac{\pi}{2}wR - \frac{wR}{2}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\cos\theta d\theta$$

$$= \frac{\pi}{2}wR - wR[\sin\theta]\,|_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{\pi}{2}wR - wR(\frac{\pi}{2}-0)$$

$$= \left(\frac{\pi}{2}-1\right)wR$$

By symmetry the reactions at A are equal to those at C

- Internal Forces can now be determined



- Shear Forces: Considering the free body diagram of the arch, and summing the forces in the radial direction ($\Sigma F_R = 0$):

$$-\underbrace{(\frac{\pi}{2}-1)wR\cos\theta}_{V_x}+\underbrace{\frac{\pi}{2}wR\sin\theta}_{V_y}-\int_{\alpha=0}^{\theta}wR\mathrm{d}\alpha\sin\theta+V=0$$

$$\Rightarrow V(\theta)=wR\left[(\tfrac{\pi}{2}-1)\cos\theta+(\theta-\tfrac{\pi}{2})\sin\theta\right]$$

- Axial Forces: Similarly, if we consider the summation of forces in the axial direction ($\Sigma F_\theta = 0$):

$$\underbrace{(\frac{\pi}{2}-1)wR\sin\theta}_{N_x}+\underbrace{\frac{\pi}{2}wR\cos\theta}_{N_y}-\int_{\alpha=0}^{\theta}wR\mathrm{d}\alpha\cos\theta+N=0$$

$$\Rightarrow N(\theta)=wR\left[(\theta-\tfrac{\pi}{2})\cos\theta-(\tfrac{\pi}{2}-1)\sin\theta\right]$$

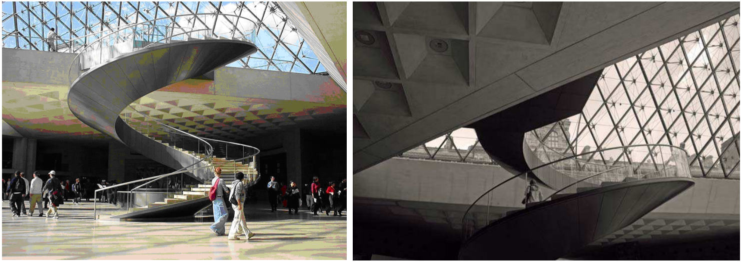
- Moment: Now we can consider the third equation of equilibrium ($\Sigma M_z = 0$):

$$\left(+\circlearrowleft\right) \Sigma M \qquad \underbrace{(\frac{\pi}{2} - 1)wR}_{C_x} \cdot R\sin\theta - \underbrace{\frac{\pi}{2}wR}_{C_y} R(1-\cos\theta)$$

$$+\int_{\alpha=0}^{\theta} wR\mathrm{d}\alpha \cdot R(\cos\alpha - \cos\theta) + M = 0$$

$$\Rightarrow M(\theta) = wR^2\left[\frac{\pi}{2}(1-\sin\theta) + (\theta - \frac{\pi}{2})\cos\theta\right]$$

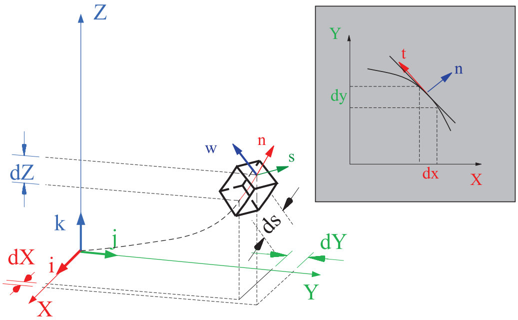
- Because space structures may have complicated geometry, we must resort to vector analysis[1] to determine the internal forces.
- In general we have six internal forces (forces and moments) acting at any section.



Louvre Museum Entrance

- In general, the geometry of the structure is most conveniently described by a parametric set of equations

$$x = f_1(\theta); \qquad y = f_2(\theta); \qquad z = f_3(\theta) \tag{46}$$



the global coordinate system is denoted by $X - Y - Z$, and its unit vectors are denoted[2] $\mathtt{i}, \mathtt{j}, \mathtt{k}$.

- The section on which the internal forces are required is cut and the **principal** axes are identified as $N - S - W$ which correspond to the normal force, and bending axes with respect to the *Strong* and *Weak* axes. The corresponding unit vectors are $\mathtt{n}, \mathtt{s}, \mathtt{w}$.
- The unit normal vector (which is tangent to the curve) at any section is given by

$$\mathtt{n} = \frac{dx\mathtt{i} + dy\mathtt{j} + dz\mathtt{k}}{ds} = \frac{dx\mathtt{i} + dy\mathtt{j} + dz\mathtt{k}}{(dx^2 + dy^2 + dz^2)^{1/2}} \tag{47}$$

- The principal bending axes must be defined, that is if the strong bending axis is parallel to the *XY* plane, or horizontal (as is generally the case for gravity load), then this axis is normal to both the *N* and *Z* axes, and its unit vector is

$$\mathbf{s} = \mathbf{n} \times \mathbf{k} \tag{48}$$

- The weak bending axis is normal to both *N* and *S*, and thus its unit vector is determined from

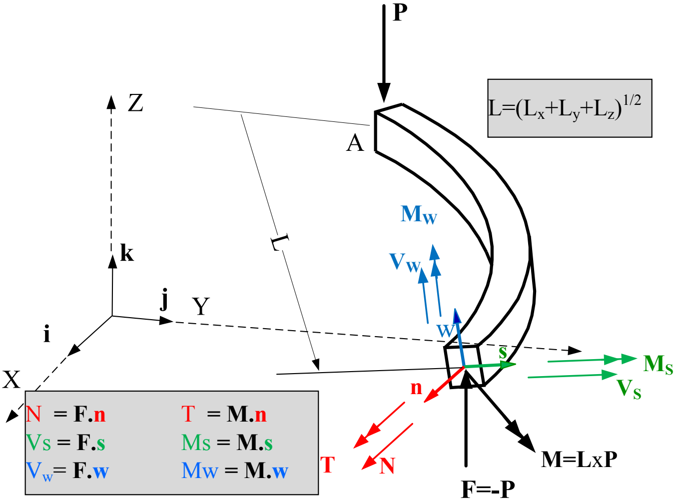
$$\mathbf{w} = \mathbf{n} \times \mathbf{s} \tag{49}$$

  Note that by now both $\mathbf{n}$ and $\mathbf{s}$ have been normalized.

[1] To which you have already been exposed at an early stage, yet have very seldom used it so far in mechanics!

[2] All vectorial quantities are denoted by a bold faced character.

- For the equilibrium equations, we consider the free body diagram.



an applied load $\mathtt{P}$ is acting at point A. The resultant force vector $\mathtt{F}$ and resultant moment vector $\mathtt{M}$ acting on the cut section B are determined from equilibrium

$$\Sigma\mathtt{F} = 0; \qquad \mathtt{P} + \mathtt{F} = 0; \qquad \mathtt{F} = -\mathtt{P} \tag{50}$$

$$\Sigma\mathtt{M}^B = 0; \quad \mathtt{L}\times\mathtt{P} + \mathtt{M} = 0; \quad \mathtt{M} = -\mathtt{L}\times\mathtt{P} \tag{51}$$

where $\mathtt{L}$ is the lever arm vector from B to A.

- The axial and shear forces N, V_s and V_w are all three components of the force vector $\mathtt{F}$ along the N, S, and W axes and can be found by dot product with the appropriate unit vectors:

$$N = \mathtt{F}\cdot\mathtt{n} \tag{52}$$

$$V_s = \mathtt{F}\cdot\mathtt{s} \tag{53}$$

$$V_w = \mathtt{F}\cdot\mathtt{w} \tag{54}$$

- Similarly the torsional and bending moments T, M_s and M_w are also components of the moment vector $\mathtt{M}$ and are determined from

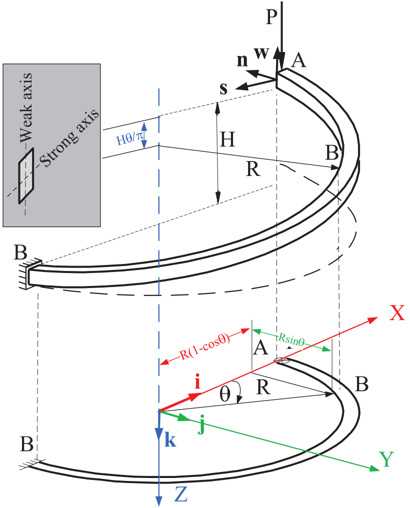
$$T = \mathtt{M}\cdot\mathtt{n} \tag{55}$$

$$M_s = \mathtt{M}\cdot\mathtt{s} \tag{56}$$

$$M_w = \mathtt{M}\cdot\mathtt{w} \tag{57}$$

- Hence, we do have a mean to determine the internal forces. In case of applied loads we sum, and for distributed load we integrate.

- We seek to determine the internal forces N, V_s, and V_w and the internal moments T, M_s and M_w along the helicoidal cantilevered girder due to a vertical load P at its free end.



- We first determine the geometry in terms of the angle θ

$$x = R\cos\theta; \qquad y = R\sin\theta; \qquad z = \frac{H}{\pi}\theta \tag{58}$$

- To determine the unit vector $\mathbf{n}$ at any point we need the derivatives:

$$dx = -R\sin\theta d\theta; \qquad dy = R\cos\theta d\theta; \qquad dz = \frac{H}{\pi}d\theta \tag{59}$$

and then insert into Eq. 47

$$\mathbf{n} = \frac{-R\sin\theta\mathbf{i} + R\cos\theta\mathbf{j} + H/\pi\mathbf{k}}{\left[R^2\sin^2\theta + R^2\cos^2\theta + (H/\pi)^2\right]^{1/2}} \tag{60}$$

$$= \underbrace{\frac{1}{\left[1 + (H/\pi R)^2\right]^{1/2}}}_{K}\left[-\sin\theta\mathbf{i} + \cos\theta\mathbf{j} + (H/\pi R)\mathbf{k}\right] \tag{61}$$

Since the denominator depends only on the geometry, it will be designated by K.

- The strong bending axis lies in a horizontal plane, and its unit vector can thus be determined from Eq. 48:

$$\mathbf{n}\times\mathbf{k} = \frac{1}{K}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & \frac{H}{\pi R} \\ 0 & 0 & 1 \end{vmatrix} \quad (62)$$

$$= \frac{1}{K}(\cos\theta\mathbf{i}+\sin\theta\mathbf{j}) \quad (63)$$

and the absolute magnitude of this vector $|\mathbf{k}\times\mathbf{n}|=\frac{1}{K}$, and thus

$$\mathbf{s}=\cos\theta\mathbf{i}+\sin\theta\mathbf{j} \quad (64)$$

- The unit vector along the weak axis is determined from Eq. 49

$$\mathbf{w}=\mathbf{s}\times\mathbf{n} = \frac{1}{K}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & \frac{H}{\pi R} \end{vmatrix} \quad (65)$$

$$= \frac{1}{K}\left(\frac{H}{\pi R}\sin\theta\mathbf{i}-\frac{H}{\pi R}\cos\theta\mathbf{j}+\mathbf{k}\right) \quad (66)$$

- With the geometry definition completed, we now examine the equilibrium equations. Eq. 50 and 51.

$$\Sigma F = 0; \quad \mathbf{F} = -\mathbf{P} \tag{67}$$

$$\Sigma M_b = 0; \quad \mathbf{M} = -\mathbf{L}\times\mathbf{P} \tag{68}$$

where

$$\mathbf{L} = (R - R\cos\theta)\mathbf{i} + (0 - R\sin\theta)\mathbf{j} + \left(0 - \frac{\theta}{\pi}H\right)\mathbf{k} \tag{69}$$

and

$$\mathbf{M} = \mathbf{L}\times\mathbf{P} = R\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (1-\cos\theta) & -\sin\theta & -\frac{\theta}{\pi}\frac{H}{R} \\ 0 & 0 & P \end{vmatrix} \tag{70}$$

$$= PR[-\sin\theta\mathbf{i} - (1-\cos\theta)\mathbf{j}] \tag{71}$$

and

$$\mathbf{M} = PR[\sin\theta\mathbf{i} + (1-\cos\theta)\mathbf{j}] \tag{72}$$

- Finally, the components of the force $\mathtt{F} = -P\mathtt{k}$ and the moment $\mathtt{M}$ are obtained by appropriate dot products with the unit vectors

$$N = \mathtt{F}{\cdot}\mathtt{n} = \boxed{-\frac{1}{K}P\frac{H}{\pi R}} \tag{73}$$

$$V_s = \mathtt{F}{\cdot}\mathtt{s} = \boxed{0} \tag{74}$$

$$V_w = \mathtt{F}{\cdot}\mathtt{w} = \boxed{-\frac{1}{K}P} \tag{75}$$

$$T = \mathtt{M}{\cdot}\mathtt{n} = \boxed{-\frac{PR}{K}(1-\cos\theta)} \tag{76}$$

$$M_s = \mathtt{M}{\cdot}\mathtt{s} = \boxed{PR\sin\theta} \tag{77}$$

$$M_w = \mathtt{M}{\cdot}\mathtt{w} = \boxed{\frac{PH}{\pi K}(1-\cos\theta)} \tag{78}$$

Structural Analysis

Approximate Analyses

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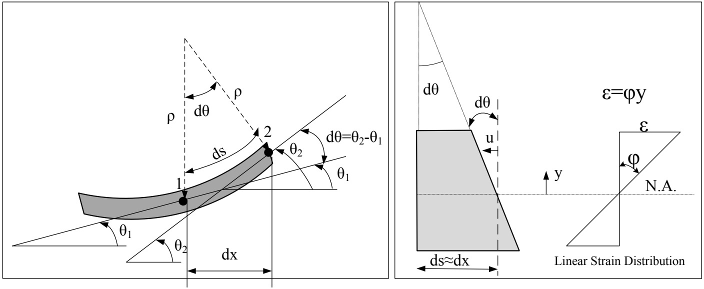
Spring 2019

Table of Contents I

Table of Contents II

- M diagram necessary for design and deflection calculations.
- Approximate method Essential for back of the envelope verification and preliminary dimensioning
- Developing a basic understanding of how structures behave under applied loads is an important part of structural engineering.
- That is how they displace and deform and how stresses develop and propagate in their members.
- The ability to visualize or having a qualitative understanding of the behavior of a structure contribute to the development of the an insight that shape structural engineering judgment.
- Excellent review of Mechanics and Statics.
- Turn a statically indeterminate structure into a statically determinate one by identifying point of inflection based on a proper sketch of the deflected shape.
- Engineers use computers to draw shear/moment diagrams, value of hand calculations very limited (but essential to understand theory).
- More important to develop a "feel" for structural engineering, and develop the capability of quickly sketching deflected shapes and moment diagrams.
- You are the first class with which I will be experimenting this approach!.



- The slope is denoted by θ, the change in slope per unit length is the curvature ϕ, the radius of curvature is ρ. From the figure we have the following relations

$$\phi = \frac{1}{\rho} = \frac{d\theta}{ds}$$

- For small displacements, and as a first order approximation, with $ds \approx dx$ and $\theta = \frac{dy}{dx}$

$$\phi = \frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

- A positive $d\theta$ at a positive y (upper fibers) will cause a *shortening* of the upper fibers.
- From the figure: $du = -yd\theta$, Dividing both sides by dx, and for linear elastic systems:

$$\underbrace{\frac{du}{\mathrm{d}x}}_{\varepsilon} = -y\frac{d\theta}{\mathrm{d}x} = -\frac{My}{EI}$$

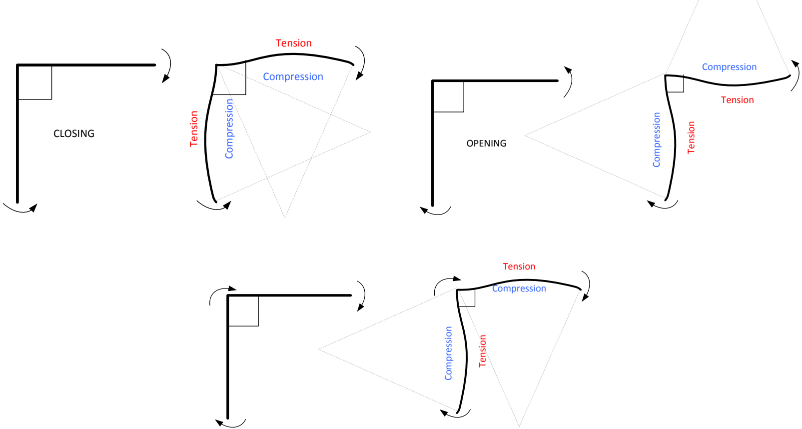
- Combining this the previous equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{M}{EI}$$

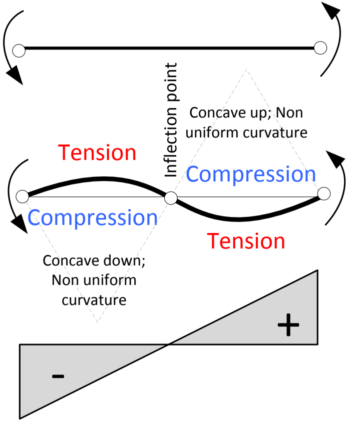
- Main conclusion for this chapter: zero moment occurs when the inflexion point (where $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$)

- No axial deformation.
- If lateral load, apply a small lateral displacement.
- Compatible corner rotations (90 degree bends stay at 90 degrees; Fixed ends remain fixed).
- Continuous smooth curves (except at internal hinges).
- Careful about inflection points (specially for the moment diagram).

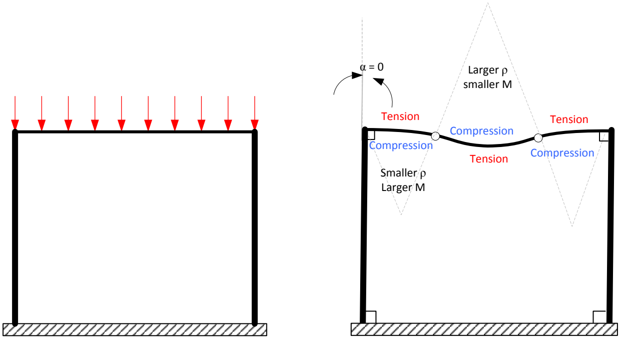
⚠ This is only for the corner and not for elements (infinitesimal size)



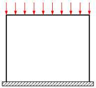
Self explanatory!

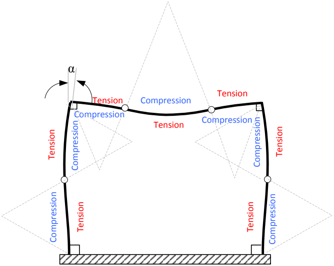


- For very rigid columns. Note the shorter the radius, the larger the moment.



- For very rigid beam: small flexure in beam, large axial in the columns
- For intermediary situation




α
Tension
Compression
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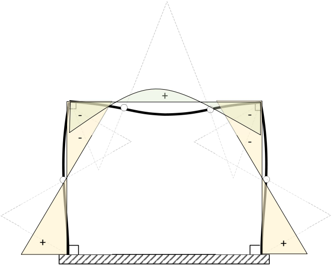
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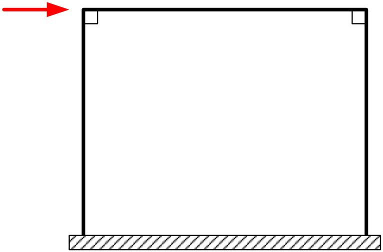
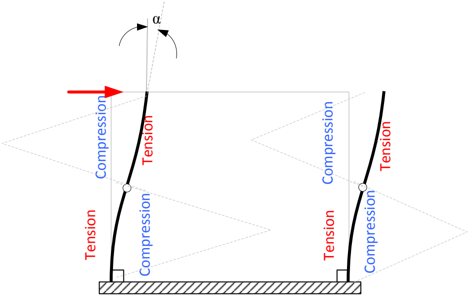
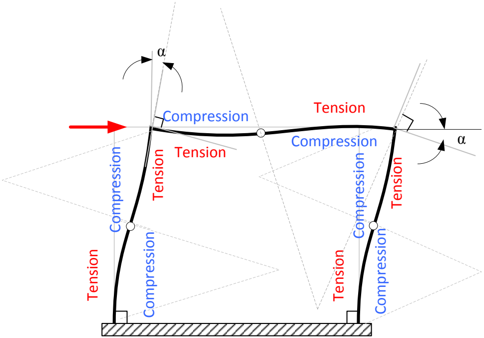


1. Draw the deflected shape of the portal frame.
2. Draw the corresponding moment diagram.
3. Show tension/compression zones.

- Draw the deflected shapes of the left column keeping in mind:
  - Bottom tangent is 90 ^{o}.
  - Tension has to be on the outside (thus concave inside).
  - The column is restrained in the top. If the cross beam is
    - Infinity rigid: the tangent to the top of the column would be also 90 ^{o}. (there will be an inflection point)
    - Very flexible: there will be no inflection point, and concavity will be entirely inside.
    - Finite stiffness: non zero rotation at the top, and an inflection point.

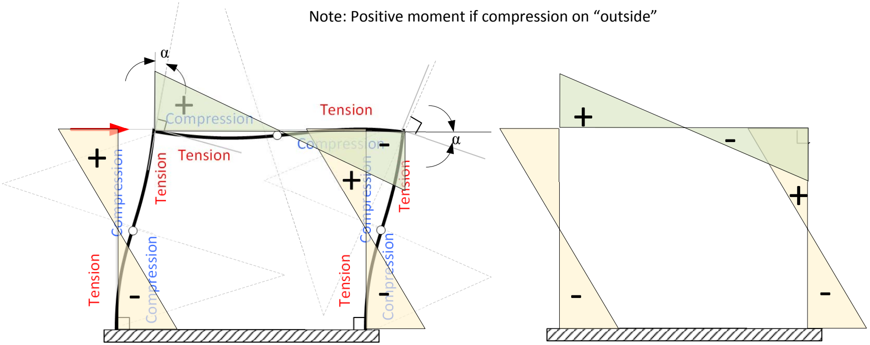


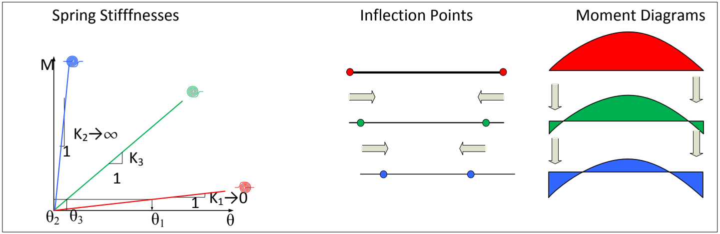
- Column in the right will have identical shape.
- Draw deflected shape of beam. There is no concentrated moment at the corner, so tensions and compressions must be continuous.

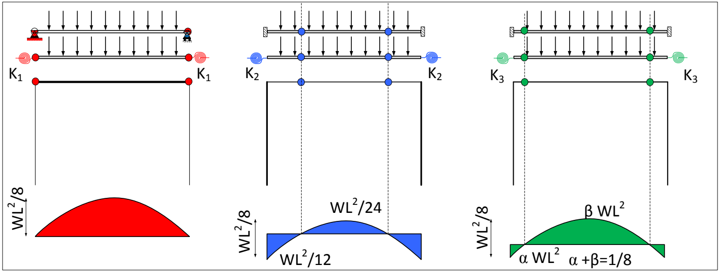


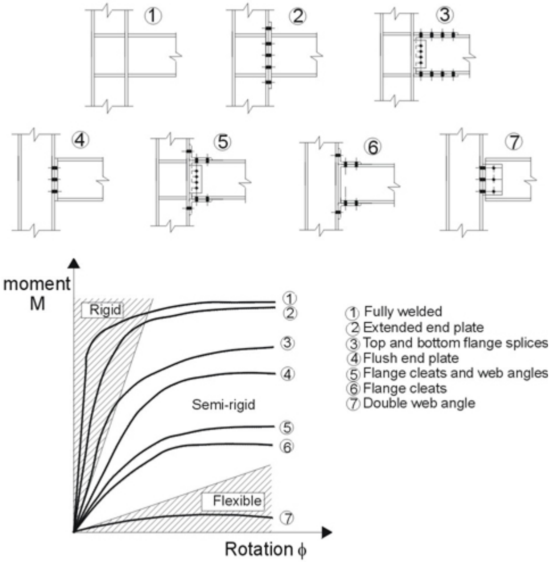
- Draw the moment diagram.

Note: Positive moment if compression on "outside"




Spring Stifffnesses
Inflection Points
Moment Diagrams
M
K2→∞
1
K3
1
1
K1→0
θ2
θ3
θ1
θ


K1
K1
K2
K2
K3
K3
WL2/8
WL2/24
WL2/8
WL2/12
β WL2
WL2/8
α WL2
α +β=1/8

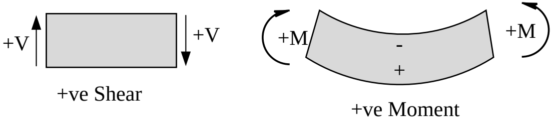


1
2
3
4
5
6
7
moment
M
Rigid
Semi-rigid
Flexible
Rotation ϕ
1 Fully welded
2 Extended end plate
3 Top and bottom flange splices
4 Flush end plate
5 Flange cleats and web angles
6 Flange cleats
7 Double web angle

- Rectangular building frames are highly statically indeterminate.
- Can be analysed by
  - computers exact solution, but time consuming.
  - approximate solution of any floor on a back of the enveloppe approach to yield quick and decent results.

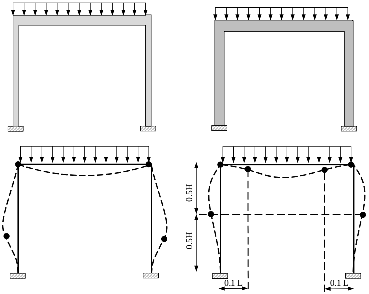
.

- Vertical loads are treated separately from the horizontal ones.
- For horizontal loads two methods:
  - Portal method for low rise buildings with predominant shear deflections.
  - Cantilever for high rise buildings.
- Design sign convention for moments (+ve tension below), and for shear (ccw +ve).
- Assume girders to be numbered from left to right.
- In all free body diagrams assume positive forces/moments, and take algebraic sums.
- Sign convention



- For a a multi-bay/multi-storey frame, girders are assumed to be continuous beams, and columns are assumed to resist the resulting unbalanced moments from the girders. Assume

  1. Girders at each floor act as continuous beams supporting a uniform load.
  2. Inflection points are assumed to be at
     1. One tenth the span from both ends of each girder.
     2. Mid-height of the columns.
  3. Guess location of inflection points.



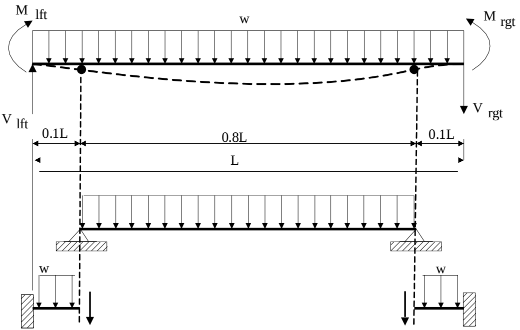
  4. Ignore axial effects.

5. Unbalanced end moments from the girders at each joint is distributed to the columns above and below.

- Then, all beams are statically determinate and have a span, L_s equal to 0.8 the original length of the girder, L. This assumes that the inflection point is at $0.1L$ which is between 0 for simply supported beam, and $0.21L$ for rigidly connected beams.
- Sequence of calculation (very important)
  1. Girder positive moment
  2. Girder negative moment
  3. Girder shear
  4. column axial forces
  5. column moments
  6. Column shears
  7. Girder axial forces

The procedure is outlined below, no need to memorize any equation, just use equations of equilibrium and free body diagrams where all end member shear and moments are assumed to be positive.

1. Maximum positive moment at center of each beam



$$M^{+} = \frac{1}{8} w L_s^2 = w \frac{1}{8} (0.8)^2 L^2 = 0.08 w L^2$$

Note that $wL^2/24 = 0.041667$.

2. Maximum negative moment at each end of the girder

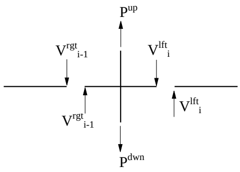
$$M^{left} = M^{rgt} = -\frac{w}{2}(0.1L)^2 - \frac{w}{2}(0.8L)(0.1L) = -0.045wL^2$$

3. Girder Shear are obtained from

$$V^{lft} = \frac{wL}{2} \qquad V^{rgt} = -\frac{wL}{2}$$

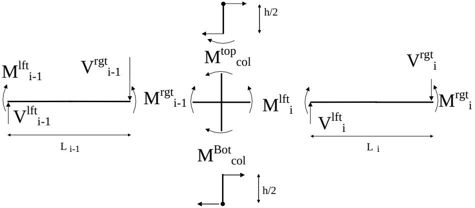
Note thhe sign of the shear forces. Graphically, they will always be shown positive.

4. Column axial force: summ all the girder shears to the axial force transmitted by the column above it.



$$P^{dwn} = P^{up} + V_{i-1}^{rgt} - V_i^{lft}$$

5. Column Moment are obtained from the free body diagram of the joints. From symmetry, left and right moments are equal and opposite, thus



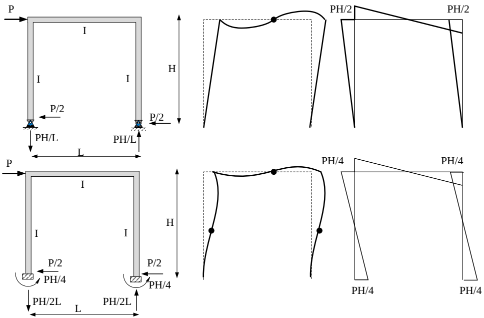
$$M^{bot} = -M^{top}$$

6. Column Shear Points of inflection are at mid-height, with possible exception when the columns on the first floor are hinged at the base

$$V = \frac{M^{top}}{\frac{h}{2}}$$

7. Girder axial forces are assumed to be negligible

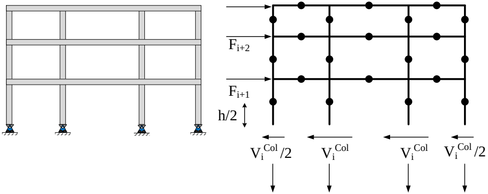
- Single bay/storey frame, depending on the boundary conditions, we will have different locations for the inflection points.



- For a multi-bays/multi-storeys frame, must differentiate between low and high rise buildings.
  - Low rise buidlings height is at least smaller than the horizontal dimension, the deflected shape is characterized by shear deformations. Use Portal Method.
  - High rise buildings height is several times greater than its least horizontal dimension, the deflected shape is dominated by overall flexural deformation.

- Low rise buildings under lateral loads predominantly shear deformations is dominant. Assume:
  1. Inflection points at
     1. Mid-height of all columns above the second floor.
     2. Mid-height of floor columns if rigid support, or at the base if hinged.
     3. At the center of each girder.
  2. Total horizontal shear at the mid-height of all columns at any floor level will be distributed among these columns so that each of the two exterior columns carry half as much horizontal shear as each interior columns of the frame.
- Sequence of calculations:
  1. Column shear
  2. Column moments
  3. Girder moments
  4. Girder shears
  5. Column axial force
  6. Girder axial force
- Note that it is the reverse order of the vertical load case.

1. Column Shear is obtained by passing a horizontal section through the mid-height of the columns at each floor and summing the lateral forces above it.



External columns take half the shear force of the interior ones.

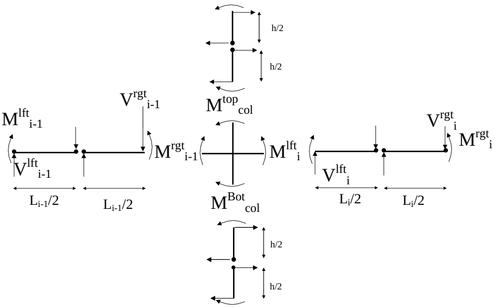
$$V_{ext}^{col} = \frac{\sum F^{lateral}}{2 \text{ No. of bays}} \qquad V_{int}^{col} = 2V_{ext}^{col}$$

2. Column Moments at the end of each column is equal to the shear at the column times half the height of the corresponding column

$$M^{top}_{col} = V^{col}\frac{h}{2} \qquad M^{bot} = -M^{top}$$

Careful exterior columns, the moments are half the ones of the interior.

3. Girder Moments is obtained from the free body diagram of the connection

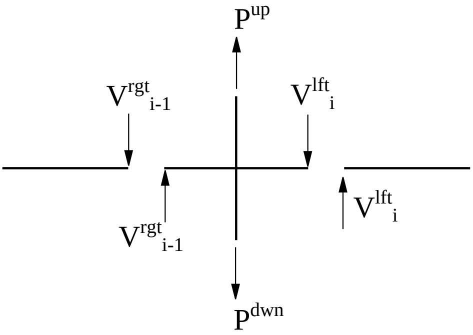


$$M_i^{lft} = M_{col}^{Top} - M_{col}^{Bot} + M_{i-1}^{rgt} \qquad M_i^{rgt} = -M_i^{lft}$$

4. Girder Shears Since there is an inflection point at the center of the girder, the girder shear is obtained by considering the sum of moments about that point

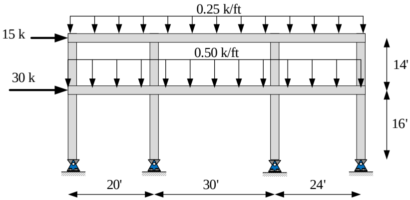
$$V^{lft} = -\frac{2M}{L} \qquad V^{rgt} = V^{lft}$$

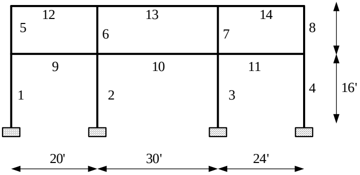
5. Column Axial Forces are obtained by summing girder shears and the axial force from the column above



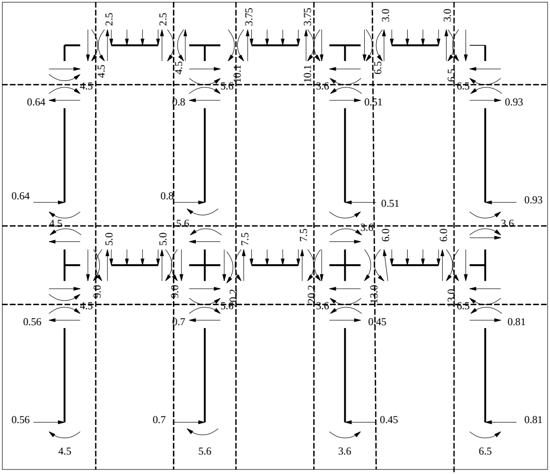
$$P = P^{above} - P^{rgt} + P^{lft}$$

6. blueGirder axial force are assumed to be negligible.


0.25 k/ft
15 k
0.50 k/ft
30 k
14'
16'
20'
30'
24'


12
13
14
5
6
7
8
9
10
11
1
2
3
4
16'
20'
30'
24'

Free body diagram



1. Top Girder Moments

$$
\begin{aligned}
M_{12}^{lft} &= -0.045 w_{12} L_{12}^2 = -(0.045)(0.25)(20)^2 &&= -\ 4.5 \text{ k.ft}\\
M_{12}^{cnt} &= 0.08 w_{12} L_{12}^2 = (0.08)(0.25)(20)^2 &&= \ 8.0 \text{ k.ft}\\
M_{12}^{rgt} &= M_{12}^{lft} &&= -\ 4.5 \text{ k.ft}\\
M_{13}^{lft} &= -0.045 w_{13} L_{13}^2 = -(0.045)(0.25)(30)^2 &&= -\ 10.1 \text{ k.ft}\\
M_{13}^{cnt} &= 0.08 w_{13} L_{13}^2 = (0.08)(0.25)(30)^2 &&= \ 18.0 \text{ k.ft}\\
M_{13}^{rgt} &= M_{13}^{lft} &&= -\ 10.1 \text{ k.ft}\\
M_{14}^{lft} &= -0.045 w_{14} L_{14}^2 = -(0.045)(0.25)(24)^2 &&= -\ 6.5 \text{ k.ft}\\
M_{14}^{cnt} &= 0.08 w_{14} L_{14}^2 = (0.08)(0.25)(24)^2 &&= \ 11.5 \text{ k.ft}\\
M_{14}^{rgt} &= M_{14}^{lft} &&= -\ 6.5 \text{ k.ft}
\end{aligned}
$$

2. Bottom Girder Moments

$$
\begin{aligned}
M_{9}^{lft} &= -0.045 w_{9} L_{9}^2 = -(0.045)(0.5)(20)^2 &&= -\ 9.0 \text{ k.ft}\\
M_{9}^{cnt} &= 0.08 w_{9} L_{9}^2 = (0.08)(0.5)(20)^2 &&= \ 16.0 \text{ k.ft}\\
M_{9}^{rgt} &= M_{9}^{lft} &&= -\ 9.0 \text{ k.ft}\\
M_{10}^{lft} &= -0.045 w_{10} L_{10}^2 = -(0.045)(0.5)(30)^2 &&= -\ 20.3 \text{ k.ft}\\
M_{10}^{cnt} &= 0.08 w_{10} L_{10}^2 = (0.08)(0.5)(30)^2 &&= \ 36.0 \text{ k.ft}\\
M_{10}^{rgt} &= M_{11}^{lft} &&= -\ 20.3 \text{ k.ft}\\
M_{11}^{lft} &= -0.045 w_{12} L_{12}^2 = -(0.045)(0.5)(24)^2 &&= -\ 13.0 \text{ k.ft}\\
M_{11}^{cnt} &= 0.08 w_{12} L_{12}^2 = (0.08)(0.5)(24)^2 &&= \ 23.0 \text{ k.ft}\\
M_{11}^{rgt} &= M_{12}^{lft} &&= -\ 13.0 \text{ k.ft}
\end{aligned}
$$

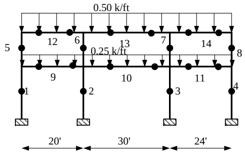
3. Top Column Moments

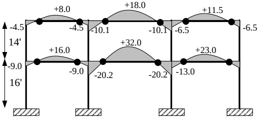
$$
\begin{array}{lllll}
M_5^{top} & = & +M_{12}^{lft} & = - & 4.5 \text{ k.ft} \\
M_5^{bot} & = & -M_5^{top} & = & 4.5 \text{ k.ft} \\
M_6^{top} & = & -M_{12}^{rgt} + M_{13}^{lft} = -(-4.5) + (-10.1) & = - & 5.6 \text{ k.ft} \\
M_6^{bot} & = & -M_6^{top} & = & 5.6 \text{ k.ft} \\
M_7^{top} & = & -M_{13}^{rgt} + M_{14}^{lft} = -(-10.1) + (-6.5) & = - & 3.6 \text{ k.ft} \\
M_7^{bot} & = & -M_7^{top} & = & 3.6 \text{ k.ft} \\
M_8^{top} & = & -M_{14}^{rgt} = -(-6.5) & = & 6.5 \text{ k.ft} \\
M_8^{bot} & = & -M_8^{top} & = - & 6.5 \text{ k.ft}
\end{array}
$$

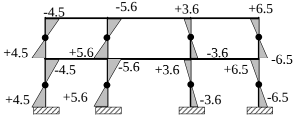
4. Bottom Column Moments

$$
\begin{array}{lllll}
M_1^{top} & = & +M_5^{bot} + M_9^{lft} = 4.5 - 9.0 & = - & 4.5 \text{ k.ft} \\
M_1^{bot} & = & -M_1^{top} & = & 4.5 \text{ k.ft} \\
M_2^{top} & = & +M_6^{bot} - M_9^{rgt} + M_{10}^{lft} = 5.6 - (-9.0) + (-20.3) & = - & 5.6 \text{ k.ft} \\
M_2^{bot} & = & -M_2^{top} & = & 5.6 \text{ k.ft} \\
M_3^{top} & = & +M_7^{bot} - M_{10}^{rgt} + M_{11}^{lft} = -3.6 - (-20.3) + (-13.0) & = & 3.6 \text{ k.ft} \\
M_3^{bot} & = & -M_3^{top} & = - & 3.6 \text{ k.ft} \\
M_4^{top} & = & +M_8^{bot} - M_{11}^{rgt} = -6.5 - (-13.0) & = & 6.5 \text{ k.ft} \\
M_4^{bot} & = & -M_4^{top} & = - & 6.5 \text{ k.ft}
\end{array}
$$

5. Moment Diagrams







6. Top Girder Shear

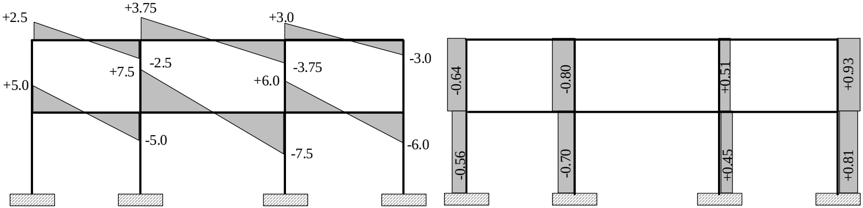
$$
\begin{aligned}
V_{12}^{lft} &= \frac{w_{12}L_{12}}{2} = \frac{(0.25)(20)}{2} &&= 2.5 \text{ k}\\
V_{12}^{rgt} &= -V_{12}^{lft} &&= -\ 2.5 \text{ k}\\
V_{13}^{lft} &= \frac{w_{13}L_{13}}{2} = \frac{(0.25)(30)}{2} &&= 3.75 \text{ k}\\
V_{13}^{rgt} &= -V_{13}^{lft} &&= -\ 3.75 \text{ k}\\
V_{14}^{lft} &= \frac{w_{14}L_{14}}{2} = \frac{(0.25)(24)}{2} &&= 3.0 \text{ k}\\
V_{14}^{rgt} &= -V_{14}^{lft} &&= -\ 3.0 \text{ k}
\end{aligned}
$$

7. Bottom Girder Shear

$$
\begin{aligned}
V_{9}^{lft} &= \frac{w_{9}L_{9}}{2} = \frac{(0.5)(20)}{2} &&= 5.00 \text{ k}\\
V_{9}^{rgt} &= -V_{9}^{lft} &&= -\ 5.00 \text{ k}\\
V_{10}^{lft} &= \frac{w_{10}L_{10}}{2} = \frac{(0.5)(30)}{2} &&= 7.50 \text{ k}\\
V_{10}^{rgt} &= -V_{10}^{lft} &&= -\ 7.50 \text{ k}\\
V_{11}^{lft} &= \frac{w_{11}L_{11}}{2} = \frac{(0.5)(24)}{2} &&= 6.00 \text{ k}\\
V_{11}^{rgt} &= -V_{11}^{lft} &&= -\ 6.00 \text{ k}
\end{aligned}
$$

8. Column Shears

$$
\begin{aligned}
V_5 &= \frac{M_5^{top}}{\frac{H_5}{2}} = \frac{-4.5}{\frac{14}{2}} &&= - & 0.64\text{ k}\\
V_6 &= \frac{M_6^{top}}{\frac{H_6}{2}} = \frac{-5.6}{\frac{14}{2}} &&= - & 0.80\text{ k}\\
V_7 &= \frac{M_7^{top}}{\frac{H_7}{2}} = \frac{3.6}{\frac{14}{2}} &&= & 0.52\text{ k}\\
V_8 &= \frac{M_8^{top}}{\frac{H_8}{2}} = \frac{6.5}{\frac{14}{2}} &&= & 0.93\text{ k}\\
V_1 &= \frac{M_1^{top}}{\frac{H_1}{2}} = \frac{-4.5}{\frac{16}{2}} &&= - & 0.56\text{ k}\\
V_2 &= \frac{M_2^{top}}{\frac{H_2}{2}} = \frac{-5.6}{\frac{16}{2}} &&= - & 0.70\text{ k}\\
V_3 &= \frac{M_3^{top}}{\frac{H_3}{2}} = \frac{3.6}{\frac{16}{2}} &&= & 0.46\text{ k}\\
V_4 &= \frac{M_4^{top}}{\frac{H_4}{2}} = \frac{6.5}{\frac{16}{2}} &&= & 0.81\text{ k}
\end{aligned}
$$



9 Top Column Axial Forces

$$
\begin{aligned}
P_5 &= V_{12}^{lft} && = 2.50 \text{ k}\\
P_6 &= -V_{12}^{rgt} + V_{13}^{lft} = -(-2.50) + 3.75 && = 6.25 \text{ k}\\
P_7 &= -V_{13}^{rgt} + V_{14}^{lft} = -(-3.75) + 3.00 && = 6.75 \text{ k}\\
P_8 &= -V_{14}^{rgt} && = 3.00 \text{ k}
\end{aligned}
$$

10 Bottom Column Axial Forces

$$
\begin{aligned}
P_1 &= P_5 + V_9^{lft} = 2.50 + 5.0 && = 7.5 \text{ k}\\
P_2 &= P_6 - V_{10}^{rgt} + V_9^{lft} = 6.25 - (-5.00) + 7.50 && = 18.75 \text{ k}\\
P_3 &= P_7 - V_{11}^{rgt} + V_{10}^{lft} = 6.75 - (-7.50) + 6.0 && = 20.25 \text{ k}\\
P_4 &= P_8 - V_{11}^{rgt} = 3.00 - (-6.00) && = 9.00 \text{ k}
\end{aligned}
$$

VERTICAL LOADS

Height	Span		L1		L2		L3	
Height	Span		20		30		24	
14	Load		0.25		0.25		0.25	
16	Load		0.5		0.5		0.5	

MOMENTS

Bay 1					Bay 2				Bay 3			
Col	Beam			Column	Beam			Column	Beam			Col
	Lft	Cnt	Rgt		Lft	Cnr	Rgt		Lft	Cnt	Rgt	
	-4.5	8.0	-4.5		-10.1	18.0	-10.1		-6.5	11.5	-6.5	
-4.5				-5.6				3.6				6.5
4.5				5.6				-3.6				-6.5
	-9.0	16.0	-9.0		-20.3	36.0	-20.3		-13.0	23.0	-13.0	
-4.5				-5.6				3.6				6.5
4.5				5.6				-3.6				-6.5

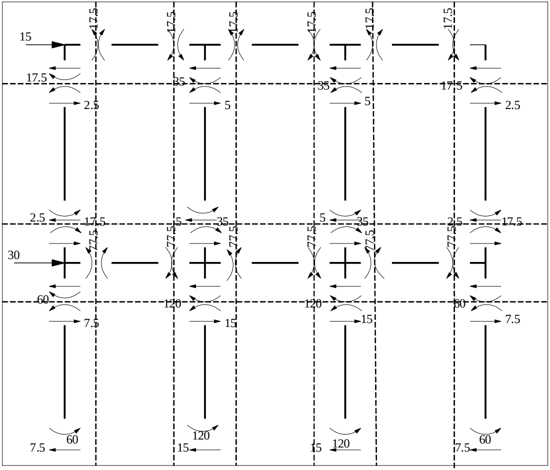
SHEAR

Bay 1					Bay 2				Bay 3			
Col	Beam			Column	Beam			Column	Beam			Col
	Lft		Rgt		Lft		Rgt		Lft		Rgt	
	2.50		-2.50		3.75		-3.75		3.00		-3.00	
-0.64				-0.80				0.52				0.93
	5.00		-5.00		7.50		-7.50		6.00		-6.00	
-0.56				-0.70				0.46				0.81

AXIAL FORCE

Bay 1			Bay 2		Bay 3	
Col	Beam	Column	Beam	Column	Beam	Col
	0.00		0.00		0.00	
2.50		6.25		6.75		3.00
	0.00		0.00		0.00	
7.50		18.75		20.25		9.00

Free body diagram



1. Column Shears

$$
\begin{aligned}
V_5 &= \frac{15}{(2)(3)} && = 2.5\text{ k}\\
V_6 &= 2(V_5) = (2)(2.5) && = 5\text{ k}\\
V_7 &= 2(V_5) = (2)(2.5) && = 5\text{ k}\\
V_8 &= V_5 && = 2.5\text{ k}\\
V_1 &= \frac{15+30}{(2)(3)} && = 7.5\text{ k}\\
V_2 &= 2(V_1) = (2)(7.5) && = 15\text{ k}\\
V_3 &= 2(V_1) = (2)(2.5) && = 15\text{ k}\\
V_4 &= V_1 && = 7.5\text{ k}
\end{aligned}
$$

2. Top Column Moments

$$
\begin{aligned}
M_5^{top} &= \frac{V_1 H_5}{2} = \frac{(2.5)(14)}{2} && = 17.5\text{ k.ft}\\
M_5^{bot} &= -M_5^{top} && = -17.5\text{ k.ft}\\
M_6^{top} &= \frac{V_6 H_6}{2} = \frac{(5)(14)}{2} && = 35.0\text{ k.ft}\\
M_6^{bot} &= -M_6^{top} && = -35.0\text{ k.ft}\\
M_7^{top} &= \frac{V_7^{up} H_7}{2} = \frac{(5)(14)}{2} && = 35.0\text{ k.ft}\\
M_7^{bot} &= -M_7^{top} && = -35.0\text{ k.ft}\\
M_8^{top} &= \frac{V_8^{up} H_8}{2} = \frac{(2.5)(14)}{2} && = 17.5\text{ k.ft}\\
M_8^{bot} &= -M_8^{top} && = -17.5\text{ k.ft}
\end{aligned}
$$

3. Bottom Column Moments

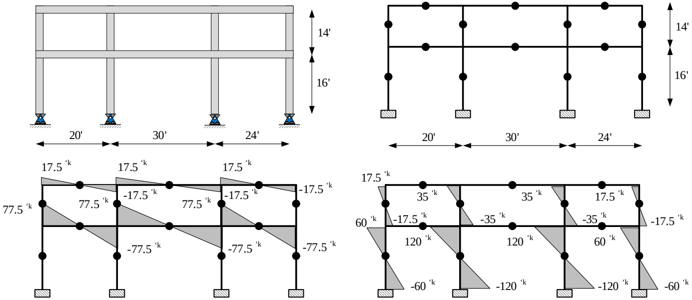
$$
\begin{aligned}
M_1^{top} &= \frac{V_1^{dwn} H_1}{2} = \frac{(7.5)(16)}{2} &&= 60 \text{ k.ft} \\
M_1^{bot} &= -M_1^{top} &&= - 60 \text{ k.ft} \\
M_2^{top} &= \frac{V_2^{dwn} H_2}{2} = \frac{(15)(16)}{2} &&= 120 \text{ k.ft} \\
M_2^{bot} &= -M_2^{top} &&= - 120 \text{ k.ft} \\
M_3^{top} &= \frac{V_3^{dwn} H_3}{2} = \frac{(15)(16)}{2} &&= 120 \text{ k.ft} \\
M_3^{bot} &= -M_3^{top} &&= - 120 \text{ k.ft} \\
M_4^{top} &= \frac{V_4^{dwn} H_4}{2} = \frac{(7.5)(16)}{2} &&= 60 \text{ k.ft} \\
M_4^{bot} &= -M_4^{top} &&= - 60 \text{ k.ft}
\end{aligned}
$$

4. Top Girder Moments

$$
\begin{aligned}
M_{12}^{lft} &= M_5^{top} &&= 17.5 \text{ k.ft} \\
M_{12}^{rgt} &= -M_{12}^{lft} &&= - 17.5 \text{ k.ft} \\
M_{13}^{lft} &= M_{12}^{rgt} + M_6^{top} = -17.5 + 35 &&= 17.5 \text{ k.ft} \\
M_{13}^{rgt} &= -M_{13}^{lft} &&= - 17.5 \text{ k.ft} \\
M_{14}^{lft} &= M_{13}^{rgt} + M_7^{top} = -17.5 + 35 &&= 17.5 \text{ k.ft} \\
M_{14}^{rgt} &= -M_{14}^{lft} &&= - 17.5 \text{ k.ft}
\end{aligned}
$$

5. Bottom Girder Moments

$$
\begin{aligned}
M_9^{lft} &= M_1^{top} - M_5^{bot} = 60 - (-17.5) &&= 77.5 \text{ k.ft} \\
M_9^{rgt} &= -M_9^{lft} &&= -77.5 \text{ k.ft} \\
M_{10}^{lft} &= M_9^{rgt} + M_2^{top} - M_6^{bot} = -77.5 + 120 - (-35) &&= 77.5 \text{ k.ft} \\
M_{10}^{rgt} &= -M_{10}^{lft} &&= -77.5 \text{ k.ft} \\
M_{11}^{lft} &= M_{10}^{rgt} + M_3^{top} - M_7^{bot} = -77.5 + 120 - (-35) &&= 77.5 \text{ k.ft} \\
M_{11}^{rgt} &= -M_{11}^{lft} &&= -77.5 \text{ k.ft}
\end{aligned}
$$



6. Top Girder Shear

$$
\begin{aligned}
V_{12}^{lft} &= -\frac{2M_{12}^{lft}}{L_{12}} = -\frac{(2)(17.5)}{20} &= -1.75 \text{ k} \\
V_{12}^{rgt} &= +V_{12}^{lft} &= -1.75 \text{ k} \\
V_{13}^{lft} &= -\frac{2M_{13}^{lft}}{L_{13}} = -\frac{(2)(17.5)}{30} &= -1.17 \text{ k} \\
V_{13}^{rgt} &= +V_{13}^{lft} &= -1.17 \text{ k} \\
V_{14}^{lft} &= -\frac{2M_{14}^{lft}}{L_{14}} = -\frac{(2)(17.5)}{24} &= -1.46 \text{ k} \\
V_{14}^{rgt} &= +V_{14}^{lft} &= -1.46 \text{ k}
\end{aligned}
$$

7. Bottom Girder Shear

$$
\begin{aligned}
V_{9}^{lft} &= -\frac{2M_{12}^{lft}}{L_{9}} = -\frac{(2)(77.5)}{20} &= -7.75 \text{ k} \\
V_{9}^{rgt} &= +V_{9}^{lft} &= -7.75 \text{ k} \\
V_{10}^{lft} &= -\frac{2M_{10}^{lft}}{L_{10}} = -\frac{(2)(77.5)}{30} &= -5.17 \text{ k} \\
V_{10}^{rgt} &= +V_{10}^{lft} &= -5.17 \text{ k} \\
V_{11}^{lft} &= -\frac{2M_{11}^{lft}}{L_{11}} = -\frac{(2)(77.5)}{24} &= -6.46 \text{ k} \\
V_{11}^{rgt} &= +V_{11}^{lft} &= -6.46 \text{ k}
\end{aligned}
$$

8. Top Column Axial Forces (+ve tension, -ve compression)

$$
\begin{aligned}
P_5 &= -V_{12}^{lft} & &= -(-1.75) \text{ k} \\
P_6 &= +V_{12}^{rgt} - V_{13}^{lft} = -1.75 - (-1.17) & &= -0.58 \text{ k} \\
P_7 &= +V_{13}^{rgt} - V_{14}^{lft} = -1.17 - (-1.46) & &= 0.29 \text{ k} \\
P_8 &= V_{14}^{rgt} = -1.46 \text{ k}
\end{aligned}
$$

9. Bottom Column Axial Forces (+ve tension, -ve compression)

$$
\begin{aligned}
P_1 &= P_5 + V_9^{lft} = 1.75 - (-7.75) & &= 9.5 \text{ k} \\
P_2 &= P_6 + V_{10}^{rgt} + V_9^{lft} = -0.58 - 7.75 - (-5.17) & &= -3.16 \text{ k} \\
P_3 &= P_7 + V_{11}^{rgt} + V_{10}^{lft} = 0.29 - 5.17 - (-6.46) & &= 1.58 \text{ k} \\
P_4 &= P_8 + V_{11}^{rgt} = -1.46 - 6.46 & &= -7.66 \text{ k}
\end{aligned}
$$

# of Bays	3

# of Storeys					2
		Force		Shear	
	H	Lat.	Tot	Ext	Int
H1	14	15	15	2.5	5
H2	16	30	45	7.5	15

HORIZONTAL LOAD

	L1			L2			L3		
	20			30			24		
MOMENTS									
Bay 1			**Bay 2**			**Bay 3**			
Col	Beam		Column	Beam		Column	Beam		Col
	Lft	Rgt		Lft	Rgt		Lft	Rgt	
	17.5	-17.5		17.5	-17.5		17.5	-17.5	
17.5			35.0			35.0			17.5
-17.5			-35.0			-35.0			-17.5
	77.5	-77.5		77.5	-77.5		77.5	-77.5	
60.0			120.0			120.0			60.0
-60.0			-120.0			-120.0			-60.0

SHEAR									
Bay 1			**Bay 2**			**Bay 3**			
Col	Beam		Column	Beam		Column	Beam		Col
	Lft	Rgt		Lft	Rgt		Lft	Rgt	
	-1.75	-1.75		-1.17	-1.17		-1.46	-1.46	
2.50			5.00			5.00			2.50
2.50			5.00			5.00			2.50
	-7.75	-7.75		-5.17	-5.17		-6.46	-6.46	
7.50			15.00			15.00			7.50
7.50			15.00			15.00			7.50

AXIAL FORCE						
Bay 1		**Bay 2**		**Bay 3**		
Col	Beam	Column	Beam	Column	Beam	Col
	0.00		0.00		0.00	
1.75		-0.58		0.29		-1.46
	0.00		0.00		0.00	
9.50		-3.17		1.58		-7.92

- Design Parameters On the basis of the two approximate analyses, vertical and lateral load, we now seek the design parameters for the frame.
- Columns

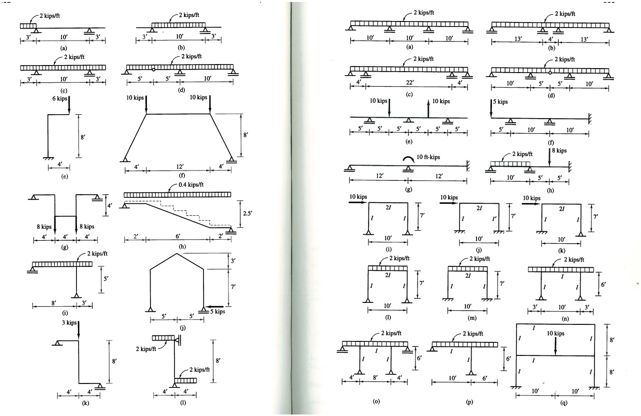
COLUMNS				
Mem.		Vert.	Hor.	**Design Values**
1	Moment	4.50	60.00	**64.50**
	Axial	7.50	9.50	**17.00**
	Shear	0.56	7.50	**8.06**
2	Moment	5.60	120.00	**125.60**
	Axial	18.75	15.83	**34.58**
	Shear	0.70	15.00	**15.70**
3	Moment	3.60	120.00	**123.60**
	Axial	20.25	14.25	**34.50**
	Shear	0.45	15.00	**15.45**
4	Moment	6.50	60.00	**66.50**
	Axial	9.00	7.92	**16.92**
	Shear	0.81	7.50	**8.31**
5	Moment	4.50	17.50	**22.00**
	Axial	2.50	1.75	**4.25**
	Shear	0.64	2.50	**3.14**
6	Moment	5.60	35.00	**40.60**
	Axial	6.25	2.92	**9.17**
	Shear	0.80	5.00	**5.80**
7	Moment	3.60	35.00	**38.60**
	Axial	6.75	2.63	**9.38**
	Shear	0.51	5.00	**5.51**
8	Moment	6.50	17.50	**24.00**
	Axial	3.00	1.46	**4.46**
	Shear	0.93	2.50	**3.43**

- Beams

BEAMS				
Mem.		Vert.	Hor.	**Design Values**
9	-ve Moment	9.00	77.50	**86.50**
	+ve Moment	16.00	0.00	**16.00**
	Shear	5.00	7.75	**12.75**
10	-ve Moment	20.20	77.50	**97.70**
	+ve Moment	36.00	0.00	**36.00**
	Shear	7.50	5.17	**12.67**
11	-ve Moment	13.0	77.50	**90.50**
	+ve Moment	23.00	0.00	**23.00**
	Shear	6.00	6.46	**12.46**
12	-ve Moment	4.50	17.50	**22.00**
	+ve Moment	8.00	0.00	**8.00**
	Shear	2.50	1.75	**4.25**
13	-ve Moment	10.10	17.50	**27.60**
	+ve Moment	18.00	0.00	**18.00**
	Shear	3.75	1.17	**4.92**
14	-ve Moment	6.50	17.50	**24.00**
	+ve Moment	11.50	0.00	**11.50**
	Shear	3.00	1.46	**4.46**

1. Qualitatively draw deflected shapes, indicate positions of inflection points.
2. Draw corresponding moment diagram.
3. Redraw structure with finite member depths, and indicate location of reinforcement.
4. compute exact moments and compare.

From pg 331 book by Meyer



Structural Analysis

Deflections; Elastic Curve

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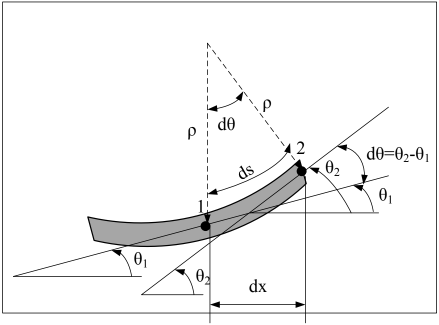
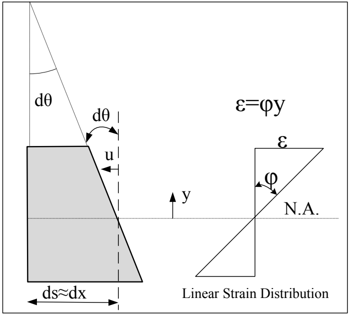
Spring 2019

Introduction

Reactions → Internal Forces → **Deflections** → Flexibility Method (SIS)

- Deflections of structures must be determined in order to satisfy serviceability requirements i.e. limit deflections under *service* loads to acceptable values (such as $\frac{\Delta}{L} \leq 360$).
- Later on, we will see that deflection calculations play an important role in the analysis of statically indeterminate structures.
- We shall focus on flexural deformation, however the end of this chapter will review axial and torsional deformations as well.
- Most of this chapter will be a *review* of subjects covered in *Strength of Materials*.
- This chapter will examine deflections of structures based on geometric considerations. Later on, we will present a more powerful method based on energy considerations.





- The **slope** is denoted by θ, the change in slope per unit length is the **curvature** ϕ, the **radius of curvature** is ρ. From *Strength of Materials* we have the following relations

$$\phi = \frac{1}{\rho} = \frac{d\theta}{ds} \tag{1}$$

- For small displacements, and as a first order approximation, with $ds \approx dx$ and $\theta = \frac{dy}{dx}$ Eq. 1 becomes

$$\phi = \frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2} \tag{2}$$

- A positive $d\theta$ at a positive y (upper fibers) will cause a *shortening* of the upper fibers $du = -yd\theta$, Dividing both sides by dx,

$$\underbrace{\frac{du}{dx}}_{\varepsilon} = -y\frac{d\theta}{dx}$$

- Combining this with Eq. 2

$$\frac{1}{\rho} = \phi = -\frac{\varepsilon}{y}$$

This is the fundamental relationship between curvature (ϕ), elastic curve (i.e. displacement) (y), and linear strain (ε).

- Note that so far we made no assumptions about material properties, i.e. it can be elastic or inelastic.
- For the elastic case:

$$\left.\begin{array}{rcl} \varepsilon & = & \frac{\sigma}{E} \\ \sigma & = & -\frac{My}{I} \end{array}\right\} \varepsilon = -\frac{My}{EI} \tag{3}$$

- Combining this last equation with Eq. 1 yields

$$\phi = \frac{1}{\rho} = \frac{d\theta}{\mathrm{d}x} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{M}{EI}$$

  This fundamental differential equation governing for beam. Similar equations will be derived later for cables and beam-columns.

- Combining this equation with the moment-shear-force relations determined in the previous chapter

$$\left.\begin{array}{rcl} \frac{dV}{\mathrm{d}x} & = & w(x) \\ \frac{dM}{\mathrm{d}x} & = & V(x) \end{array}\right\} \frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = w(x)$$

  we obtain

$$\frac{w(x)}{EI} = \frac{d^4 y}{\mathrm{d}x^4}$$

- † Next, we shall (re)derive the **exact** expression for the curvature.

$$\tan\theta = \frac{dy}{dx} \tag{4}$$

- Defining t as $t = \frac{dy}{dx}$ and combining with Eq. 4 we obtain $\theta = \tan^{-1} t$
- Applying the chain rule to $\phi = \frac{d\theta}{ds}$ we have

$$\phi = \frac{d\theta}{dt}\frac{dt}{ds} \tag{5}$$

ds can be rewritten as

$$\left.\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx \\ t &= \frac{dy}{dx} \end{aligned}\right\} ds = \sqrt{1+t^2}dx \tag{6}$$

- Next combining Eq. 5 and 6 we obtain

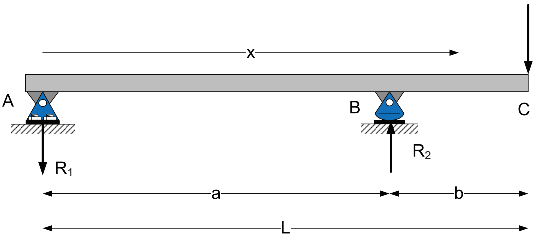
$$\left.\begin{aligned} \phi &= \frac{d\theta}{dt}\frac{dt}{\sqrt{1+t^2}dx} \\ \theta &= \tan^{-1} t \\ \frac{d\theta}{dt} &= \frac{1}{1+t^2} \end{aligned}\right\} \left.\begin{aligned} \phi &= \frac{1}{1+t^2}\frac{1}{\sqrt{1+t^2}}\frac{dt}{dx} \\ \frac{dt}{dx} &= \frac{d^2y}{dx^2} \end{aligned}\right\} \phi = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \tag{7}$$

- Thus the slope θ, curvature ϕ, radius of curvature ρ are related to the y displacement at a point x along a flexural member by

$$\phi = \frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

- If the displacements are very small, we will have $\frac{dy}{dx} << 1$, thus the last equation reduces to $\phi = \frac{d^2y}{dx^2} = \frac{1}{\rho}$

Example I



- Reactions

$$
\begin{aligned}
\left(+\circlearrowleft\right) \Sigma M_z^B = 0; \Rightarrow aR_1 - bP &= 0 \Rightarrow R_1 = \frac{b}{a}P \\
\left(+\circlearrowleft\right) \Sigma M_z^A = 0; \Rightarrow aR_2 - PL &= 0 \Rightarrow R_2 = \frac{L}{a}P
\end{aligned}
$$

Example II

- Differential equation

$$
\begin{aligned}
EIy'' &= -\frac{b}{a}Px + \frac{L}{a}P < x - a > \\
EIy' &= -\frac{b}{2a}Px^2 + \frac{L}{2a}P < x - a >^2 + C_1 \\
EIy &= -\frac{b}{6a}Px^3 + \frac{L}{6a}P < x - a >^3 + C_1 x + C_2
\end{aligned}
$$

- Apply the boundary conditions, at $x = 0$, $y = 0$ therefore $C_2 = 0$, and at $x = a$, $y = 0$, thus $0 = -[b/(6a)]Pa^3 + aC_1$ or $C_1 = (ab/6)P$
- Slope under the load (note $x = a + b = L$)

$$
\begin{aligned}
EIy' &= -\frac{b}{2a}P(a+b)^2 + \frac{a+b}{2a}Pb^2 + \frac{ab}{6}P \\
&= -\frac{b}{2a}P(a^2 + 2ab + b^2) + \frac{ab^2 + b^3}{2a}P + \frac{ab}{6}P \\
&= \ldots \\
&= -\frac{1}{6}b(2L + b)P
\end{aligned}
$$

Example III

- Deflection under the load P:

$$\begin{aligned} EIy &= -\frac{b}{6a}P(a+b)^3 + \frac{a+b}{6a}Pb^3 + \frac{ab}{6}P(a+b) \\ &= ... \\ &= -\frac{1}{3}Lb^2P \end{aligned}$$

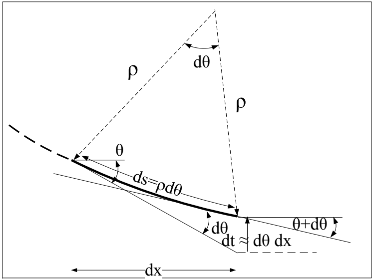
- Maximum deflection between the supports will occur where $y' = 0$.

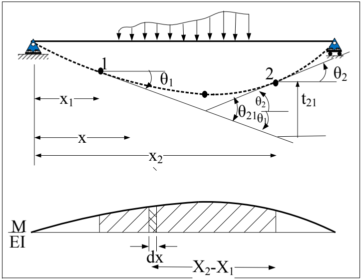
$$EIy' = -\frac{b}{2a}Px^2 + \frac{L}{2a} < x - a >^2 + \frac{ab}{6}P$$

at $y' = 0$, , $x - a >$ does not exist, thus $0 = -\frac{b}{2a}Px^2 + \frac{ab}{6}P$ solving for a, $a = \frac{1}{\sqrt{3}}a$, thus we can write

$$\begin{aligned} EIy_{max} &= -\frac{b}{6a}P\left(\frac{1}{\sqrt{3}}a\right)^3 + \frac{ab}{6}P\left(\frac{1}{\sqrt{3}}a\right) \\ &= ... \\ &= \frac{a^2b}{9\sqrt{3}}P \end{aligned}$$

Curvature Area Method; First Moment Area Theorem





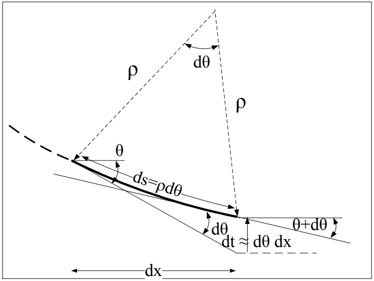
- From equation 4 we have $\frac{d\theta}{\mathrm{d}x} = \frac{M}{EI}$ this can be rewritten as (note similarity with $\frac{\mathrm{d}V}{\mathrm{d}x} = w(x)$).

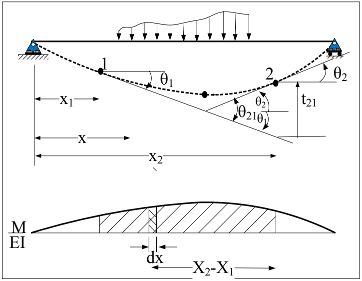
$$\theta_{21} = \theta_2 - \theta_1 = \int_{x_1}^{x_2} d\theta = \int_{x_1}^{x_2} \frac{M}{EI}\mathrm{d}x \tag{8}$$

First Area Moment Theorem:

The change in slope from point 1 to point 2 on a beam is equal to the area under the M/EI curvature diagram between those two points.

Curvature Area Method; Second Moment Area Theorem I





- We define by t_{21} the distance between point 2 and the tangent at point 1. For an infinitesimal distance $ds = \rho d\theta$ and for small displacements

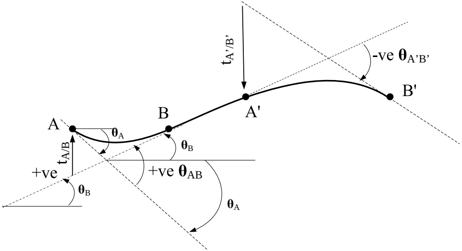
$$\left.\begin{array}{rcl} \mathrm{d}t & = & \mathrm{d}\theta(x_2 - x_1) \\ \frac{\mathrm{d}\theta}{\mathrm{d}x} & = & \frac{M}{EI} \end{array}\right\} \mathrm{d}t = \frac{M}{EI}(x_2 - x_1)\mathrm{d}x$$

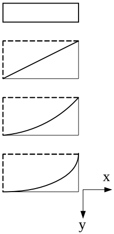
$$t_{21} = \int_{x_1}^{x_2} dt = \int_{x_1}^{x_2} \frac{M}{EI}(x_2 - x_1)\mathrm{d}x$$

Curvature Area Method; Second Moment Area Theorem II

Second Moment Area Theorem: The tangent distance t_{21} between a point, 2, on the beam and the tangent of another point, 1, is equal to the moment of the M/EI diagram between points 1 and 2, with respect to point 2.

Curvature Area Method; Misc.





degree	AREA (solid)	AREA (dashed)	CENTROID (solid)	CENTROID (dashed)
0	xy	—	$\frac{x}{2}$	—
1	$\frac{xy}{2}$	$\frac{xy}{2}$	$\frac{2}{3}x$	$\frac{x}{3}$
2	$\frac{xy}{3}$	$\frac{2}{3}xy$	$\frac{3}{4}x$	$\frac{3x}{8}$
3	$\frac{xy}{4}$	$\frac{3}{4}xy$	$\frac{4}{5}x$	$\frac{2x}{5}$
n	$\frac{xy}{n+1}$	$\frac{nxy}{n+1}$	$\frac{x}{n+2}$	$\frac{(n+1)x}{2(n+2)}$

Elastic Weight/Conjugate Beams I

- There is a strong analogy between the two sets of relationships:
  1. Load (w), shear (V) and moment (M).
  2. Curvature ($1/\rho = M/EI$), slope (θ) and displacement (y).

  those are summarized in the following table

V and M		θ and y	
$V_{21} = \int_{x_1}^{x_2} w dx$	$V = \int w dx + C_1$	$\theta_{21} = \int_{x_1}^{x_2} \frac{1}{\rho} dx$	$\theta = \int \frac{1}{\rho} dx + C_1$
$M_{21} = \int_{x_1}^{x_2} V dx$	$M = \int V dx + C_2$	$t_{21} = \int_{x_1}^{x_2} \theta dx$	$y = \int \theta dx + C_2$

- Since we know how to draw the shear and moment diagrams for actual load, we can apply the same methodology to *elastic weight* and similarly determine slope and deflection.

$$\textit{Load } q \equiv \text{curvature } \frac{1}{\rho} = \phi = \frac{M}{EI} \quad (9)$$

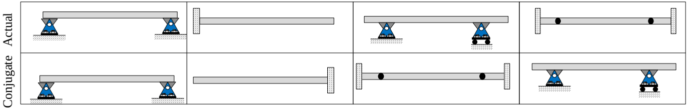
$$\textit{Shear } V \equiv \text{slope } \theta \quad (10)$$

$$\textit{Moment } M \equiv \text{deflection } y \quad (11)$$

Elastic Weight/Conjugate Beams II

- Since V & M can be conjugated from statics, by analogy θ & y can be thought of as the V & M of a fictitious beam (or **conjugate beam**) loaded by $\frac{M}{EI}$ **elastic weight**.
- Boundary conditions are determined from

Actual Beam			Conjugate Beam		
Hinge	$\theta \neq 0$	$y = 0$	$V \neq 0$	$M = 0$	"Hinge"
Fixed End	$\theta = 0$	$y = 0$	$V = 0$	$M = 0$	Free end
Free End	$\theta \neq 0$	$y \neq 0$	$V \neq 0$	$M \neq 0$	Fixed end
Interior Hinge	$\theta \neq 0$	$y \neq 0$	$V \neq 0$	$M \neq 0$	Interior support
Interior Support	$\theta \neq 0$	$y = 0$	$V \neq 0$	$M = 0$	Interior hinge

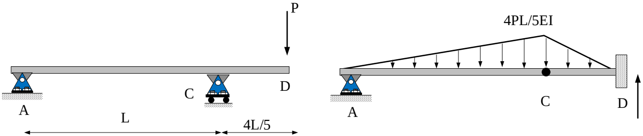


- Whereas the curvature area method has a well defined basis, its direct application can be sometimes confusing.

Elastic Weight/Conjugate Beams III

- Alternatively, the curvature area method was derived from the moment area method, and is a far simpler method to remember and use in practice when simple “back of the envelope” calculations are required.
- Note that we can only have distributed load, and that the load is positive for a positive moment, and negative for a negative moment. “Shear” and “Moment” diagrams should be drawn accordingly.
- **Units** of the “distributed load” w^* are $\frac{FL}{EI}$ (force time length divided by EI). Thus the “Shear” would have units of $w^* \times L$ or $\frac{FL^2}{EI}$ and the “moment” would have units of $(w^* \times L) \times L$ or $\frac{FL^3}{EI}$. Recalling that EI has units of $FL^{-2}L^4 = FL^2$, we observe that indeed the “shear” corresponds to a rotation in radians and the “moment” to a displacement.

Example 1; I



- 3 equations of equilibrium and 1 equation of condition = 4 = number of reactions. Deflection at D = Shear at D of the corresponding conjugate beam (Reaction at D) Take AC and ΣM with respect to C

$$\left(+\circlearrowleft\right)\Sigma M_z^C = 0 \Rightarrow R_A(L) - \left(\frac{4PL}{5EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right) = 0 \tag{12}$$

$$\Rightarrow R_A = \frac{2PL^2}{15EI} \tag{13}$$

which is the slope in real beam at A As computed before!

Example 1; II

- Let us draw the Moment Diagram for the conjugate beam at a point x away from A. From A to C

$$M = \frac{P}{EI}\left[\frac{2}{15}L^2x - \left(\frac{4}{5}x\right)\left(\frac{x}{2}\right)\left(\frac{x}{3}\right)\right] \quad (14)$$

$$= \frac{P}{EI}\left(\frac{2}{15}L^2x - \frac{2}{15}x^3\right) \quad (15)$$

$$= \frac{2P}{15EI}\left(L^2x - x^3\right) \quad (16)$$

- Point of Maximum Moment (Δ_{max}) occurs when $\frac{\mathrm{d}M}{\mathrm{d}x} = 0$

$$\frac{dM}{dx} = \frac{2P}{15EI}(L^2 - 3x^2) = 0 \Rightarrow 3x^2 = L^2 \Rightarrow x = \frac{L}{\sqrt{3}} \quad (17)$$

Example 1; III

as previously determined

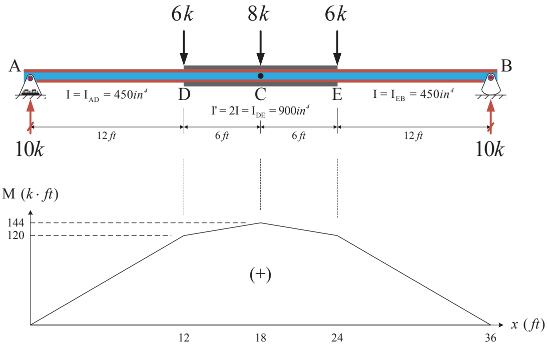
$$x = \frac{L}{\sqrt{3}} \tag{18}$$

$$\Rightarrow M = \frac{2P}{15EI}\left(\frac{L^2 L}{\sqrt{3}} - \frac{L^3}{3\sqrt{3}}\right) \tag{19}$$

$$= \frac{4PL^3}{45\sqrt{3}EI} \tag{20}$$

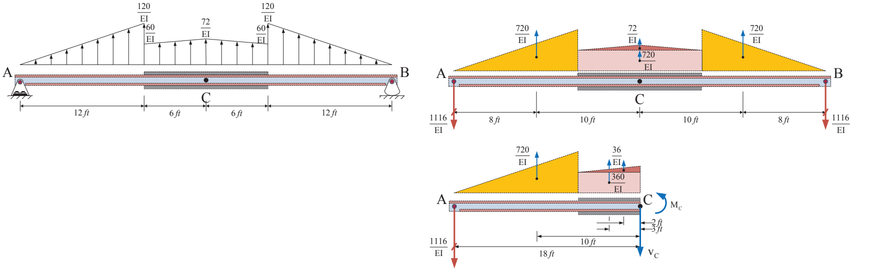
as before.

Example 2; I



- From simple observation, the reactions at A and B are equal to 10 k. The elastic load on the conjugate beam is then shown below.

Example 2; II



- We next seek to determine the internal moment at C' in the conjugate beam, it is obtained from equilibrium:

$$\left(+\curvearrowleft\right)\ \Sigma M_z^B = -;\ \frac{1,116}{EI}(18) - \frac{720}{EI}(10) - \frac{360}{EI}(3) - \frac{36}{EI}(2) + M_C = 0 \Rightarrow M_{C'} = -\frac{11,736\text{k.ft}^3}{EI} \quad (21)$$

Substituting

$$\Delta_C = M_{C'} = -\frac{11,736\text{k.ft}^3(12^3)\text{in}^3/\text{ft}^3}{(29\times10^3)\text{k/in}^2(450)\text{in}^4} = -1.55" \quad (22)$$

Structural Analysis

Virtual Work

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Spring 2022

Table of Contents I

Introduction

Reactions → Internal Forces → **Deflections** → Flexibility Method (SIS)

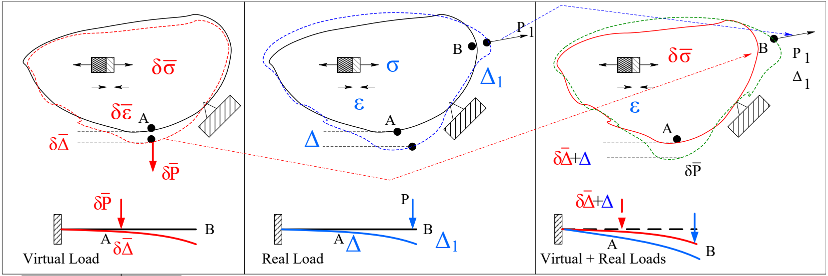
- Determination of displacements is critical in structural analysis:
  - Deflections are need to assess stiffness of a structure (i.e all design codes impose a maximum allowable displacement)
  - Must be determined to analyses statically indeterminate structures by the flexibility method.
- Many methods are available to compute deflections. However, we will focus on the most efficient and powerful one based on the Principle of Virtual Force (PVf).
- Strictly speaking (as shall be seen later) this is the Principle of Complementary Virtual Work.
- In the context of this chapter, we will refer to it as Principle of Virtual Work (PVW).
- This is the only unified method that allows us to compute deflections in all types of structures, under a variety of loads (including thermal and initial deformation), and for both linear and nonlinear structures.
- The method will be revisited in more advanced courses (Matrix Structural Analysis or Finite Element Analysis).
- In terms of notation one is confronted with a small dilemma:

- Use the simplified notation of the textbook, however this will ill-prepare you for subsequent courses.
- Use right away the more rigorous notation that is understandable across all courses.

- We will proceed with the second. Virtual quantities will be preceded by a δ and will have an overbar above (such as $\delta\overline{P}$).
- The correspondence between the two notations (as in the textbook by *Leet*) is as follows

Variable	Textbook	These notes
Dummy Force	Q	$\delta\overline{P}$
Virtual element Bar Force	F_Q	$\delta\overline{P}^{(e)}$
Real element Bar Force	F_P	$P^{(e)}$
Virtual Moment	M_Q	$\delta\overline{M}$
Real Moment	M_P	M
Virtual Strain Energy	U_Q	$\delta\overline{U}^*$
Virtual Work	W_Q	$\delta\overline{W}^*$

- This chapter will begin with a simplified derivation of the PVW at its most elementary level (in terms of stress and strain internally), and then generalize to truss, beam, frames.



$$\underbrace{\frac{1}{2}\delta\bar{P}\,\Delta}_{Ext} = \underbrace{\frac{1}{2}\int \delta\bar{\sigma}\,\delta\varepsilon\ dVol}_{Int}$$

$$\underbrace{\frac{1}{2}P_1\Delta_1}_{Ext} = \underbrace{\frac{1}{2}\int \sigma\varepsilon\ dVol}_{Int}$$

$$\frac{1}{2}\delta\bar{P}\Delta + \frac{1}{2}P_1\Delta_1 + \delta\bar{P}\,\Delta = \frac{1}{2}\int \delta\bar{\sigma}\,\delta\varepsilon\ dVol + \frac{1}{2}\int \sigma\varepsilon\ dVol + \int \delta\bar{\sigma}\ \varepsilon\ dVol$$

$$\underbrace{\delta\bar{P}\ \Delta}_{Ext} = \underbrace{\int \delta\bar{\sigma}\ \varepsilon\ dVol}_{Int}$$

- Consider an arbitrary structure and load. or the sake of simplicity, let us assume (or consider) that this structure develops only internal axial stresses and strains (σ and ε).
- The structure will be subjected to two types of loads:

Virtual load applied at the location A and along the direction (degree of freedom) where we want to compute the displacement (or rotation). It is virtual, and its value is irrelevant, but is often assumed to be unity.

Real corresponding to the actual externally applied load, B.

- We are going to load the structure in two different sequences:
  - Apply virtual load only at A (obtain Eq. 1); OR apply real load at B only (obtain Eq. 2).
  - Apply virtual load at A AND then apply real load at B (while virtual load is still on).

This will result in:

1. Application of virtual load at A; external work must be equal to the internal strain energy over the entire volume, then:

$$\underbrace{\frac{1}{2}\delta\overline{P}\delta\overline{\Delta}}_{\text{External virtual work}} = \underbrace{\frac{1}{2}\int_{\text{dVol}}\delta\overline{\sigma}\delta\overline{\varepsilon}\text{dVol}}_{\text{Internal virtual strain energy}} \tag{1}$$

   - The 1/2 stems from the fact that the load is gradually applied ramping from 0 to it full value linearly. The area under the curve represents the external work (likewise for the internal strain energy).
   - Strain energy is the internal work and is integrated over the volume

2. Application of real load at B. Again, external work must be equal to the internal strain energy over the entire volume, then:

$$\underbrace{\frac{1}{2}P_1\Delta_1}_{\text{External real work}} = \underbrace{\frac{1}{2}\int_{\text{dVol}}\sigma\epsilon\text{dVol}}_{\text{Internal real strain energy}} \tag{2}$$

3. Now, virtual load is first applied (resulting in $\frac{1}{2}\delta\overline{P}\delta\overline{\Delta} = \frac{1}{2}\int_{\text{dVol}}\delta\overline{\sigma}\delta\overline{\varepsilon}\text{dVol}$) and we then apply the real (actual) load on top of the deformed system (resulting in $\frac{1}{2}P_1\Delta_1 = \frac{1}{2}\int_{\text{dVol}}\sigma\epsilon\text{dVol}$). However, as we applied the second real load, the $\delta\overline{P}$ remained constant but underwent an additional displacement Δ. So that there is an additional external work equal to $\delta\overline{P}\Delta$ at that location equal to

$$\delta\overline{P}\Delta = \int_{\text{Vol}}\delta\overline{\sigma}\varepsilon\text{dVol} \tag{3}$$

(note absence of 1/2 term). Hence, the total work done becomes

4. Summing Eq. 1, 2 and 3 we obtain:

$$\underbrace{\overbrace{\frac{1}{2}\delta\overline{P}\delta\overline{\Delta}}^{A} + \overbrace{\frac{1}{2}P_1\Delta_1}^{B}}_{\text{Uncoupled}} + \underbrace{\overbrace{\delta\overline{P}\Delta}^{A-B}}_{\text{Coupled}} = \underbrace{\overbrace{\frac{1}{2}\int_{\text{Vol}}\delta\overline{\sigma}\delta\overline{\varepsilon}\text{dVol}}^{A} + \overbrace{\frac{1}{2}\int_{\text{Vol}}\sigma\varepsilon\text{dVol}}^{B}}_{\text{Uncoupled}} + \underbrace{\overbrace{\int_{\text{Vol}}\delta\overline{\sigma}\varepsilon\text{dVol}}^{A-B}}_{\text{Coupled}} \tag{4}$$

5. Since the strain energy and work done must be the same whether the loads are applied together or separately, we obtain, from subtracting the sum of Eqs. 2 and 1 from 4 and generalizing, we obtain

$$\underbrace{\int \delta\overline{\sigma}\varepsilon d\mathrm{Vol}}_{\delta\overline{U}^*} = \underbrace{\delta\overline{P}\Delta}_{\delta\overline{W}^*}$$

- This last equation is the key to the method of virtual forces. The left hand side is the internal virtual strain energy $\delta\overline{U}^*$[1]. Similarly the right hand side is the external virtual work.
- A variation of this derivation would lead to the so-called Maxwell-Betti reciprocal theorem which states that *If two load sets act on a linearly elastic structure, work done by the first set of loads in acting through the displacements produced by the second set of loads is equal to the work done by the second set of loads in acting through displacements produced by the first set.*

$$P_1\delta_{12} = P_2\delta_{21}$$

[1] We use the * to distinguish it from the internal virtual strain energy obtained from the **virtual displacement** method $\delta\overline{U}$.

$$\delta\overline{W}^* = \sum\nolimits_{i=1}^{n}(\Delta_i)\delta\overline{P}_i + \sum\nolimits_{i=1}^{n}(\theta_i)\delta\overline{M}_i$$

- Note that there is no such a thing as distributed virtual load.
- Recall that all overbar quantities are virtual and the other ones are the real.

- The general expression for the internal virtual work is

$$\delta\overline{U}^* = \int \delta\overline{\sigma}\varepsilon \mathrm{dVol}$$

- We skip the general formulation for inelastic systems.
- Should we have a linear elastic material $\sigma = E\varepsilon$.
- One has to be extremely careful in properly handling the units

Axial Members:

$$\left.\begin{array}{rcl}
\delta\overline{U}^* & = & \int_{\mathrm{dVol}} \varepsilon(x)\delta\overline{\sigma}(x)d\mathrm{Vol} \\
\delta\overline{\sigma}(x) & = & \frac{\delta\overline{P}^{(e)}(x)}{A} \\
\varepsilon(x) & = & \frac{P^{(e)}(x)}{AE} \\
dV & = & Adx
\end{array}\right\} \delta\overline{U}^* = \int_0^L \delta\overline{P}^{(e)}(x) \underbrace{\frac{P^{(e)}(x)}{AE}}_{\Delta} dx$$

Note that for a truss where we have n members, the above expression becomes

$$\delta\overline{U}^* = \Sigma_1^n \delta\overline{P}^{(i)} \frac{P^{(i)} L_i}{A_i E_i}$$

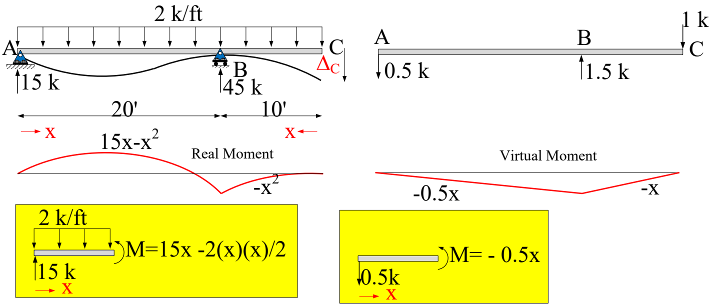
Note that P is the axial force caused by the real load and $\delta\overline{P}$ is the axial force caused by the virtual load. Those forces are determined from a truss analysis.

Flexural Members:

$$\left.\begin{array}{rcl}
\delta\overline{U}^* & = & \int_{\text{Vol}} \varepsilon(x) \underbrace{E\delta\overline{\varepsilon}(x)}_{\delta\overline{\sigma}(x)} \text{dVol} \\
\delta\overline{\sigma}_x(x) & = & \frac{\delta\overline{M}_z(x)y}{I_z} \\
\varepsilon(x) & = & \frac{M_z(x)y}{EI_z} \\
d\text{Vol} & = & dA\,dx \\
I_z & = & \int_A y^2 dA
\end{array}\right\} \delta\overline{U}^* = \int_0^L \delta\overline{M}(x) \underbrace{\frac{M(x)}{EI_z}}_{\Phi(x)} dx$$

Again, Note that $M(x)$ is the moment diagram caused by the real load and $\delta\overline{M}(x)$ is the moment diagram caused by the virtual load.
Which is why we need to have analytical expressions for the moments.

Determine the deflection at point C. $E = 29,000$ ksi, $I = 100$ in^4.



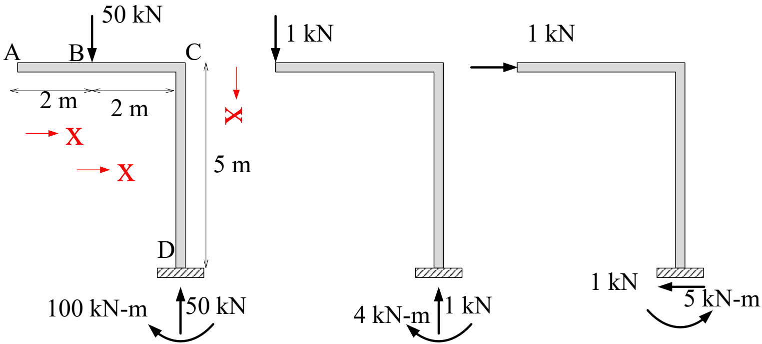
Element	$x=0$	$M(x)$	$\delta\overline{M}(x)$
AB	A	$15x-x^2$	$-0.5x$
BC	C	$-x^2$	$-x$

Units: k & in.

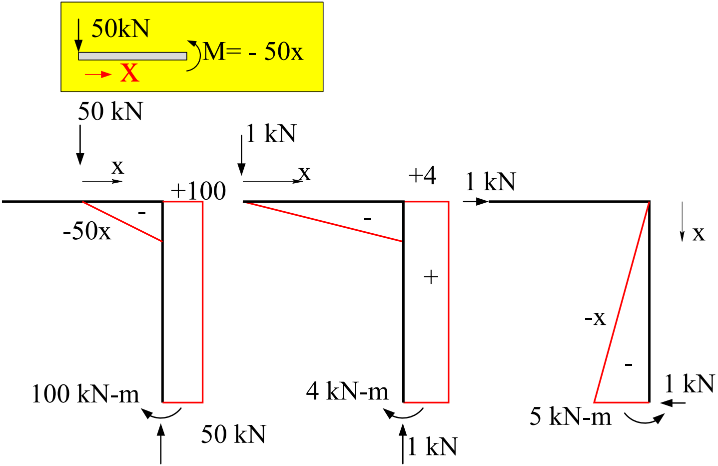
Applying the principle of virtual work, we obtain

$$
\begin{aligned}
\underbrace{\Delta_C \delta \overline{P}}_{\delta \overline{W}^*} &= \underbrace{\int_0^L \delta \overline{M}(x) \frac{M(x)}{EI_z} dx}_{\delta \overline{U}^*} \\
(1)k(\Delta_C)ft &= \int_0^{20} (-0.5x) \frac{(15x - x^2)}{EI} dx + \int_0^{10} (-x) \frac{-x^2}{EI} dx \\
&= \frac{2,500}{EI} \\
(1)k(\Delta_C) \text{ in} &= \frac{(2,500) \text{ k}^2 - \text{ ft}^4 (1,728) \text{ in}^3 / \text{ ft}^3}{(29,000) \text{ k/ in}^2 (100) \text{ in}^4} \\
&= \boxed{1.49 \text{ in}}
\end{aligned}
$$

Determine both the vertical and horizontal deflection at A. $E = 200 \times 10^6$ kN/ m^2, $I = 200 \times 10^6$ mm^4.



- To analyse this frame we must determine analytical expressions for the moments along each member for the real load and the two virtual ones. One virtual load is a unit horizontal load at A, and the other a unit vertical one at A also.



Element	$x = 0$	M	$\delta \overline{M}_v$	$\delta \overline{M}_h$
AB	A	0	$-x$	0
BC	B	$-50x$	$-2 - x$	0
CD	C	100	4	$-x$

- Note that moments are considered positive when they produce compression on the inside of the frame.
- Units: kN & m

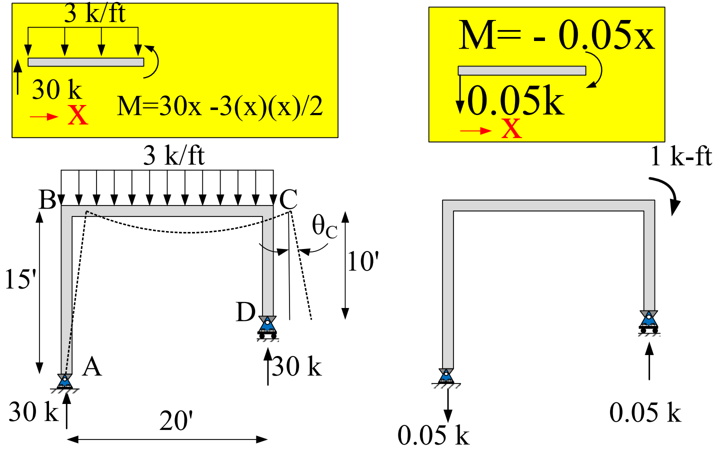
$$
\begin{aligned}
\underbrace{\Delta_v \delta\overline{P}}_{\delta\overline{W}^*} &= \underbrace{\int_0^L \delta\overline{M}(x)\frac{M(x)}{EI_z}dx}_{\delta\overline{U}^*} \\
(1)kN(\Delta_v)m &= \int_0^2 (-x)\frac{(0)}{EI}dx + \int_0^2 (-2-x)\frac{-50x}{EI}dx + \int_0^5 (4)\frac{100}{EI}dx \\
&= \frac{2,333 \text{ kN}^2 \text{ m}^4}{EI} \\
&= \frac{(2,333) \text{ kN}^2 \text{ m}^4 (10^3)^4 \text{ mm}^4/\text{ m}^4}{(200\times 10^6) \text{ kN}/\text{ m}^2 (200\times 10^6) \text{ mm}^4} \\
&= 0.058 \text{ m} = \boxed{5.8 \text{ cm}}
\end{aligned}
$$

- Similarly for the horizontal displacement:

$$\begin{aligned}
\underbrace{\Delta_h \delta \overline{P}}_{\delta \overline{W}^*} &= \underbrace{\int_0^L \delta \overline{M}(x) \frac{M(x)}{EI_z} dx}_{\delta \overline{U}^*} \\
(1)kN(\Delta_h)m &= \int_0^2 (0) \frac{(0)}{EI} dx + \int_0^2 (0) \frac{-50x}{EI} dx + \int_0^5 (-x) \frac{100}{EI} dx \\
&= \frac{-1,250 \text{ kN}^2 \text{ m}^4}{EI} \\
&= \frac{(-1,250) \text{ kN}^2 \text{ m}^4 (10^3)^4 \text{ mm}^4/\text{ m}^4}{(200 \times 10^6) \text{ kN}/\text{ m}^2 (200 \times 10^6) \text{ mm}^4} \\
&= -0.031 \text{ m} = \boxed{-3.1 \text{ cm}}
\end{aligned}$$

- Note that the horizontal deflection is to the left (opposite to the direction of the virtual force).

Determine the rotation of joint C. $E = 29,000$ ksi, $I = 240$ in^4.



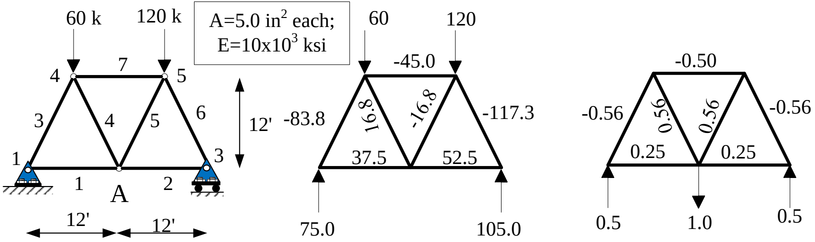
In this problem the virtual force is a unit moment applied at joint C, $\delta\overline{M}_e$. It will cause an internal moment $\delta\overline{M}_i$

Element	$x = 0$	M	$\delta\overline{M}$
AB	A	0	0
BC	B	$30x - 1.5x^2$	$-0.05x$
CD	D	0	0

Note that moments are considered positive when they produce compression on the outside of the frame. Substitution yields:

$$\begin{aligned}
\underbrace{\theta_C \delta\overline{M}_e}_{\delta\overline{W}^*} &= \underbrace{\int_0^L \delta\overline{M}\frac{M}{EI_z}dx}_{\delta\overline{U}^*} \\
(1)k - ft(\theta_C)\text{rad} &= \int_0^{20}(-0.05x)\frac{(30x - 1.5x^2)}{EI}dx \quad \text{k}^2\ \text{ft}^3 \\
&= -\frac{(1,000)k^2\ \text{ft}^2(144)\ \text{in}^2/\ \text{ft}^2}{(29,000)\ \text{k}/\ \text{in}^2(240)\ \text{in}^4} \\
&= \boxed{-0.021\ \text{radians}}
\end{aligned}$$

Determine the deflection at node A for the truss.

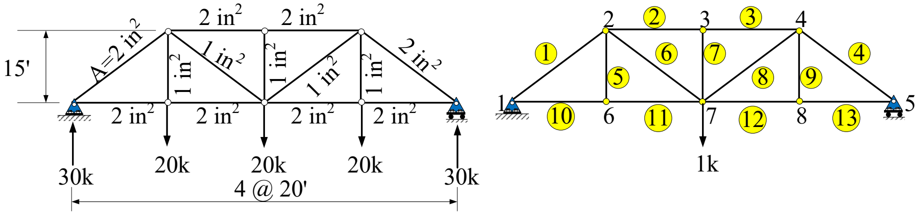


Member	$\delta\overline{P}^{(e)}$ kips	$P^{(e)}$, kips	L, ft	A, in^2	E, ksi	$\delta\overline{P}^{(e)}\frac{P^{(e)}L}{AE}$
1	+0.25	+37.5	12	5.0	10×10^3	$+22.5\times10^{-4}$
2	+0.25	+52.5	12	5.0	10×10^3	$+31.5\times10^{-4}$
3	-0.56	-83.8	13.42	5.0	10×10^3	$+125.9\times10^{-4}$
4	+0.56	+16.8	13.42	5.0	10×10^3	$+25.3\times10^{-4}$
5	+0.56	-16.8	13.42	5.0	10×10^3	-25.3×10^{-4}
6	-0.56	-117.3	13.42	5.0	10×10^3	$+176.6\times10^{-4}$
7	-0.50	-45.0	12	5.0	10×10^3	$+54.0\times10^{-4}$
						$+410.5\times10^{-4}$

The deflection is thus given by

$$
\begin{aligned}
(\delta\overline{P})\ \text{k}(\Delta)\ \text{in} &= \sum_{1}^{7} \delta\overline{P}^{(e)} \frac{PL}{AE} \\
(1)\ \text{k}(\Delta)\ \text{in} &= (410.5 \times 10^{-4}) \frac{\text{k}^2\ \text{ft}}{\text{in}^2\ \text{k}/\ \text{in}^2} (12\ \text{in}/\ \text{ft}) = \boxed{0.493\ \text{in}}
\end{aligned}
$$

Determine the vertical deflection of joint 7. $E = 30,000$ ksi.



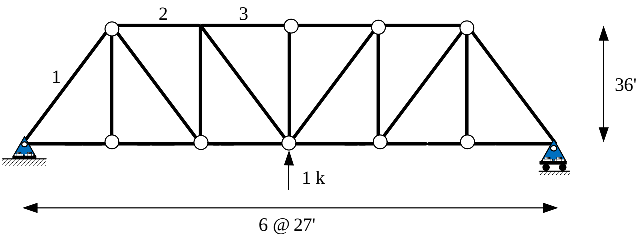
- Two analyses are required. One with the real load, and the other using a unit vertical load at joint 7. Results for those analysis are summarized below. Note that advantage was taken of the symmetric load and structure.

Member	A in^2	L ft	$P^{(e)}$ k	$\delta\overline{P}^{(e)}$ k	$\frac{\delta\overline{P}^{(e)}P^{(e)}L}{A}$ k.ft/ in^2	n	$\frac{n\delta\overline{P}^{(e)}P^{(e)}L}{A}$ k.ft/ in^2
1 & 4	2	25	-50	-0.083	518.75	2	1,037.5
10 & 13	2	20	40	0.67	268.0	2	536.0
11 & 12	2	20	40	0.67	268.0	2	536.0
5 & 9	1	15	20	0	0	2	0
6 & 8	1	25	16.7	0.83	346.5	2	693.0
2& 3	2	20	-53.3	-1.33	708.9	2	1,417.8
7	1	15	0	0	0	1	0
Total							4,220.3

- Thus from Eq. 10 we have:

$$
\begin{aligned}
\underbrace{\Delta\delta\overline{P}}_{\delta\overline{W}^*} &= \underbrace{\int_0^L \delta\overline{P}\frac{P}{AE}dx}_{\delta\overline{U}^*} \\
&= \Sigma\delta\overline{P}^{(e)}\frac{P^{(e)}L}{AE} \\
(1)\text{ k}(\Delta)\text{ in} &= \frac{(4,220.3)\text{ k}^2\text{ ft}/\text{ in}^2(12)\text{ in/ft}}{30,000\text{ ksi}} \\
&= \boxed{1.69\text{ in}}
\end{aligned}
$$

It is desired to provide 3 in. of camber at the center of the truss shown below



by fabricating the endposts and top chord members additionally long. How much should the length of each endpost and each panel of the top chord be increased?

- Assume that each endpost and each section of top chord is increased 0.1 in.

Member	$\delta\overline{P}_{int}^{(e)}$	ΔL	$\delta\overline{P}_{int}^{(e)}\Delta L$
1	+0.625	+0.1	+0.0625
2	+0.750	+0.1	+0.0750
3	+1.125	+0.1	+0.1125
			+0.2500

Thus,

$$(1)\ \mathrm{k}(\Delta)\ \mathrm{in} = (2)(0.250)\ \mathrm{k\ in} \Rightarrow \Delta = 0.50\ \mathrm{in}$$

- Since the structure is linear and elastic, the required increase of length for each section will be

$$\left(\frac{3.0}{0.50}\right)(0.1) = 0.60\ \mathrm{in}$$

- If we use the practical value of 0.625 in., the theoretical camber will be

$$\frac{(6.25)(0.50)}{0.1} = \boxed{3.125\ \mathrm{in}}$$

insert detials of the paper by maxwell (stored in victor-research-pdf-library-truss-design) and maxwell wrote FL which is really FL/AE, and his theorem is nothing but the internal work equal external wok. Prepare a new handout, and address optimization

Structural Analysis

Flexibility Method

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Spring 2022

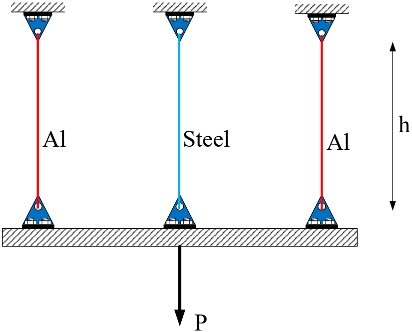
Table of Contents I

Reactions → Internal Forces → Deflections → **Flexibility Method (SIS)**

- A statically indeterminate structure has more unknowns than equations of equilibrium (and equations of conditions if applicable).
- The advantages of a statically indeterminate structures are:
  1. Lower internal forces
  2. Safety in redundancy, i.e. if a support or members fails, the structure can *redistribute* its internal forces to accommodate the changing B.C. without resulting in a sudden failure.
- Only disadvantage is that it is more complicated to analyze.
- Analysis methods of statically indeterminate structures *must satisfy* three requirements

  Equilibrium
  Force-displacement (or stress-strain) relations (linear elastic in this course).
  Compatibility of displacements (i.e. no discontinuity)
- This can be achieved through two classes of solution

  Force or Flexibility method;
  Displacement or Stiffness method



1. Three unknowns, two independent equations of equilibrium $\Rightarrow$ statically indeterminate to the first degree.
2. Equations of equilibrium
$$\Sigma M_z = 0; \quad \Rightarrow \quad P_{Al}^{\text{left}} = P_{Al}^{\text{right}}$$
$$\Sigma F_y = 0; \quad \Rightarrow \quad 2P_{Al} + P_{St} = P$$
two unknowns and one equation.
3. Need a third equation. Obtained from compatibility of displacements $\Delta L_{Al} = \Delta L_{St}$
4. Force-Displacement relations: $\Delta L = \frac{PL}{AE}$ or $\underbrace{\frac{P_{Al}L}{E_{Al}A_{Al}}}_{\Delta_{Al}} = \underbrace{\frac{P_{St}L}{E_{St}A_{St}}}_{\Delta_{St}} \Rightarrow \frac{P_{Al}}{P_{St}} = \frac{(EA)_{Al}}{(EA)_{St}}$ or
$$-(EA)_{St}P_{Al} + (EA)_{Al}P_{St} = 0$$

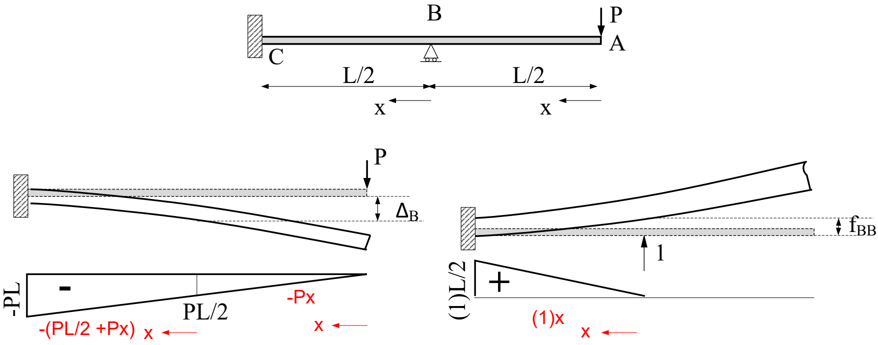
5. In matrix form:

$$\begin{bmatrix} 2 & 1 \\ -(EA)_{St} & (EA)_{Al} \end{bmatrix} \begin{Bmatrix} P_{Al} \\ P_{St} \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} P_{Al} \\ P_{St} \end{Bmatrix} = \begin{bmatrix} 2 & 1 \\ -(EA)_{St} & (EA)_{Al} \end{bmatrix}^{-1} \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

$$= \frac{1}{\underbrace{2(EA)_{Al} + (EA)_{St}}_{\text{Determinant}}} \begin{bmatrix} (EA)_{Al} & -1 \\ (EA)_{St} & 2 \end{bmatrix} \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

6. We observe that the solution of this problem, contrarily to statically determinate ones, depends on the elastic properties.



Primary Structure Under Actual Load

Primary Structure Under Redundant Loading

Note similarity with the derivation of the virtual force principle.

1. Remove roller support, and have a primary structure.
2. Deflection at B due to the applied load P using the virtual force method

$$
\begin{aligned}
1.\Delta &= \int \delta\overline{M}\frac{M}{EI}dx = \int_0^{L/2} 0\frac{-Px}{EI}dx \\
&\quad + \int_0^{L/2}(x)\frac{-\left(\frac{PL}{2}+Px\right)}{EI}dx \\
&= -\frac{1}{EI}\int_0^{L/2}\left(\frac{PL}{2}x + Px^2\right)dx \\
&= -\frac{1}{EI}\left[\frac{PLx^2}{4}+\frac{Px^3}{3}\right]\Bigg|_0^{L/2} = -\frac{5}{48}\frac{PL^3}{EI}
\end{aligned}
$$

3. Apply a unit load at point B and solve for the displacement at B using the PVF

$$
\begin{aligned}
1f_{BB} &= \int \delta\overline{M}\frac{M}{EI}dx \\
&= \int_0^{L/2}(x)\frac{x}{EI}dx = \frac{(1)L^3}{24EI}
\end{aligned}
$$

4. Displacement at B is zero $\Rightarrow$, f_{BB} should be multiplied by $R_B^?$ such that $R_B^? f_{BB} = \Delta$ to ensure compatibility of displacements, hence

$$
\begin{aligned}
Rf_{BB} + \Delta &= 0 \\
\Rightarrow R &= -\frac{\Delta}{f_{BB}} = -\frac{-\frac{5}{48}\frac{PL^3}{EI}}{\frac{(1)L^3}{24EI}} \\
&= \boxed{\frac{5}{2}P}
\end{aligned}
$$

Note that EI cancels out.

- A degree of freedom is an independent displacement or rotation of a point.
- Method:
  1. Identify degree of static indeterminacy (exterior and/or interior) n.
  2. Select n redundant unknown forces and/or couples in the loaded structure along with n corresponding releases (angular or translation): primary structure.
  3. Determine the n displacements in the primary structure (with the load applied) corresponding to the releases, Δ_i.
  4. Apply a unit force at each of the releases j on the primary structure (without the external load) and determine the displacements in all releases i: flexibility coefficients, f_{ij}, i.e. displacement at release i due to a unit force at j. Direction is irrelevant; If reaction is positive it will be along the specified direction,if negatie, otherwise.
  5. Write the compatibility of displacement relation

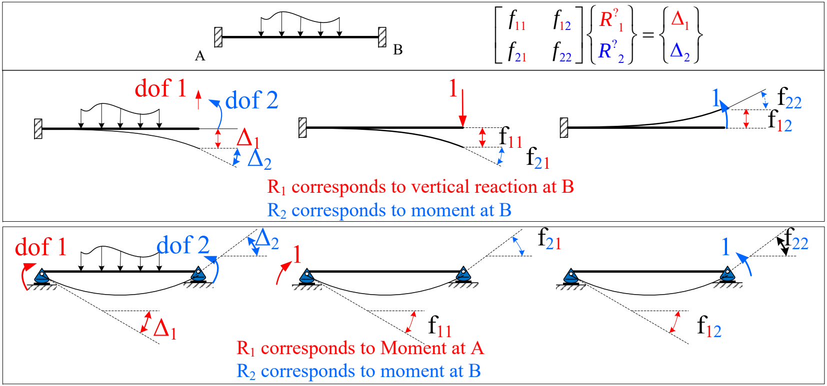
$$\underbrace{\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}}_{[\mathrm{f}]} \underbrace{\left\{ \begin{array}{c} R_1 \\ R_2 \\ \cdots \\ R_n \end{array} \right\}}_{\mathrm{R}} + \underbrace{\left\{ \begin{array}{c} \Delta_1 \\ \Delta_2 \\ \cdots \\ \Delta_n \end{array} \right\}}_{\Delta} = \underbrace{\left\{ \begin{array}{c} \Delta_1^0 \\ \Delta_2^0 \\ \cdots \\ \Delta_n^0 \end{array} \right\}}_{\Delta^0}$$

and

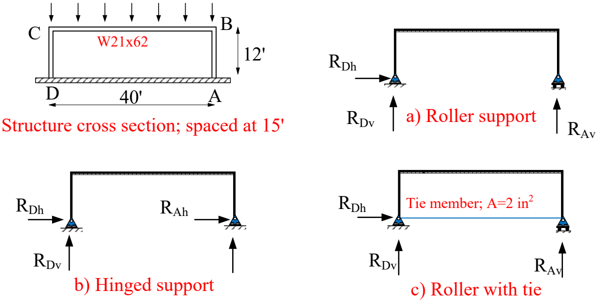
$$[\mathrm{R}] = [\mathrm{f}]^{-1} \left\{ \Delta + \Delta^0 \right\}$$

Δ_i^0 vector of initial displacements, which is usually zero unless we have an initial displacement of the support (such as support settlement).

6. Reactions are obtained by simply inverting the flexibility matrix.



- Recall that f_{ij}, i.e.displacement at release i due to a unit force at j.
- Displacement at dof i due to point load at i: $1.\Delta_i = \int \delta\overline{M}_i \frac{M_i}{EI} dx$
- Displacement at dof i due to a unit force at j is: $f_{ij} = \int \delta\overline{M}_i \frac{M_j}{EI} dx$
- Displacement at dof j due to a unit force at i: $f_{ji} = \int \delta\overline{M}_j \frac{M_i}{EI} dx$
- Both virtual loads and real loads are unit: $\delta\overline{M}_i = M_i$, $\delta\overline{M}_j = M_j$
- or $f_{ij} = f_{ji}$ Which is Maxwell-Betti's reciprocal theorem, and results in a positive definite symmetric matrix. Positive definite because f_{ii} is always positive.



Frames, 15 ft apart must support snow load: 30 psf, dead load: 20 psf. Sections: (W 21 $\times$ 62). Cable $A = 2$ in.^{2}. Consider three designs, analyze and compare.

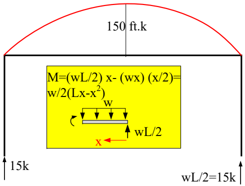
a) Poor soil conditions foundations may not be able to develop horizontal forces $\Rightarrow$ hinge at one of the bases and a roller at the other;

b) Excellent soil conditions hinges at both points A and D.

c) Intermediate case steel cable between A and D. The foundations would not be expected to provide any horizontal restraint for this latter case.

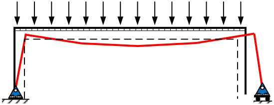
Solution:

- Design load: 15 (30 + 20) = 750 lb/ft.
    - Structure a: Statically determinate as there are three unknown reactions and three equations of equilibrium.

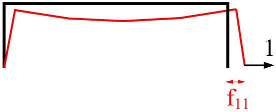


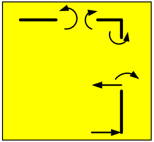
- Structure b Statically indeterminate to the first degree (one redundant).

1. Apply release at A, redundant shear force R_1. Δ_1:

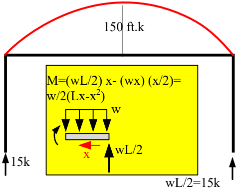


a) Primary structure

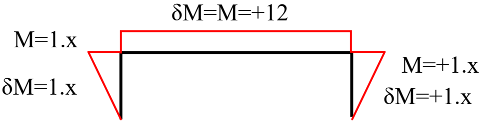




b) Load for determining f_{11}



c) Moments produced by real load



d) Moments produced by virtual forces and unit redundant

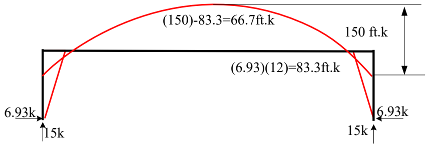
2. Solve for Δ_1 and f_{11}:

$$1(\text{ k})\cdot\Delta_1(\text{ ft}) = \int_0^L (12)\frac{w}{2}\frac{LX-x^2}{EI}dx = \int_0^{40}(12)\frac{(1/2)(.75)(40x-x^2)}{EI} = \frac{48,000}{EI}\text{ k}^2\text{ ft}^3$$

$$1(\text{ k})\cdot f_{11}(\text{ ft}) = 2\left[\int_0^{12} x\frac{x dx}{EI} + \int_0^{20} 12\frac{12dx}{EI}\right] = \frac{6,912}{EI}\text{ k}^2\text{ ft}^3$$

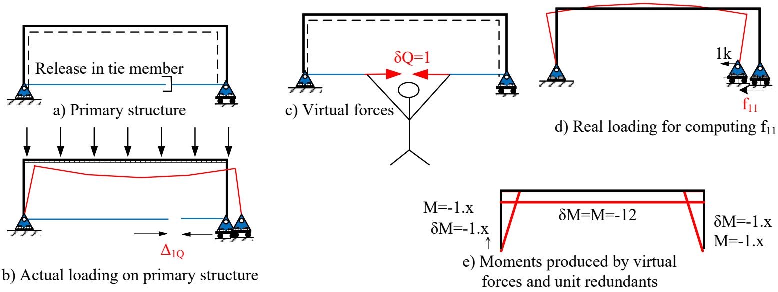
3. Solving for R_1

$$\frac{1}{EI}[48,000+6,912R_1] = 0 \Rightarrow R_1 = \boxed{-6.93\text{ k}\leftarrow}$$



Structure c Three unknown external forces, but structure is statically indeterminate to the first degree since the tie member provides one degree of internal redundancy.

1. Release the tie member, with its associated longitudinal displacement and axial force.



2. Compatibility equation: relative displacement of the two sections of the tie at the point of release must be zero, or $\Delta_1 + f_{11}R_1 = 0$ where

Δ_1	=	displacement at release 1 in the primary structure
f_{11}	=	relative displacement at release 1 for a unit axial force in the tie member,
R_1	=	force in the tie member in the original structure.

3. Δ_1 is determined from case b:

$$\Delta_1 = \frac{(48,000)\text{ k ft}^3(1,728)\text{ in}^3/\text{ ft}^3}{(30\cdot 10^3)\text{ ksi}(1,327)\text{ in}^4} = 2.08\text{ in}_{\text{rgt}}$$

4. f_{11} is caused by both flexural and axial deformations

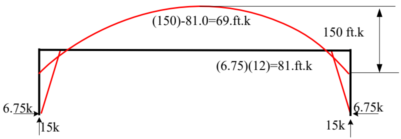
$$
\begin{aligned}
1 \cdot f_{11} &= \underbrace{2\left[\int_0^{12}(-x)\frac{(-x)\mathrm{d}x}{EI} + \int_0^{20}(-12)\frac{(-12)\mathrm{d}x}{EI}\right]}_{\text{Flexure}} + \underbrace{\delta\overline{P}\frac{PL}{EA}}_{\text{Axial}} \\
&= \frac{6,912}{EI} + \frac{1(1)(40)}{EA} \\
&= \frac{(6,912)\text{ k ft}^3(1,728)\text{ in}^3/\text{ ft}^3}{(30\cdot 10^3)\text{ ksi}(1,327)\text{ in}^4} + \frac{(40)\text{ ft}(12)\text{ in}/\text{ ft}}{(30\cdot 10^3)\text{ ksi}(2)} \\
&= 0.300 + 0.008 = 0.308
\end{aligned}
$$

thus f_{11} = 0.308 in./k

5. Consistent deformation:

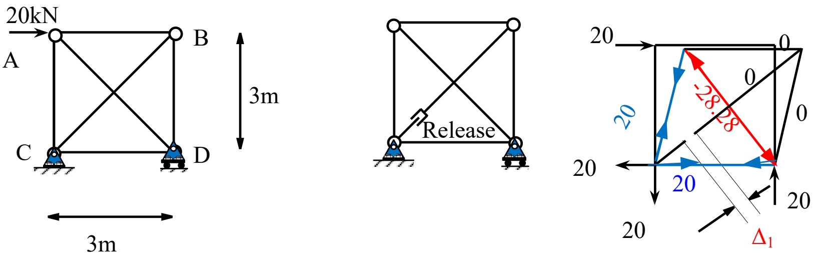
$$\Delta_1 + f_{11}R_1 = 0 \Rightarrow 2.08 + .308R_1 = 0 \Rightarrow R_1 = -6.75 \text{ k}$$

6. The two displacement terms in the equation must carry opposite signs to account for their difference in direction.



Comments

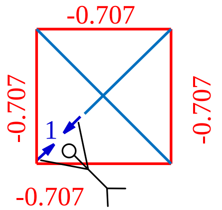
- M diagram in c, very close to M for b. Cable was very stiff. Reducing the area of the cable will increase the moment.
- Frames with tie members are used widely in industrial buildings.
- Maximum moment frames (b) and (c) is about 55% of (a). Continuity causes a decrease in the positive moment and an increase in the negative one. More optimal design.
- Vertical reactions are not affected by the horizontal support conditions.

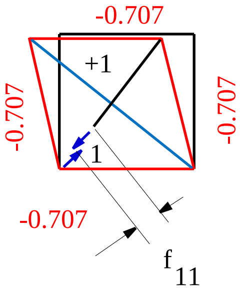


1. Check: $2 \times 4 = 8$ equations, 6 members $+$ 3 reactions $\Rightarrow$ one degree of indeterminacy. A longitudinal release in any of the six bars may be chosen.
2. Release diagonal member *BC* for release.
3. Δ_1 Relative displacement of joint *B* with respect to joint *C*.

4. Equation of compatibility along CD

$$\begin{aligned}\Delta_1 + f_{11}F_1 &= 0\\ 1\cdot\Delta_1 &= \Sigma\delta\overline{P}\frac{PL}{AE}\\ f_{11} &= \Sigma\delta\overline{P}\frac{\overline{P}L}{AE}\end{aligned}$$





5. Evaluating these summations in tabular form:

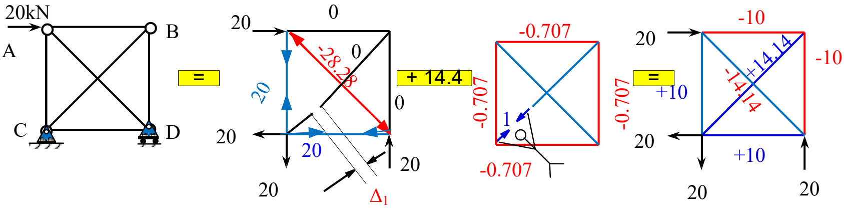
Member	P	$\delta\overline{P}$	L	$\delta\overline{P}PL$ (Δ_1)	$\delta\overline{P}\overline{P}L$ (f_{11})
AB	0	-0.707	3	0	1.5
BD	0	-0.707	3	0	1.5
CD	$+20$	-0.707	3	-42.42	1.5
AC	$+20$	-0.707	3	-42.42	1.5
AD	-28.28	$+1$	4.242	-119.96	4.242
BC	0	$+1$	4.242	0	4.242
				-204.8	14.484

6. Since A is constant for each member

$$\begin{aligned}
\Delta_1 &= \Sigma\delta\overline{P}\frac{PL}{AE} = -\frac{-204.8}{AE}\text{m.kN}^2 \\
f_{11} &= \frac{14.484}{AE}\text{m.kN}^2 \\
0 &= \frac{1}{AE}[-204.8 + 14.484F_1] \\
F_1 &= 14.14\text{ kN}
\end{aligned}$$

.

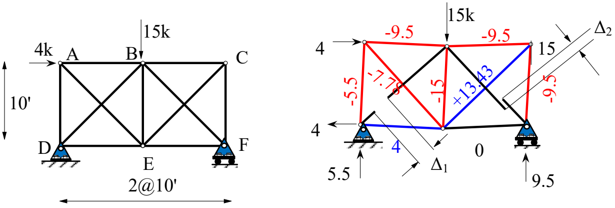
7. Final forces are obtained by superimposing forces due to the redundant and the forces due to the real loading.
8. Redundant force effect: multiply member forces by 14.14(redundant force)



Member	$\delta\overline{P}$	$F_1\delta\overline{P}$	P	P_{total}
AB	-0.707	-10.0	0.0	-10.0
BD	-0.707	-10.0	0.0	-10.0
CD	-0.707	-10.0	$+20.0$	$+10.0$
AC	-0.707	-10.0	$+20.0$	$+10.0$
AD	$+1.0$	$+14.14$	-28.28	-14.14
BC	$+1.00$	$+14.14$	0	$+14.14$

Another panel with a second redundant member is added to the truss of the preceding example

1. Release two diagonals (*DB* and *BF*).
2. The member forces and required displacements for the real loading and for



the two redundant forces in members *DB* and *BF*.

3. Although the real loading ordinarily stresses all members of the entire truss, we see that the unit forces corresponding to the redundants stress only those members in the panel that contains the redundant; all other bar forces are zero.
4. Recognizing this fact enables us to solve the double diagonal truss problem more rapidly than a frame with multiple redundants.

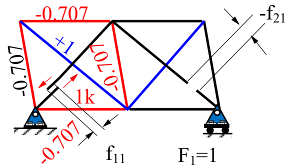
5. The virtual work equations for computing the six required displacements (two due to load and four flexibilities) are

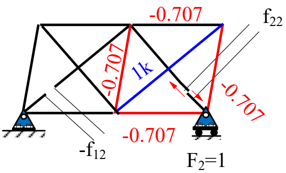
$$1 \cdot \Delta_1 = \Sigma\delta\overline{P}_1\left(\frac{PL}{AE}\right) \qquad 1 \cdot \Delta_2 = \Sigma\delta\overline{P}_2\left(\frac{PL}{AE}\right)$$

$$1 \cdot f_{11} = \Sigma\delta\overline{P}_1\left(\frac{\overline{P}_1 L}{AE}\right) \qquad 1 \cdot f_{21} = \Sigma\delta\overline{P}_2\left(\frac{\overline{P}_1 L}{AE}\right)$$

$$f_{12} = f_{21} \text{ by the reciprocal theorem} \qquad 1 \cdot f_{22} = \Sigma\delta\overline{P}_2\frac{\overline{P}_2 L}{AE}$$

6. Assume tensile unit forces (positive).





7. Tabulate:

Member	P	$\overline{P}_1$	$\overline{P}_2$	L	Displacements Δ_1 $\delta P_1 PL$	Displacements Δ_2 $\delta P_2 PL$	Flexibilities f_{11} $\delta P_1 P_1 L$	Flexibilities f_{21} $\delta P_2 P_1 L$	Flexibilities f_{22} $\delta P_2 P_2 L$
AB	-9.5	-0.707	0	120	+806	0	60	0	0
BC	-9.5	0	-0.707	120	0	+806	0	0	60
CF	-9.5	0	-0.707	120	0	+806	0	0	60
EF	0	0	-0.707	120	0	0	0	0	60
DE	+4	-0.707	0	120	-340	0	60	0	0
AD	-5.5	-0.707	0	120	+466	0	60	0	0
AE	+7.78	+1	0	170	+1,322	0	170	0	0
BE	-15.0	-0.707	-0.707	120	+1,272	+1272	60	60	60
CE	+13.43	0	+1	170	0	+2,280	0	0	170
BD	0	+1	0	170	0	0	170	0	0
BF	0	0	+1	170	0	0	0	0	170
					+3,528	+5,164	+580	+60	+580

8. Compatibility equations:

$$\begin{aligned} \Delta_1 + f_{11}F_1 + f_{12}F_2 &= 0 \\ \Delta_2 + f_{21}F_1 + f_{22}F_2 &= 0 \end{aligned}$$

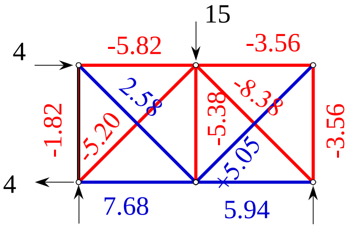
or

$$\frac{1}{AE}\begin{bmatrix} 580 & 60 \\ 60 & 580 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = -\frac{1}{AE}\begin{bmatrix} 3,528 \\ 5,164 \end{bmatrix} \quad (1)$$

and $F_1 = \boxed{-5.20 \text{ k}}$ and $F_2 = \boxed{-8.38 \text{ k}}$

9. Final set of forces: add for each member the three separate effects: $F = P + F_1\overline{P}_1 + F_2\overline{P}_2$

Member	P	$\overline{P}_1$	$\overline{P}_2$	$F_1\overline{P}_1$	$F_2\overline{P}_2$	P_{tot}
AB	-9.5	-0.707	0.0	3.676	0.0	-5.82
BC	-9.5	0.0	-0.707	0	5.925	-3.56
CF	-9.5	0.0	-0.707	0	5.925	-3.56
EF	0.0	0.0	-0.707	0	5.925	5.94
DE	+4	-0.707	0.0	3.676	0.0	7.68
AD	-5.5	-0.707	0.0	3.676	0.0	-1.82
AE	+7.78	+1	0.0	-5.20	0.0	2.58
BE	-15.0	-0.707	-0.707	3.676	5.925	-5.38
CE	+13.43	0.0	+1	0.0	-8.38	5.05
BD	0.0	+1	0.0	-5.20	0.0	-5.20
BF	0.0	0.0	+1	0.0	-8.38	-8.38



Structural Analysis

Introduction to Stiffness Method

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Spring 2022

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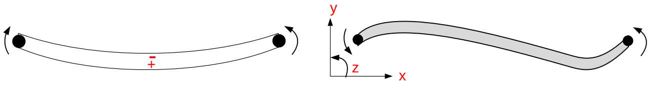
Table of Contents II

- There are two classes of structural analysis methods

	Flexibility	**Stiffness**
Primary Variable (d.o.f.)	Forces	Displacements
Indeterminancy	Static	Kinematic
Force-Displacement	Displacement(Force)/Structure	Force(Displacement)/Element
Governing Relations	Compatibility of displacement	Equilibrium

- Flexibility method: 1) release redundant force(s) $\Rightarrow$ structure statically determinate; 2) Apply unit forces determine f_{ij}; 3) kinematic constraint equation.
- Stiffness method: 1) Constrain all displacements $\Rightarrow$ kinematically determinate; 2)Release one constraint at a time, apply unit displacement determine k_{ij}; 3) Write equilibrium equation.

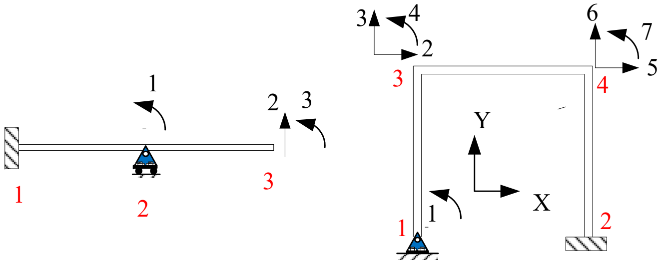
- In the stiffness method the sign convention adopted is consistent with the local element coordinate system. Hence, we define a positive moment as one which is counter-clockwise.
- Note that this is opposite to the convention in some introductory textbooks!.



Design (US)

Analysis

- A degree of freedom (d.o.f.) is an independent generalized nodal displacement (translation or rotation) at a node.
- The displacements must be linearly independent (of coordinate system) and thus not related to each other.
- An element dof is defined wrt its own local coordinate system. A structural dof is defined wrt a global coordinate system.



Type		Node 1	Node 2	$[\mathrm{k}^{(e)}]$	$[\mathrm{K}^{(e)}]$
				(Local)	(Global)
1 Dimensional					
Beam	$\{\mathrm{p}\}$	F_{y1}, M_{z2}	F_{y3}, M_{z4}	4×4	4×4
	$\{\delta\}$	v_1, θ_2	v_3, θ_4		
2 Dimensional					
Truss	$\{\mathrm{p}\}$	F_{x1}	F_{x2}	2×2	4×4
	$\{\delta\}$	u_1	u_2		
Frame	$\{\mathrm{p}\}$	F_{x1}, F_{y2}, M_{z3}	F_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\delta\}$	u_1, v_2, θ_3	u_4, v_5, θ_6		
Grid	$\{\mathrm{p}\}$	T_{x1}, F_{y2}, M_{z3}	T_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\delta\}$	θ_1, v_2, θ_3	θ_4, v_5, θ_6		
3 Dimensional					
Truss	$\{\mathrm{p}\}$	$F_{x1},$	F_{x2}	2×2	6×6
	$\{\delta\}$	$u_1,$	u_2		
Frame	$\{\mathrm{p}\}$	$F_{x1}, F_{y2}, F_{y3},$ $T_{x4}\ M_{y5}, M_{z6}$	$F_{x7}, F_{y8}, F_{y9},$ $T_{x10}\ M_{y11}, M_{z12}$	12×12	12×12
	$\{\delta\}$	$u_1, v_2, w_3,$ $\theta_4, \theta_5\ \theta_6$	$u_7, v_8, w_9,$ $\theta_{10}, \theta_{11}\ \theta_{12}$		

Slope Deflection: (Mohr, 1892) n linear equations with n unknowns, where n is the degree of kinematic indeterminancy (i.e. total number of independent displacements/rotation).

Moment Distribution: (Cross, 1930) Iterative method to solve for the n displacements and corresponding internal forces in flexural structures.

Direct Stiffness method: ($\simeq$1960) formal statement of the stiffness method and cast in matrix form is by far the most powerful method of structural analysis.

The first two methods lend themselves to hand calculation, and the third to a computer based analysis.

- Flexibility $\Delta(F)$ at the structure level (used virtual work equations).
- Stiffness $F(\Delta)$ at the structure or element level (to be derived next).

- From strength of materials:

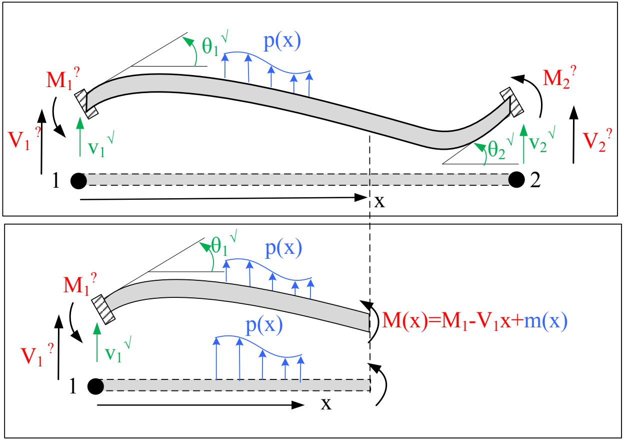
$$\sigma = E\epsilon \Rightarrow \underbrace{A\sigma}_{P} = \frac{AE}{L}\underbrace{\Delta}_{1}$$

- For a unit displacement, applied force should be equal to $\frac{AE}{L}$.
- From statics, force at other end must be equal and opposite.

- Objective: solve for forces in terms of known displacements in a beam: Four unknowns forces ($V_1^?$, $V_2^?$, $M_1^?$ and $M_2^?$) in terms of four known displacements ($v_1^{\surd}$, $v_2^{\surd}$, $\theta_1^{\surd}$ and $\theta_2^{\surd}$)

$$\begin{array}{rclrcl} V_1^? & = & V_1^?(v_1^{\surd},\theta_1^{\surd},v_2^{\surd},\theta_2^{\surd}) & M_1^? & = & M_1^?(v_1^{\surd},\theta_1^{\surd},v_2^{\surd},\theta_2^{\surd}) \\ V_2^? & = & V_2^?(v_1^{\surd},\theta_1^{\surd},v_2^{\surd},\theta_2^{\surd}) & M_2^? & = & M_2^?(v_1^{\surd},\theta_1^{\surd},v_2^{\surd},\theta_2^{\surd}) \end{array} \quad (1)$$

- Four unknowns, need four equations. Two provided by the second order linear differential equation governing flexure, and two from the two equations of equilibrium.



- A. Differential equation

$$M = \underbrace{-EI\frac{d^2v}{dx^2}}_{\text{Diff Eq.}} = \underbrace{M_1^? - V_1^? x + m(x)}_{\text{Statics}} \tag{2}$$

- $m(x)$ moment due to applied load $q(x)$ at section x (for uniformly distributed load: $m(x) = -\frac{1}{2}wx^2$)
- Integrating twice

$$-EIv' = M_1^? x - \frac{1}{2}V_1^? x^2 + f(x) + C_1 \tag{3}$$

$$-EIv = \frac{1}{2}M_1^? x^2 - \frac{1}{6}V_1^? x^3 + g(x) + C_1 x + C_2 \tag{4}$$

where $f(x) = \int m(x)dx$, and $g(x) = \int f(x)dx$.

- Boundary conditions at $x = 0$

$$\left.\begin{array}{rcl} v' & = & \theta_1^{\surd} \\ v & = & v_1^{\surd} \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} C_1 & = & -EI\theta_1^{\surd} \\ C_2 & = & -EIv_1^{\surd} \end{array}\right. \tag{5}$$

- Boundary conditions at $x = L$ and combining with C_1 and C_2

$$\left.\begin{array}{rcl} v' & = & \theta_2^{\surd} \\ v & = & v_2^{\surd} \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} -EI\theta_2^{\surd} & = & M_1^? L - \frac{1}{2}V_1^? L^2 + f(L) - EI\theta_1^{\surd} \\ -EIv_2^{\surd} & = & \frac{1}{2}M_1^? L^2 - \frac{1}{6}V_1^? L^3 + g(L) - EI\theta_1^{\surd} L - EIv_1^{\surd} \end{array}\right. \tag{6}$$

- Though we could solve for $M_1^?$ and $V_1^?$ in terms of $v_1^{\surd}$, $v_2^{\surd}$, $\theta_1^{\surd}$ and $\theta_2^{\surd}$, we proceed with

- B. Equilibrium

$$V_1^? + q + V_2^? = 0 \qquad M_1^? - V_1^? L + m(L) + M_2^? = 0 \tag{7}$$

where $q = \int_0^L w(x)dx$,

- thus

$$V_1^? = \frac{(M_1^? + M_2^?)}{L} + \frac{1}{L} m(L) \qquad V_2^? = -(V_1^? + q) \tag{8}$$

- Substituting V_1 into θ_2 and v_2 (Eq. 6)

$$\left\{ \begin{array}{rl} M_1^? - M_2^? & = \frac{2EI_z}{L}\theta_1^{\surd} + \frac{2EI_z}{L}\theta_2^{\surd} + m(L) - \frac{2}{L}f(L) \\ 2M_1^? - M_2^? & = \frac{6EI_z}{L}\theta_1^{\surd} - \frac{6EI_z}{L^2}v_1^{\surd} - \frac{6EI_z}{L^2}v_2^{\surd} + m(L) - \frac{6}{L^2}g(L) \end{array} \right. \tag{9}$$

- Solve for the moments

$$M_1 = \underbrace{\frac{2EI_z}{L}\left(2\theta_1^{\surd} + \theta_2^{\surd}\right) - \frac{6EI_z}{L^2}\left(v_2^{\surd} - v_1^{\surd}\right)}_{I} + \underbrace{FEM_{12}}_{II} \tag{10}$$

$$M_2 = \underbrace{\frac{2EI_z}{L}\left(\theta_1^{\surd} + 2\theta_2^{\surd}\right) - \frac{6EI_z}{L^2}\left(v_2^{\surd} - v_1^{\surd}\right)}_{I} + \underbrace{FEM_{21}}_{II} \tag{11}$$

where

- In Eq. 10 and 11 if we let $\Delta = v_2 - v_1$ (relative displacement), $\psi = \Delta/L$ (rotation of the chord of the member), and $K = I/L$ (stiffness factor[1]) then the end equations are:

$$M_1 = 2EK(2\theta_1 + \theta_2 - 3\psi) + FEM_1 \quad (12)$$

$$M_2 = 2EK(\theta_1 + 2\theta_2 - 3\psi) + FEM_2 \quad (13)$$

- Note that ψ will be positive if counterclockwise, negative otherwise.
- From Eq. 12 and 13, if a node has a displacement Δ, then both moments in the adjacent elements will have the same sign. However, the moments in elements on each side of the node will have different signs.

$$\Psi_{21} = \frac{v_2 - v_1}{2} \quad (14)$$

$$K = \frac{I}{L} \text{ Relative stiffness} \quad (15)$$

$$FEM_1 = \frac{2}{L^2}\left[Lf(L) - 3g(L)\right] \quad (16)$$

$$FEM_2 = -\frac{1}{L^2}\left[L^2 m(L) - 4Lf(L) + 6g(L)\right] \quad (17)$$

$$(18)$$

- FEM_1 and FEM_2 are the fixed end moments for $\theta_1 = \theta_2 = 0$ and $v_1 = v_2 = 0$.

Load	FEM_1	FEM_2
Uniform load w	$\frac{wL^2}{12}$	$-\frac{wL^2}{12}$
Center Point load	$\frac{PL}{8}$	$-\frac{PL}{8}$

Recall that in our notation, (-ve) moment means clockwise

- In Eq. 10 and 11 we observe that the moments developed at the end of a member are caused by: I) end rotation and displacements; and II) fixed end members.
- We can substitute those expressions in Eq. 8 and solve for the shear forces:

$$V_1 = \underbrace{\frac{6EI_z}{L^2}\left(\theta_1^{\surd} + \theta_2^{\surd}\right) - \frac{12EI_z}{L^3}\left(v_2^{\surd} - v_1^{\surd}\right)}_{I} + \underbrace{V_1^F}_{II} \quad (19)$$

$$V_2 = \underbrace{-\frac{6EI_z}{L^2}\left(\theta_1^{\surd} + \theta_2^{\surd}\right) + \frac{12EI_z}{L^3}\left(v_2^{\surd} - v_1^{\surd}\right)}_{I} + \underbrace{V_2^F}_{II} \quad (20)$$

where

$$V_1^F = \frac{6}{L^3}\left[Lf(L) - 2g(L)\right] \tag{21}$$

$$V_2^F = -\left[\frac{6}{L^3}\left[Lf(L) - 2g(L)\right] + q\right] \tag{22}$$

- It is very important to note that the derived equations are based on:
  1. Equilibrium
  2. Stress-strain
  3. Compatibility
- The relationships just derived enable us now to determine the stiffness matrix of a beam element.

$$\left\{\begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \end{array}\right\} = \underbrace{\begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \left[\begin{array}{cccc} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{array}\right] \end{array}}_{\mathtt{k}^e} \underbrace{\left\{\begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}\right\}}_{\Delta} + \underbrace{\left\{\begin{array}{c} V_1^F \\ M_1^F \\ V_2^F \\ M_2^F \end{array}\right\}}_{NEL} \tag{23}$$

where NEL: Nodal Equivalent Load (negative of the fixed end actions)

[1] K will be defined as $K = 4EI/L$ in the moment distribution method, and as a matrix in the direct stiffness method.

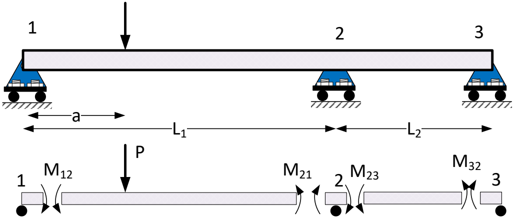
- In the presence of thermal load (or initial strains), nodal equivalent forces can be readily determined as follows:
- Trusss

$$F_1^T = -AE\alpha\Delta T \qquad F_2^T = AE\alpha\Delta T \tag{24}$$

- Beam

$$\begin{aligned} F_1^T &= -AE\alpha\Delta T^{avg} & F_2^T &= AE\alpha\Delta T^{avg} \\ M_1^T &= \frac{EI\alpha(\Delta T^{top} - \Delta T^{bot})}{h} & M_2^T &= -\frac{EI\alpha(\Delta T^{top} - \Delta T^{bot})}{h} \end{aligned} \tag{25}$$

  where α is the coefficient of thermal expansion, $T^{avg} = \frac{\Delta T^{top} + \Delta T^{bot}}{2}$.
- For initial forces (such as prestressed members) one needs to simply specify $\alpha\Delta T$ for the initial strain induced by prestressing
- In the load input data file one simply needs to specify $\alpha\Delta T$ for the thermally loaded truss, and $\alpha(\Delta T^{top} - \Delta T^{bot})$ and h for beams.



- Identify degree of kinematic indeterminacy: three rotations θ_1, θ_2, and θ_3 (i.e. three degrees of freedom) at the supports.
- Separating the spans from the support, draw free body diagrams and assume positive moments at the end of the beams.
- Moments can be expressed in terms of the three unknown rotations.
- Using equations 12 and 13 we obtain

$$\begin{aligned} M_{12} &= 2EK_{12}(2\theta_1 + \theta_2) + FEM_{12}; & M_{21} &= 2EK_{12}(\theta_1 + 2\theta_2) + FEM_{21}; \\ M_{23} &= 2EK_{23}(2\theta_2 + \theta_3); & M_{32} &= 2EK_{23}(\theta_2 + 2\theta_3); \end{aligned} \tag{26}$$

- We have 3 unknowns θ_1, θ_2, and θ_3 and we need three equations of equilibrium.
- Write one equilibrium equations for each support

$$\begin{aligned} M_{12} &= 0 \\ M_{21} + M_{23} &= 0 \\ M_{32} &= 0 \end{aligned}$$

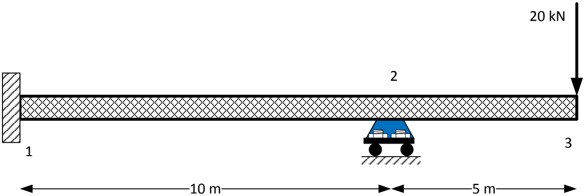
- Substituting, we obtain:

$$\underbrace{\underbrace{\begin{bmatrix} 4K_{12} & 2K_{12} & 0 \\ 2K_{12} & 4(K_{12}+K_{23}) & 2K_{23} \\ 0 & 2K_{23} & 4K_{23} \end{bmatrix}}_{\text{Stiffness Matrix}} \begin{Bmatrix} \theta_1^? \\ \theta_2^? \\ \theta_3^? \end{Bmatrix}}_{F_{int}} + \underbrace{\begin{Bmatrix} -\frac{FEM_{12}}{2EK_{12}} \\ -\frac{FEM_{21}}{2E} \\ 0 \end{Bmatrix}}_{F_{ext}} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Where the fixed end moment can be separately determined.

- This is an equation of global equilibrium which satisfies Newton's third law.
- Solve for rotations
- Substitute rotations in Eq. 26 to determine moments at each end of a beam segment.
- Computational requirements far less than for the flexibility method (or method of consistent deformation) because we implicitly accounted for the force displacement relationships. (though we are comparing static and kinematic unknowns).

1. Sketch deflected shape.
2. Identify unknown support degrees of freedom (rotations and deflections).
3. Write the equilibrium equations at all the supports in terms of the end moments.
4. Express the end moments in terms of the support rotations, deflections and fixed end moments.
5. Substitute the expressions obtained in the previous step in the equilibrium equations.
6. Solve equilibrium equations to determine the unknown support rotation and/or deflections.
7. Use the slope deflection equations to determine end moments.
8. Draw the moment diagram, careful about the difference in sign convention between the slope deflection moments and the moment diagram.



1. The beam is kinematically indeterminate to the third degree $(\theta_2,\ \Delta_3,\ \theta_3)$, however by replacing the the overhang by a fixed end moment equal to +100 kN.m at support 2, we reduce the degree of kinematic indeterminacy to one (θ_2).
2. The equilibrium relation is $M_{21} + M_{23} = 0$ or $M_{21} + 100 = 0$
3. The members end moments in terms of the rotations are (Eq. 12 and 13)

$$M_{21} = 2EK_{12}(\theta_1 + 2\theta_2) = \frac{4}{10}EI\theta_2 \quad \text{To solve for } \theta_2$$

$$M_{12} = 2EK_{12}(2\theta_1 + \theta_2) = \frac{2}{10}EI\theta_2 \quad \text{To solve for the end moment once } \theta_2 \text{ determined}$$

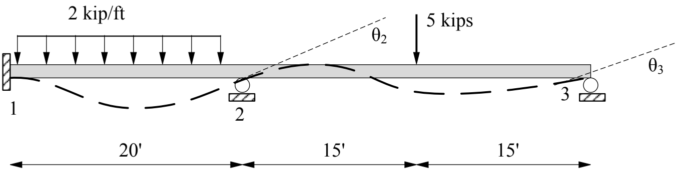
4. Substituting into the equilibrium equations, solve for θ_2.

$$\begin{aligned}\theta_2 &= \frac{10}{4EI}M_{21} \\ &= \frac{10}{4EI}(-100) = -\frac{250}{EI}\end{aligned}$$

(clockwise rotation: -ve)

5. Substitute and solve for M_{12}

$$M_{12} = \frac{2}{10}EI\theta_2 = -\frac{2}{10}EI\frac{250}{EI} = -50 \text{ kN.m}$$



1. The unknowns are θ_2, and θ_3
2. The equilibrium relations are $M_{21} + M_{23} = 0$ and $M_{32} = 0$
3. The fixed end moments are

$$FEM_{12} = -FEM_{21} = \frac{wL^2}{12} = \frac{(2)(20)^2}{12} = 66.67 \text{ k.ft}$$

$$FEM_{23} = -FEM_{32} = -\frac{PL}{8} = \frac{(5)(30)}{8} = 18.75 \text{ k.ft}$$

4. The members end moments in terms of the rotations are (Eq. 12 and 13)

$$
\begin{aligned}
M_{12} &= 2EK_{12}(\theta_2) + FEM_{12} = \frac{2EI}{L_1}\theta_2 + FEM_{12} = \frac{EI}{10}\theta_2 + 66.67 \quad \text{Not used} \\
M_{21} &= 2EK_{12}(2\theta_2) + FEM_{21} = \frac{4EI}{L_1}\theta_2 + FEM_{21} = \frac{EI}{5}\theta_2 - 66.67 \\
M_{23} &= 2EK_{23}(2\theta_2 + \theta_3) + FEM_{23} = \frac{2EI}{L_2}(2\theta_2 + \theta_3) + FEM_{23} \\
&= \frac{EI}{7.5}\theta_2 + \frac{EI}{15}\theta_3 + 18.75 \\
M_{32} &= 2EK_{23}(\theta_2 + 2\theta_3) + FEM_{32} = \frac{2EI}{L_2}(\theta_2 + 2\theta_3) + FEM_{32} \\
&= \frac{EI}{15}\theta_2 + \frac{EI}{7.5}\theta_3 - 18.75
\end{aligned}
$$

5. Substituting into the equilibrium equations

$$\begin{aligned}\frac{EI}{5}\theta_2 - 66.67 + \frac{EI}{7.5}\theta_2 + \frac{EI}{15}\theta_3 + 18.75 &= 0\\ \frac{EI}{15}\theta_2 + \frac{EI}{7.5}\theta_3 - 18.75 &= 0\end{aligned}$$

or

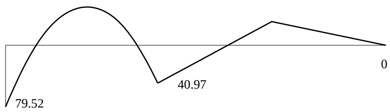
$$\underbrace{\underbrace{EI\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}}_{\text{Stiffness Matrix}}\begin{Bmatrix}\theta_2\\ \theta_3\end{Bmatrix}}_{\text{Internal Force}} - \underbrace{\begin{Bmatrix}718.8\\ 281.25\end{Bmatrix}}_{\text{External Force}} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

which will give $EI\theta_2 = 128.48$ and $EI\theta_3 = 76.38$

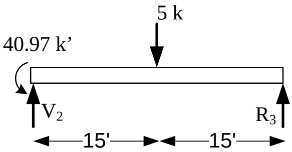
6. Substituting back for the moments

$$\begin{aligned}M_{12} &= \frac{128.48}{10} + 66.67 = 79.52 \text{ k.ft}\\ M_{21} &= \frac{128.48}{5} - 66.67 = -40.97 \text{ k.ft}\\ M_{23} &= \frac{128.48}{7.5} + \frac{76.38}{15} + 18.75 = 40.97 \text{ k.ft}\surd\\ M_{32} &= \frac{128.48}{15} + \frac{76.38}{7.5} - 18.75 = 0 \text{ k.ft}\surd\end{aligned}$$

Note the last two equations were written simply to check our calculations.

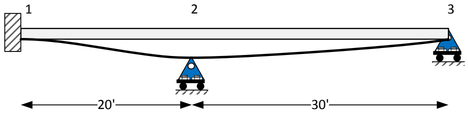


7. We note that the midspan moment has to be separately computed from the equations of equilibrium in order to complete the diagram.
8. The reaction at 3 is obtained from statics



$$
\begin{aligned}
15(5) - 30R_3 - 40.97 &= 0 \Rightarrow R_3 = 1.134 \\
V_2 = 5 - 1.134 &= 3.77
\end{aligned}
$$

Can you solve for R_2?



1. Since we are performing a linear elastic analysis, we can separately analyze the beam for support settlement, and then add then add the moments to those due to the applied loads.
2. The unknowns are θ_2, and θ_3
3. The equilibrium relations are $M_{21} + M_{23} = 0$ and $M_{32} = 0$
4. The members end moments in terms of the rotations are (Eq. 12 and 13)

$$\begin{aligned}
M_{12} &= 2EK_{12}\left(\theta_2 - 3\frac{\Delta}{L_{12}}\right) = \frac{2EI}{20}\left(\theta_2 + 3\frac{0.5}{20}\right) = \frac{EI}{10}\theta_2 + \frac{3EI}{400} \\
M_{21} &= 2EK_{12}\left(2\theta_2 - 3\frac{\Delta}{L_{12}}\right) = \frac{2EI}{20}\left(2\theta_2 + 3\frac{0.5}{20}\right) = \frac{EI}{5}\theta_2 + \frac{3EI}{400} \\
M_{23} &= 2EK_{23}\left(2\theta_2 + \theta_3 - 3\frac{\Delta}{L_{23}}\right) = \frac{2EI}{30}\left(2\theta_2 + \theta_3 - 3\frac{0.5}{30}\right) = \frac{EI}{7.5}\theta_2 + \frac{EI}{15}\theta_3 + \frac{EI}{300} \\
M_{32} &= 2EK_{23}\left(\theta_2 + 2\theta_3 - 3\frac{\Delta}{L_{23}}\right) = \frac{2EI}{30}\left(\theta_2 + 2\theta_3 - 3\frac{0.5}{30}\right) = \frac{EI}{15}\theta_2 + \frac{EI}{7.5}\theta_3 + \frac{EI}{300}
\end{aligned}$$

5. Substituting into the equilibrium equations

$$\begin{aligned} \frac{EI}{5}EI\theta_2 + \frac{3EI}{400} + \frac{EI}{15}\theta_3 + \frac{EI}{300} &= 0 \\ \frac{EI}{15}\theta_2 + \frac{EI}{7.5}\theta_3 + \frac{5EI}{300} &= 0 \end{aligned}$$

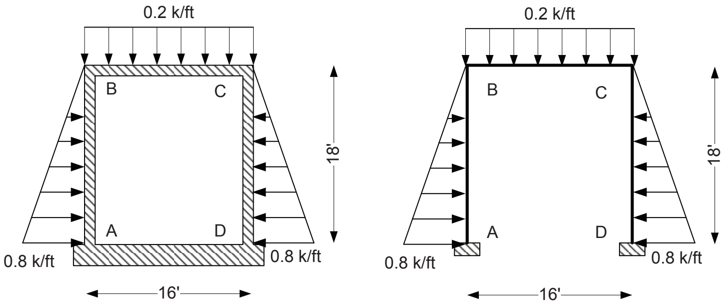
or

$$\underbrace{\underbrace{EI\begin{bmatrix} 100 & 20 \\ 20 & 40 \end{bmatrix}}_{\text{Stiffness Matrix}}\left\{\begin{array}{c} \theta_2 \\ \theta_3 \end{array}\right\}}_{\text{Internal Force}} - \underbrace{EI\left\{\begin{array}{c} -\frac{13}{4} \\ -1 \end{array}\right\}}_{\text{External Force}} = \left\{\begin{array}{c} 0 \\ 0 \end{array}\right\}$$

which will give $\theta_2 = -\frac{5.5}{180} = -0.031$ radians and $\theta_3 = \frac{-1+\frac{5.5}{9}}{40} = -0.0097$ radians

6. Thus the additional moments due to the settlement are

$$\begin{aligned} M_{12} &= \frac{EI}{10}(-0.031) + \frac{3EI}{400} = 0.0044EI \\ M_{21} &= \frac{EI}{5}(-0.031) + \frac{3EI}{400} = 0.0013EI \\ M_{23} &= \frac{EI}{7.5}(-0.031) + \frac{EI}{15}(-0.0097) + \frac{EI}{300} = -0.0013EI\surd \\ M_{32} &= \frac{EI}{15}\theta_2 + \frac{EI}{7.5}(0.0097) + \frac{EI}{300} = 0.\surd \end{aligned}$$



1. From symmetry $\theta_B = -\theta_C$, and at the base $\theta_A = \theta_D = 0$, thus we only have one unknown (no lateral displacement, no relative vertical displacement).

2. The fixed end moments are given by

$$\begin{aligned}
FEM_{BC} &= \frac{wL^2}{12} = \frac{(0.2)(16)^2}{12} = 4.267 \text{ k.ft} \\
FEM_{CB} &= -\frac{wL^2}{12} = -\frac{(0.2)(16)^2}{12} = -4.267 \text{ k.ft} \\
FEM_{AB} &= \frac{wL^2}{20} = \frac{(0.8)(18)^2}{20} = 12.96 \text{ k.ft} \\
FEM_{BA} &= \frac{wL^2}{30} = \frac{(0.8)(18)^2}{30} = -8.64 \text{ k.ft}
\end{aligned}$$

3. The moments are given by

$$\begin{aligned}
M_{BC} &= \frac{2EI}{16}(2\theta_B + \underbrace{\theta_C)}_{-\theta_B} + 4.267 = \frac{EI}{8}\theta_B + 4.267 \\
M_{BA} &= \frac{2EI}{18}(2\theta_B + 0) - 8.64 = \frac{2EI}{9}\theta_B - 8.64 \\
M_{AB} &= \frac{2EI}{18}(\theta_B) + 12.96
\end{aligned}$$

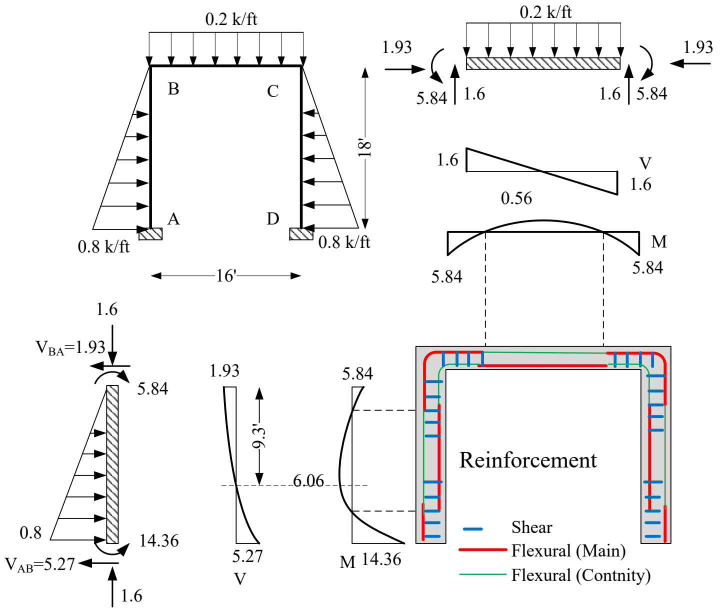
4. Equilibrium at joint B

$$\begin{aligned} M_{BA} + M_{BC} &= 0 \\ \frac{2EI}{9}\theta_B - 8.64 + \frac{EI}{8}\theta_B + 4.267 &= 0 \\ \theta_B &= -\frac{12.61}{EI} \end{aligned}$$

5. Substitute θ_B to get the moments

$$\begin{aligned} M_{BC} &= \frac{EI}{8}\left(\frac{12.61}{EI}\right) + 4.266 = 5.84 \text{ k.ft} \circlearrowleft \\ M_{AB} &= \frac{EI}{9}\left(\frac{12.61}{EI}\right) + 12.96 = 14.36 \text{ k.ft} \circlearrowleft \\ M_{BA} &= \frac{2EI}{9}\left(\frac{12.61}{EI}\right) - 8.64 = -5.84 \text{ k.ft} \circlearrowright \end{aligned}$$

6. Member forces are determined from statics. Careful, the moment diagram is now based on the so-called "design" sign convention.


0.2 k/ft
B
C
A
D
18'
0.8 k/ft
0.8 k/ft
16'
0.2 k/ft
1.93
1.93
5.84
1.6
1.6
5.84
1.6
V
1.6
0.56
M
5.84
5.84
1.6
V_{BA}=1.93
5.84
0.8
14.36
V_{AB}=5.27
1.6
1.93
9.3'
6.06
5.27
V
5.84
M 14.36
Reinforcement
Shear
Flexural (Main)
Flexural (Contnity)

- Gauss-Seidel is an indirect Method to solve a system of n equations with n unknowns (indirect means that *a priori* we do not know how many mathematical operations will be needed.
- Consider:

$$\begin{array}{ccccccc} c_{11}x_1 & + & c_{12}x_2 & + & c_{13}x_3 & = & r_1 \\ c_{21}x_1 & + & c_{22}x_2 & + & c_{23}x_3 & = & r_2 \\ c_{31}x_1 & + & c_{32}x_2 & + & c_{33}x_3 & = & r_3 \end{array}$$

- solve 1^{st} equation for x_1 using initial "guess" for x_2, x_3.

$$x_1 = \frac{r_1 - c_{12}x_2 - c_{13}x_3}{c_{11}}$$

- solve 2^{nd} equation for x_2 using the computed value of x_1 & initial guess of x_3

$$x_2 = \frac{r_2 - c_{21}x_1 - c_{23}x_3}{c_{22}}$$

- so on & so forth $\cdots$
- The iterative process can be considered to have converged if:

$$\left| \frac{\mathtt{x}^k - \mathtt{x}^{k-1}}{\mathtt{x}^k} \right| \leq \varepsilon$$

- Used to solve extremely large n (millions).
- The next method is essentially similar to this one with an initial guess of $\mathtt{x} = 0$

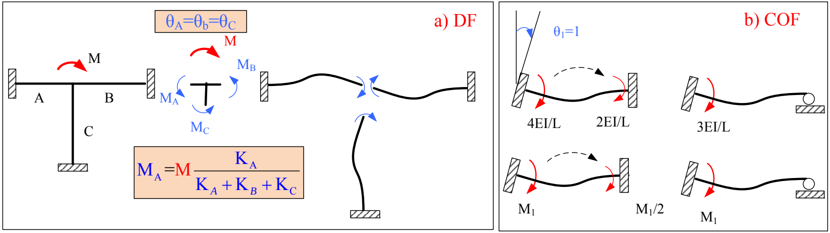
If you do not have a computer or a calculator, only a slide rule, you would like to have a simple way of solving a systme of equations:

$$\left\{ \begin{array}{rcl} x+y & = & 3 \\ 2x+y & = & 8 \end{array} \right. \Rightarrow y=-3-x; \rightarrow y=8-2x \Rightarrow \left\{ \begin{array}{rcl} x & = & 5 \\ y & = & -2 \end{array} \right.$$

Iteration	x	y	$N=x^2+y^2$	$\lvert N_i - N_{i-1}/N_i \rvert$
1	0	3	9	
2	2.5	0.5	6.5	38.00%
3	3.75	-0.75	14.625	55.56%
4	4.375	-1.375	21.03125	30.46%
5	4.6875	-1.6875	24.82031	15.27%
6	4.84375	-1.84375	26.86133	7.60%
7	4.921875	-1.92188	27.91846	3.79%
8	4.960938	-1.96094	28.45618	1.89%
9	4.980469	-1.98047	28.72733	0.94%
10	4.990234	-1.99023	28.86347	0.47%
11	4.995117	-1.99512	28.93169	0.24%
12	4.997559	-1.99756	28.96583	0.12%

- Slope deflection: had to invert the stiffness matrix to solve for rotations and then the moments.
- We will solve for the moments directly but iteratively.

- why? Slope deflection must invert an $n \times n$ matrix; When only slide rules or mechanical calculators were available, need for a simplified analysis method.
- Brief presentation as in modern times, it is of limited practical use, but very helpful to understand load paths in flexural members.
- Applicable to beams and frames only.
- A variation of the slope deflection method. Substitute direct solution of n equations by an iterative one (note analogy between Gauss-Jordan and Gauss-Seidel).
- A partial solution for a modified frame is altered systematically to lead to the correct one. 1
- Lock all the joints $\rightarrow$ unlock each joint in succession $\Rightarrow$ internal moments are "distributed" and balanced until all the joints have rotated to their final (or nearly final) equilibrium position.
- This is a relaxation technique analogous to the one of Southwell (1940).
- In order to better understand the method, some key terms must first be defined.
- Sign convention same as for slope deflection method.
- Fixed end moments same as for slope deflection method.



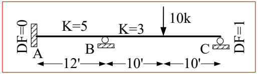
- From Eq. 10 $M_{12} = \frac{4EI}{L}\theta_1 + \frac{4EI}{L}\theta_1$
- Define stiffness factor K as moment required to rotate the end of a beam by a unit angle of one radian, while the other end is fixed i.e. $\theta_2 = v_1 = v_2 = 0$, and $\theta_1 = 1$, $\Rightarrow K = \frac{4EI}{L}$
- Slightly different than slope deflection method (I/L).
- If a moment is applied to a rigid joint where there are$n$ members, $\Rightarrow$ equilibrium: $M = M_1 + M_2 + \cdots + M_n$
- Eq. 12, and assuming the other end of the member to be fixed, then $M = K_1\theta + K_2\theta + \cdots + K_n\theta$ or $DF_i = \frac{M_i}{M} = \frac{K_i}{\Sigma K_i}$
- Note that $DF = 0$ (fixed support) acts as a sink, whereas $DF = 1$ acts like a mirror, it “bounces” back the moment.

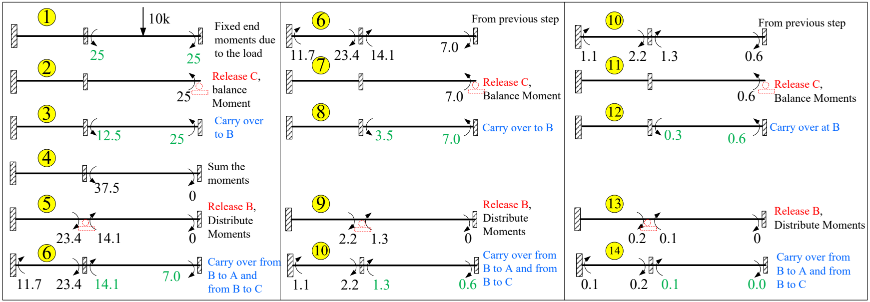
- Hence if a moment M is applied at a joint, portion of M carried by a member connected to this joint is proportional to the distribution factor, i.e. the stiffer the member (larger I, smaller L), the greater the moment carried.
- Similarly, $DF = 0$ for a fixed end, and $DF = 1$ for a pin support.
- A rigidly supported beam subjected to a moment M_1 (and corresponding rotation θ_1) at one end, and fixed at the other ($\theta_2 = 0$) $\Rightarrow M_1 = \frac{4EI}{L}\theta_1$ and $M_2 = \frac{2EI}{L}\theta_1$.
- Carry-over factor as the fraction of M that is “carried over” from the rotating end to the fixed one and $CO = \frac{1}{2}$.

1. Constrain all the rotations and translations.
2. Apply the load, and determine the fixed end moments (which may be caused by element loading, or support translation).
3. At any given joint i equilibrium is not satisfied $M^F_{left} \neq M^F_{right}$, and the net moment is M_i
4. We enforce equilibrium by applying at the node $-M_i$, in other words we balance the forces at the node.
5. How much of M_i goes to each of the elements connected to node i depends on the distribution factor.
6. But by applying a portion of $-M_i$ to the end of a beam, while the other is still constrained, from Eq. 12, half of that moment must also be carried over to the other end.
7. We then lock node i, and move on to node j where these operations are repeated
   1. Sum moments
   2. Balance moments
   3. Distribute moments (K, DF)
   4. Carry over moments (CO)
   5. lock node
8. Repeat the above operations until all nodes are balanced, then sum all moments.
9. The preceding operations can be easily carried out through a proper tabulation.

- The general procedure of the Moment Distribution method can be described as follows:
- If an end node is hinged, then we can use the reduced stiffness factor and we will not carry over moments to it.
- Analysis of frame with unsymmetric loading, will result in lateral displacements, and a two step analysis must be performed (see below).

1. Calculate the stiffness factor ($K = 4EI/L$) for all the members and the distribution factors at all the joints.
2. *If a member AB is pinned at B, then $K^{AB} = 3EI/L$, and $K^{BA} = 4EI/L$. Thus, we must apply the reduced stiffness factor to K^{AB} only and not to K^{BA}.*
3. Carry-over factor is $\frac{1}{2}$ for members with constant cross-section.
4. Compute fixed-end moments for all the members. Note that even if the end of a member is pinned, we must determine the fixed end moments as if it was fixed.
5. Start out by fixing all the joints, and release them one at a time.
6. If a node is pinned, start by balancing this particular node. If no node is pinned, start from either end of the structure.
7. Distribute the unbalanced moment at the released joint
8. Carry over the moments to the far ends of the members (unless it is pinned).
9. Fix the joint, and release the next one.
10. Continue releasing joints until distributed moments are insignificant. If the last moments carried over are small and cannot be distributed, it is better to discard them so that the joints remain in equilibrium.
11. Sum up the moments at each end of the members to obtain the final moments.



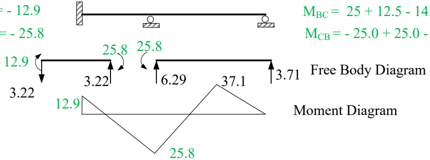


M_{AB}=- 11.7 - 1.1 - 0.1= - 12.9

M_{BA}= - 23.4 - 2.2 - 0.2= - 25.8

M_{BC} = 25 + 12.5 - 14.1 + 3.5 - 1.3 + 0.3 - 0.1 = 25.8

M_{CB} = - 25.0 + 25.0 - 7.0 + 7.0 - 0.6 + 0.6 = 0.0



1. For this example the fixed-end moments are computed as follows:

$$\begin{aligned} M^F_{BC} &= \frac{PL}{8} = \frac{(10)(20)}{8} = +25.0 \text{ k.ft} \\ M^F_{CB} &= -25.0 \text{ k.ft} \end{aligned}$$

2. Since the relative stiffness is given in each span, the distribution factors are

$$\begin{aligned} DF_{AB} &= \tfrac{K_{AB}}{\Sigma K} &= \frac{5}{\infty + 5} = 0, \\ DF_{BA} &= \tfrac{K_{BA}}{\Sigma K} &= \frac{5}{5+3} = 0.625, \\ DF_{BC} &= \tfrac{K_{BC}}{\Sigma K} &= \frac{3}{5+3} = 0.375, \\ DF_{CB} &= \tfrac{K_{CB}}{\Sigma K} &= \frac{3}{3} = 1. \end{aligned}$$

3. The balancing computations are shown below.

Joint	A	B		C	Step	Balance	CO
Member	AB	BA	BC	CB			
K	5	5	3	3			
DF	0	0.625	0.375	1			
FEM			+25.0	-25.0	①		
			+12.5	+25.0	② ③	C	BC
	-11.7	-23.4	-14.1	-7.0	④ ⑤ ⑥	B	AB; CB
			+3.5	+7.0	⑦ ⑧	C	BC
	-1.1	-2.2	-1.3	-0.6	⑨ ⑩ ⓫	B	AB; CB
			+0.3	+0.6	⓬ ⓭	C	BC
	-0.1	-0.2	-0.1		⓮	B	AB
Total	**-12.9**	**-25.8**	**+25.8**	**0**			

4. The above solution is that referred to as the ordinary method.
5. The correctness of the answers may in a sense be checked by verifying that $\Sigma M = 0$ at each joint. However, even though the final answers satisfy this equation at every joint, this in no way a check on the initial fixed-end moments. These fixed-end moments, therefore, should be checked with great care before beginning the balancing operation. Moreover, it occasionally happens that compensating errors are made in the balancing, and these errors will not be apparent when checking $\Sigma M = 0$ at each joint.

6. To draw the final shear and moment diagram, we start by drawing the free body diagram of each beam segment with the computed moments, and then solve from statics for the reactions:

$$
\begin{aligned}
12.9 + 25.8 - 12V_A = 0 \quad &\Rightarrow \quad V_A = R_A = 3.22 \text{ k} \downarrow \\
V_A + V_B^L = 0 \quad &\Rightarrow \quad V_B^L = -3.22 \text{ k} \uparrow \\
25.8 + (10)(10) - 20V_B^R = 0 \quad &\Rightarrow \quad V_B^R = 6.29 \text{ k} \uparrow \\
6.29 + V_C - 10 = 0 \quad &\Rightarrow \quad V_C = R_C 3.71 \text{ k} \uparrow \\
-V_B^L - V_B^R + R_B = 0 \quad &\Rightarrow \quad R_B = 9.51 \text{ k} \uparrow \\
\text{Check: } R_A + R_B + R_C - 10 \quad &= \quad -3.22 + 9.51 + 3.71 - 10 = 0\surd \\
M_{BC}^+ = (3.71)(10) = 37.1 \text{ k.ft} &
\end{aligned}
$$

7. Solving by slope deflection, and solve system of equations by Gauss-Seidel will yield identical intermediary steps.

- We will revisit the previous problem using the slope deflection method.
- The fixed end moments have been previously determined to be 25.
- The moments are given by

$$
\begin{aligned}
M_{AB} &= \frac{2EI}{L}(2\theta_A + \theta_B) = 2\frac{EI}{L}\theta_B = 10E\theta_B \\
M_{BA} &= \frac{2EI}{L}(2\theta_B + \theta_A) = 4\frac{EI}{L}\theta_B = 20E\theta_B \\
M_{BC} &= \frac{2EI}{L}(2\theta_B + \theta_C) + 25 = 4\frac{EI}{L}\theta_B + 2\frac{EI}{L}\theta_C + 25 = 12E\theta_B + 6E\theta_C + 25 \\
M_{CB} &= \frac{2EI}{L}(2\theta_C + \theta_B) - 25 = 4\frac{EI}{L}\theta_C + 2\frac{EI}{L}\theta_B - 25 = 12E\theta_C + 6E\theta_B - 25
\end{aligned}
$$

- We now write equations of equilibrium at each node

$$
\begin{aligned}
M_{BA} + M_{BC} &= 0 \\
M_{CB} &= 0
\end{aligned}
$$

- Substitute

$$
\left\{
\begin{aligned}
6E\theta_B + 12E\theta_C &= 25 \\
20E\theta_B + 12E\theta_B + 6E\theta_C &= -25
\end{aligned}
\right.
\tag{27}
$$

- The exact solution is $\theta_B = -\frac{75}{58}\frac{1}{E} = -\frac{1.29}{E}$ and $\theta_C = \frac{475}{174}\frac{1}{E} = \frac{2.73}{E}$.

- Substituting (results are now independent of E which cancel out) above we obtain $M_{AB} = -\frac{375}{29} = -12.93$, $M_{BA} = -\frac{750}{29} = -25.86$, $M_{BC} = \frac{750}{29} = 25.86$, and $M_{CB} = 0$. Results areexactly same results as in the moment distribution.
- Eq. 27 can now be written as

$$\begin{bmatrix} 6 & 12 \\ 32 & 6 \end{bmatrix} \begin{Bmatrix} \theta_B \\ \theta_C \end{Bmatrix} = \begin{Bmatrix} 25 \\ -25 \end{Bmatrix} \tag{28}$$

- We will now solve this by Gauss-Seidel iterative method with

$$6\theta_B + 12\theta_C = 25 \quad \Rightarrow \quad \theta_C = \frac{25}{12} - \frac{1}{2}\theta_B$$

$$32\theta_B + 6\theta_B = -25 \quad \Rightarrow \quad \theta_B = -\frac{25}{32} - \frac{6}{32}\theta_C$$

Start with $\theta_C = \theta_B = 0$, and then solve for $\theta_C \rightarrow \theta_B \rightarrow \theta_C \cdots$ until convergence and the following table summarizes each of the steps

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$E\theta_C$	0.0	2.083	2.670	2.724	2.729	2.730
$E\theta_B$	0.0	-1.172	-1.282	-1.292	-1.293	-1.293
M_{AB}	0.0	-11.72	-12.82	-12.92	-12.93	-12.93
M_{BA}	0.0	-23.44	-25.63	-25.84	-25.86	-25.86
M_{BC}	0.0	23.44	25.63	25.84	25.86	25.86
M_{CB}	0.0	-7.03	-0.658	-0.062	-0.006	0.000

- We indeed iteratively recover the previously computed moments by iteration five.
- Note that in the moment distribution, we solve directly for the moments whereas in the slope deflection method we first determine the rotations and the moments.
- We finally compare intermediary values of the moment distribution and the slope deflection method:

	Method	$n = 1$	$n = 2$
M_{AB}	MD	-11.7	-11.7-1.1=-12.8
	SD	-11.7	-12.82
M_{BA}	MD	-23.4	-23.4-2.2=-25.6
	SD	-23.44	-25.63
M_{BC}	MD	25+12.5-14.1=23.4	23.4+3.5-1.3=25.7
	SD	23.44	25.63
M_{CB}	MD	-25.0+25.0-7.0=-7.0	-7.0+7.0-0.6=-0.6
	SD	-7.03	-0.658

Clearly, the intermediary steps of the moment distribution correspond to those of the Gauss-Seidel iterative method. Similar conclusion would be drawn had we started by solving for θ_B.

Structural Analysis

Direct Stiffness Method

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Spring 2022

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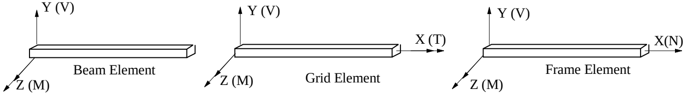
Essence of the stiffness method

1. Constrain all the degrees of freedom
2. Apply a unit displacement at each d.o.f.(while restraining all others to be zero)
3. Determine the reactions associated with all the d.o.f.

$$\{\mathrm{p}\} = [\mathrm{k}]\{\delta\} \tag{1}$$

k_{ij} will correspond to the reaction at dof i due to a unit deformation (translation or rotation) at dof j.

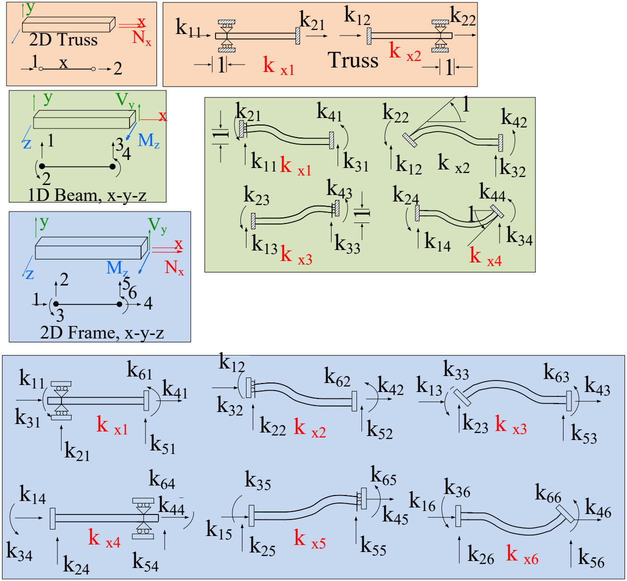
- We seek to determine forces (reactions) due an externally applied unit displacement.
- All forces are shown in the positive direction.



N: Axial; M: Moment; T: Torsion

Cartesian						
	Forces			Moments		
	x	y	z	x	y	z
Beam		V_y				M_z
2D Frame	N_x	V_y				M_z
Grid		V_y		T_x		M_z
3D Frame	N_x	V_y	V_z	T_x	M_y	M_z

Element Stiffness Matrix Revisited



- axial

$$\sigma = E\epsilon \Rightarrow \underbrace{A\sigma}_{P} = \underbrace{\frac{AE}{L}}_{k^{axial}} \underbrace{\Delta}_{1} \tag{2}$$

- Flexural

$$M_1 = \underbrace{\frac{2EI_z}{L}(2\theta_1 + \theta_2) - \frac{6EI_z}{L^2}(v_2 - v_1)}_{I} + \underbrace{M_1^F}_{II} \tag{3}$$

$$M_2 = \underbrace{\frac{2EI_z}{L}(\theta_1 + 2\theta_2) - \frac{6EI_z}{L^2}(v_2 - v_1)}_{I} + \underbrace{M_2^F}_{II} \tag{4}$$

$$V_1 = \underbrace{\frac{6EI_z}{L^2}(\theta_1 + \theta_2) - \frac{12EI_z}{L^3}(v_2 - v_1)}_{I} + \underbrace{V_1^F}_{II} \tag{5}$$

$$V_2 = \underbrace{-\frac{6EI_z}{L^2}(\theta_1 + \theta_2) + \frac{12EI_z}{L^3}(v_2 - v_1)}_{I} + \underbrace{V_2^F}_{II} \tag{6}$$

The truss element (whether in 2D or 3D) has only one degree of freedom associated with each node. Hence, from Eq. 2, we have

$$[\mathrm{k}^t] = \frac{AE}{L} \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \tag{7}$$

Using Equations 3, 4, 5 and 6 we can determine the forces associated with each unit displacement by setting all displacements equal to zero except:

$$[\mathrm{k}^b] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{array}{c} \begin{array}{cccc} \quad v_1 \quad & \quad \theta_1 \quad & \quad v_2 \quad & \quad \theta_2 \quad \end{array} \\ \left[\begin{array}{cccc} \text{Eq. } 5(v_1 = 1) & \text{Eq. } 5(\theta_1 = 1) & \text{Eq. } 5(v_2 = 1) & \text{Eq. } 5(\theta_2 = 1) \\ \text{Eq. } 3(v_1 = 1) & \text{Eq. } 3(\theta_1 = 1) & \text{Eq. } 3(v_2 = 1) & \text{Eq. } 3(\theta_2 = 1) \\ \text{Eq. } 6(v_1 = 1) & \text{Eq. } 6(\theta_1 = 1) & \text{Eq. } 6(v_2 = 1) & \text{Eq. } 6(\theta_2 = 1) \\ \text{Eq. } 4(v_1 = 1) & \text{Eq. } 4(\theta_1 = 1) & \text{Eq. } 4(v_2 = 1) & \text{Eq. } 4(\theta_2 = 1) \end{array}\right] \end{array} \tag{8}$$

or

$$[\mathrm{k}^b] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{array}{c} \begin{array}{cccc} \quad v_1 \quad & \quad \theta_1 \quad & \quad v_2 \quad & \quad \theta_2 \quad \end{array} \\ \left[\begin{array}{cccc} \frac{12EI_{zz}}{L^3} & \frac{6EI_{zz}}{L^2} & -\frac{12EI_{zz}}{L^3} & \frac{6EI_{zz}}{L^2} \\ \frac{6EI_{zz}}{L^2} & \frac{4EI_{zz}}{L} & -\frac{6EI_{zz}}{L^2} & \frac{2EI_{zz}}{L} \\ -\frac{12EI_{zz}}{L^3} & -\frac{6EI_{zz}}{L^2} & \frac{12EI_{zz}}{L^3} & -\frac{6EI_{zz}}{L^2} \\ \frac{6EI_{zz}}{L^2} & \frac{2EI_{zz}}{L} & -\frac{6EI_{zz}}{L^2} & \frac{4EI_{zz}}{L} \end{array}\right] \end{array} \tag{9}$$

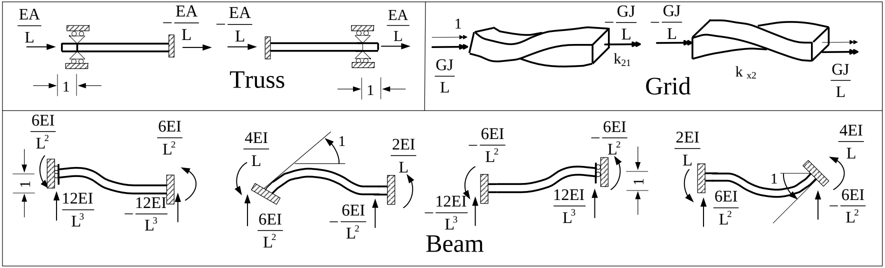
Hence, k_{32} is the shear at the right node due to a unit rotation on the left one. k_{41} is the moment at the left node due to a unit translation of the left one.

$\mathtt{k}^{2dfr} = \mathtt{k}^b \bigcup \mathtt{k}^t$, Note no coupling between the axial forces and the shear/moment.

$$[\mathtt{k}^{2dfr}] = \begin{array}{c} \\ N_{1x} \\ V_{1y} \\ M_{1z} \\ N_{2x} \\ V_{2y} \\ M_{2z} \end{array} \begin{array}{c} \begin{array}{cccccc} u_{1x} & v_{1y} & \theta_{1z} & u_{2x} & v_{2y} & \theta_{2z} \end{array} \\ \begin{bmatrix} k^t_{11} & 0 & 0 & k^t_{12} & 0 & 0 \\ 0 & k^b_{11} & k^b_{12} & 0 & k^b_{13} & k^b_{14} \\ 0 & k^b_{21} & k^b_{22} & 0 & k^b_{23} & k^b_{24} \\ k^t_{21} & 0 & 0 & k^t_{22} & 0 & 0 \\ 0 & k^b_{31} & k^b_{32} & 0 & k^b_{33} & k^b_{34} \\ 0 & k^b_{41} & k^b_{42} & 0 & k^b_{43} & k^b_{44} \end{bmatrix} \end{array} \tag{10}$$

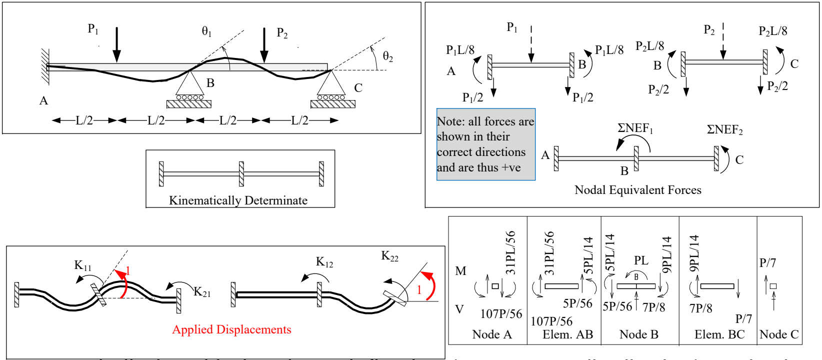
$$[\mathtt{k}^{2dfr}] = \begin{array}{c} \\ N_{1x} \\ V_{1y} \\ M_{1z} \\ N_{2x} \\ V_{2y} \\ M_{2z} \end{array} \begin{array}{c} \begin{array}{cccccc} u_{1x} & v_{1y} & \theta_{1z} & u_{2x} & v_{2y} & \theta_{2z} \end{array} \\ \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_{zz}}{L^3} & \frac{6EI_{zz}}{L^2} & 0 & -\frac{12EI_{zz}}{L^3} & \frac{6EI_{zz}}{L^2} \\ 0 & \frac{6EI_{zz}}{L^2} & \frac{4EI_{zz}}{L} & 0 & -\frac{6EI_{zz}}{L^2} & \frac{2EI_{zz}}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI_{zz}}{L^3} & -\frac{6EI_{zz}}{L^2} & 0 & \frac{12EI_{zz}}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI_{zz}}{L^2} & \frac{2EI_{zz}}{L} & 0 & -\frac{6EI_{zz}}{L^2} & \frac{4EI_{zz}}{L} \end{bmatrix} \end{array} \tag{11}$$

k_{21} is the shear in the left node due to a unit axial displacement at that same node. It is equal to zero because an axial force does not induce a shear force.

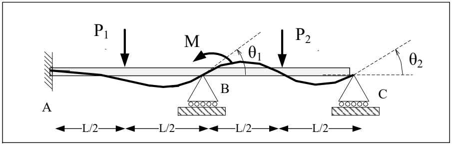

EA/L
−EA/L
1
Truss
GJ/L
−GJ/L
k_{21}
k_{x2}
Grid
6EI/L²
12EI/L³
−12EI/L³
4EI/L
2EI/L
6EI/L²
−6EI/L²
Beam
−6EI/L²
12EI/L³
−12EI/L³
4EI/L
2EI/L

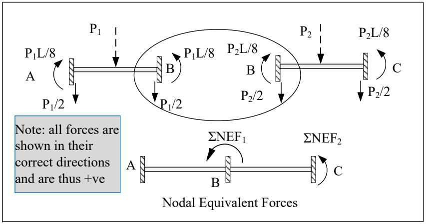
- Stiffness matrix of individual elements previously derived in local coordinate system, and assigned to it lower case letters $\mathtt{k}$.
- We need to derive the stiffness matrix of a structure in global coordinate system and will use upper cae $\mathtt{K}$.
- The direct stiffness method will be introduced in two steps:
    1. Orthogonal structures (simplified).
    2. Generalized structures. time permitting

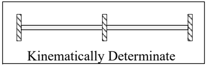
1. Determine the degree of kinematic indeterminacy.
2. Fix all the displacements, the structure is now kinematically determinate (all displacements are known and are equal to zero).
3. Determine the end nodal forces for each loaded element, sum up, and add to nodal forces.
4. Apply a unit displacement (rotation or displacement) at each free/unrestrained degree of freedom j at a time, and determine the internal reaction forces at degrees of freedom i, K_{ij}.
5. Assemble the reduced structure stiffness matrix in global coordinate system in terms of the individual element stiffness matrices also transformed in the global coordinate system. This will result in an equation of equilibrium at each node: $\mathrm{K}\Delta - \mathrm{P} = 0$. Where P includes nodal forces and nodal equivalent loads.
6. Reduced because we are not considering the restrained degrees of freedom.

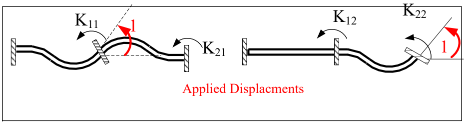


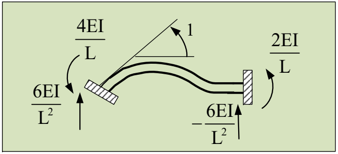
Note strong similarity with the slope-deflection (or moment distribution) methods.











$P_1 = 2P$, $M = PL$, and $P_2 = P$. Solve for the displacements.

1. Degree of kinematic indeterminacy is 2.

2. Using the previously defined sign convention, determine thenodal equivalent load (to the load applied along the member)

$$\Sigma\text{NEF}_1 = \underbrace{\frac{P_1 L}{8}}_{BA} - \underbrace{\frac{P_2 L}{8}}_{BC} = \frac{2PL}{8} - \frac{PL}{8} = \frac{PL}{8} \quad (12)$$

$$\Sigma\text{NEF}_2 = \underbrace{\frac{PL}{8}}_{CB} \quad (13)$$

3. If it takes $\frac{4EI}{L}$ (k_{44}^{BA}) to rotate BA and $\frac{4EI}{L}$ (k_{22}^{BC}) to rotate BC, it will take a total force of $\frac{8EI}{L}$ to simultaneously rotate BA and BC.

4. The sum of the rotational stiffnesses at global d.o.f. 1 is $K_{11} = \frac{8EI}{L}$; similarly, $K_{21} = \frac{2EI}{L}$ (k_{42}^{BC}).

5. If we now rotate d.o.f. 2 by a unit angle, we will have $K_{22} = \frac{4EI}{L}$ (k_{22}^{BC}) and $K_{12} = \frac{2EI}{L}$ (k_{42}^{BC}).

6. Equation of equilibrium:

$$\underbrace{\underbrace{\left\{\begin{array}{c} PL \\ 0 \end{array}\right\}}_{\mathtt{p}} + \underbrace{\left\{\begin{array}{c} \frac{PL}{8} \\ \frac{PL}{8} \end{array}\right\}}_{NEF}}_{P_{ext}} - \underbrace{\underbrace{\left[\begin{array}{cc} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array}\right]}_{\mathtt{K}} \underbrace{\left\{\begin{array}{c} \theta_1^? \\ \theta_2^? \end{array}\right\}}_{\Delta}}_{P_{int}} = \left\{\begin{array}{c} 0 \\ 0 \end{array}\right\} \tag{14}$$

7. Note that we have $P_{ext} - P_{int} = 0$ and not $P_{ext} + P_{int} = 0$ because the external forces must be resisted by the internal ones in an equal and opposite direction.

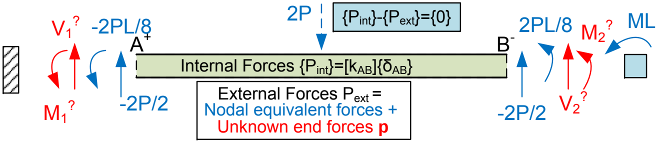
$$\left\{\begin{array}{c} PL + \frac{PL}{8} \\ +\frac{PL}{8} \end{array}\right\} = \left[\begin{array}{cc} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array}\right] \left\{\begin{array}{c} \theta_1^? \\ \theta_2^? \end{array}\right\} \tag{15}$$

Note that we will always write the equilibrium relationship as $P_{ext} - P_{int} = 0$

8. Invert the two by two matrix

$$\left\{\begin{array}{c} \theta_1 \\ \theta_2 \end{array}\right\} = \left[\begin{array}{cc} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array}\right]^{-1} \left\{\begin{array}{c} PL + \frac{PL}{8} \\ +\frac{PL}{8} \end{array}\right\} = \left\{\begin{array}{c} \frac{17}{112}\frac{PL^2}{EI} \\ -\frac{5}{112}\frac{PL^2}{EI} \end{array}\right\} \tag{16}$$

9. Recall that for each element $\{p\} = [k]\{\delta\}$, and in this case $\{p\} = \{P\}$ and $\{\delta\} = \{\Delta\}$ for element *AB*. The element stiffness matrix has been previously derived, and in this case the global and local d.o.f. are the same.
10. Next, we need to compute the element internal forces.
11. Equilibrium equation for element AB, at the element level, can be written as (note that we must include the nodal equivalent loads to maintain equilibrium):

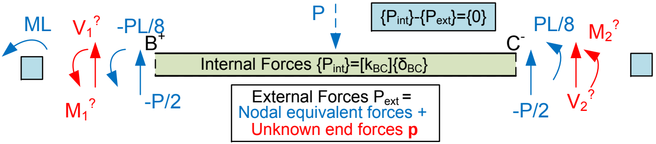


$$\underbrace{\underbrace{\begin{Bmatrix} V_1^? \\ M_1^? \\ V_2^? \\ M_2^? \end{Bmatrix}}_{\{\mathtt{p}\}} + \underbrace{\begin{Bmatrix} -\frac{2P}{2} \\ -\frac{2PL}{8} \\ -\frac{2P}{2} \\ \frac{2PL}{8} \end{Bmatrix}}_{\text{NEF}}}_{p_{ext}} - \underbrace{\underbrace{\begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}}_{[\mathtt{k}^{AB}]} \underbrace{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{17}{112}\frac{PL^2}{EI} \end{Bmatrix}}_{\{\delta^{AB}\}}}_{p_{int}} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Note: This step is called Force recovery, i.e. we determine the internal forces from the nodal displacements. Solving

$$\lfloor\ V_1 \quad M_1 \quad V_2 \quad M_2\ \rfloor = \lfloor\ \frac{107}{56}P \quad \frac{31}{56}PL \quad \frac{5}{56}P \quad \frac{5}{14}PL\ \rfloor$$

12. Similarly, for element *BC*:



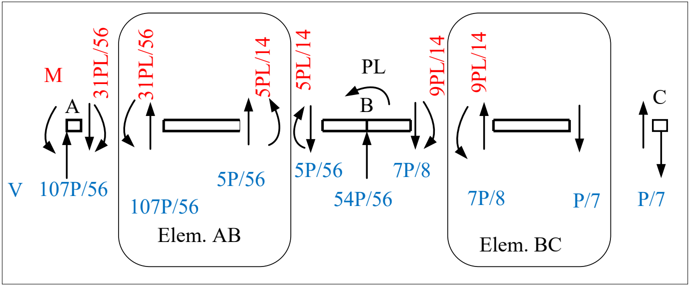
$$\underbrace{\underbrace{\left\{\begin{array}{c} V_1^? \\ M_1^? \\ V_2^? \\ M_2^? \end{array}\right\}}_{\{\mathrm{p}\}} + \underbrace{\left\{\begin{array}{c} -\frac{P}{2} \\ -\frac{PL}{8} \\ -\frac{P}{2} \\ \frac{PL}{8} \end{array}\right\}}_{\text{NEF}}}_{P_{ext}} - \underbrace{\underbrace{\left[\begin{array}{cccc} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array}\right]}_{[\mathbf{k}^{BC}]} \underbrace{\left\{\begin{array}{c} 0 \\ \frac{17}{112}\frac{PL^2}{EI} \\ 0 \\ -\frac{5}{112}\frac{PL^2}{EI} \end{array}\right\}}_{\{\delta^{BC}\}}}_{P_{int}} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right\}$$

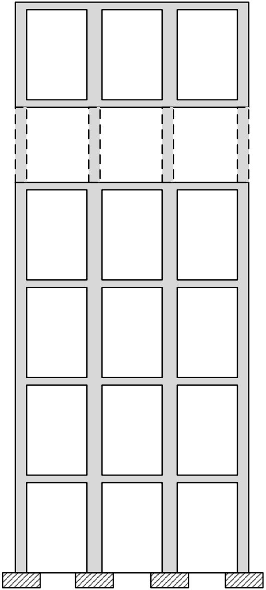
or

$$\lfloor\; p_1 \quad p_2 \quad p_3 \quad p_4 \;\rfloor = \lfloor\; \frac{7}{8}P \quad \frac{9}{14}PL \quad -\frac{P}{7} \quad 0 \;\rfloor$$

13 This simple example calls for the following observations:

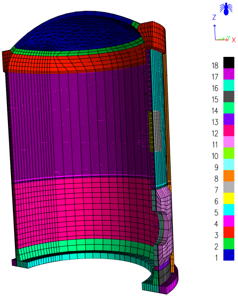
1. Node A has contributions from element *AB* only, while node B has contributions from both *AB* and *BC*.
2. We observe that $p_3^{AB} \neq p_1^{BC}$ even though they both correspond to a shear force at node B, the difference between them is equal to the reaction at B. Similarly, $p_4^{AB} \neq p_2^{BC}$ due to the externally applied moment at node B.
3. Must conclude with free body, shear and moment diagrams.

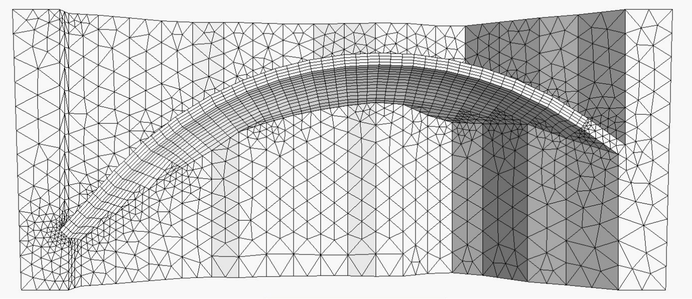

M
31PL/56
A
V
107P/56
31PL/56
5PL/14
107P/56
5P/56
Elem. AB
5PL/14
PL
B
9PL/14
5P/56
7P/8
54P/56
9PL/14
7P/8
P/7
Elem. BC
C
P/7



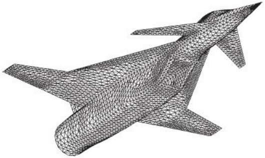
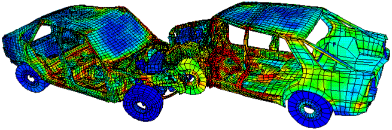
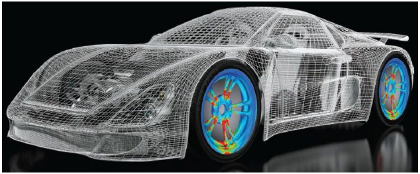
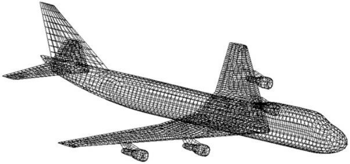
- We have already applied the direct stiffness method.
- The method can be applied to much more complex structures, and can be (relatively) easily be programmed.
- if we consider a 100 story, 3 bay frame, fixed at the base.
- The degree of static indeterminancy is $3(4) - 3 = 9 \Rightarrow [\mathrm{f}]_{9\times 9}$, i.e. we will have to invert a 9 by 9 matrix.
- The degree of kinematic indeterminacy is $100(4)(3) = 1,200 \Rightarrow [\mathrm{K}]_{1,200\times 1,200}$, i.e we will have to invert a 1,200 by 1,200 matrix.
- Because the stiffness method can be programmed, and a computer can easily invert a large matrix, this problem is best solved by the stiffness method.

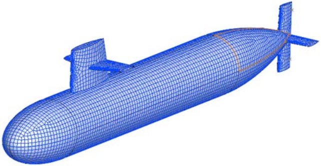
- The method just presented is actually referred to as the Finite Element Method.
- A structural engineer, well versed in the finite element analysis is thus equipped to handle the analysis of all structures that are discretized (just as our building was discretized into (4)(100)+3(100)=600 elements (400 columns and 300 beams).
- Hence, a Civil engineer well versed in structural analysis is not limited do the analysis of buildings, bridges, dams, nuclear reactors.

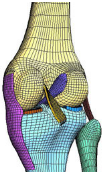

Z
Y
X
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1

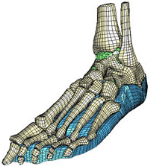


- but can find employment in automotive, aerospace, manufacturing, biomedical industry.



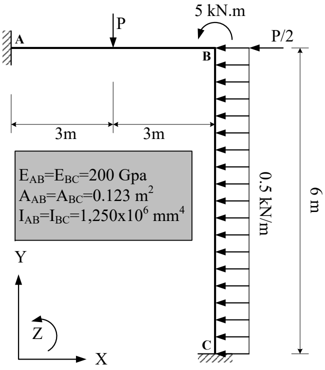


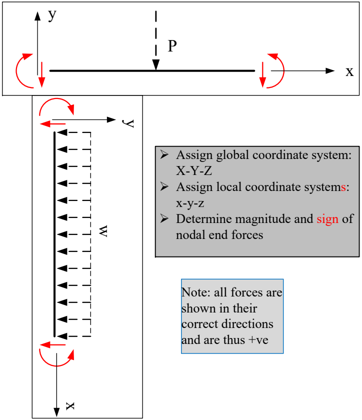


- Do not limit yourself to civil structures.
- You are better equipped than your fellow classmates from aerospace or mechanical engineering to become a Structural Analyst who go from Statics->Mechanics of Materials $\rightarrow$Finite Element.
- Civil Engineering students: Statics $\rightarrow$ Mechanics of Materials $\rightarrow$ Structural Analysis, $\rightarrow$ Matrix Analysis $\rightarrow$ Finite Element.
- Within Civil Engineering, Structural Engineering is the specialty that offers the broadest opportunities across various departments (Mechanical, Aerospace, Naval).

- Things get even more exciting if you consider that you must also understand material's response, seismic or dynamic analysis, probabilistic methods, numerical techniques, etc..
- Challenging specialty, an M.S. is a minimum.
- Need to grasp those opportunities before you finalize your fields of interest.

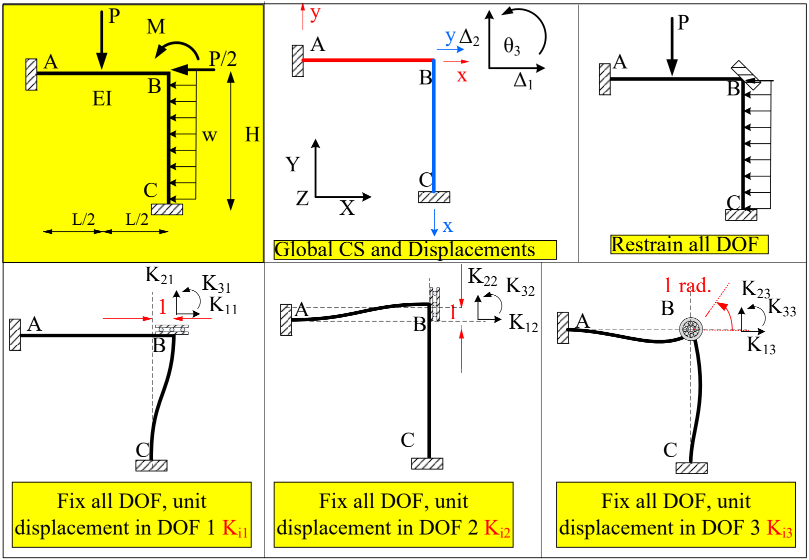
Analyse the following frame for $P = 2$ kN, $L = H = 6$ m, $M = 5$ kN.m, $w = 0.5$ kN/m, $E = 2 \times 10^8$ kPa, $A = 0.123$ m^2, and $I^b = I^c = 0.00125$ m^4



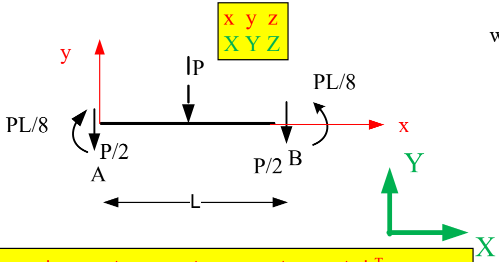


- Assign global coordinate system: X-Y-Z
- Assign local coordinate systems: x-y-z
- Determine magnitude and sign of nodal end forces

Note: all forces are shown in their correct directions and are thus +ve

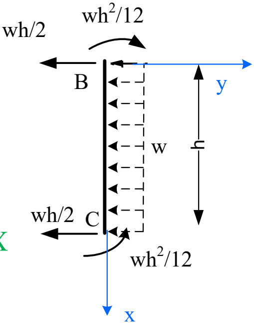

P
M
P/2
A
B
EI
w
H
C
L/2
L/2
y
A
y Δ2
θ3
x
B
Δ1
Y
Z
X
C
x
Global CS and Displacements
P
A
B
C
Restrain all DOF
K21
K31
K11
1
A
B
C
Fix all DOF, unit displacement in DOF 1 Ki1
K22
K32
A
1
K12
B
C
Fix all DOF, unit displacement in DOF 2 Ki2
1 rad.
K23
B
K33
A
K13
C
Fix all DOF, unit displacement in DOF 3 Ki3

1. Assuming axial deformations, we do have three global degrees of freedom, Δ_1, Δ_2, and θ_3.
2. Constrain all the degrees of freedom, and thus make the structure kinematically determinate.
3. Determine the nodal equivalent loads for each element in local coordinate system in its own local coordinate system (element 1 is assumed to be defined from A to B, and element 2 from B to C):



$$p_{AB} = \lfloor 0 \;\; -P/2 \;\; -PL/8 \;\; 0 \;\; -P/2 \;\; PL/8 \rfloor^T$$
$$P_{AB} = \lfloor 0 \;\; -P/2 \;\; -PL/8 \; \boxed{0 \;\; -P/2 \;\; PL/8} \rfloor^T$$

$$p_{BC} = \lfloor 0 \;\; -wh/2 \;\; -wh^2/12 \;\; 0 \;\; -wh/2 \;\; wh^2/12 \rfloor^T$$
$$P_{BC} = \lfloor \boxed{-wh/2 \;\; 0 \; -wh^2/12} \; -wh/2 \; 0 \;\; wh^2/12 \rfloor^T$$



x	y	z
-Y	X	Z

$$\underbrace{\lfloor\; p_1^A \quad p_2^A \quad p_3^A \mid p_4^B \quad p_5^B \quad p_6^B \;\rfloor}_{AB} = \lfloor\; 0 \quad -\frac{P}{2} \quad -\frac{PL}{8} \mid 0 \quad -\frac{P}{2} \quad \frac{PL}{8} \;\rfloor$$

$$= \lfloor\; 0 \quad -\frac{2}{2} \quad -\frac{(2)(6)}{8} \mid 0 \quad -\frac{2}{2} \quad \frac{(2)(6)}{8} \;\rfloor$$

$$= \lfloor\; 0 \quad -1.0 \quad -1.5 \mid 0 \quad -1.0 \quad 1.5 \;\rfloor$$

$$\underbrace{\lfloor\; p_1^B \quad p_2^B \quad p_3^B \mid p_4^C \quad p_5^C \quad p_6^C \;\rfloor}_{BC} = \lfloor\; 0 \quad -\frac{wH}{2} \quad -\frac{wH^2}{12} \mid 0 \quad -\frac{wH}{2} \quad \frac{wH^2}{12} \;\rfloor$$

$$= \lfloor\; 0 \quad -\frac{(0.5)(6)}{2} \quad -\frac{(0.5)(6)^2}{12} \mid 0 \quad -\frac{(0.5)(6)}{2}$$

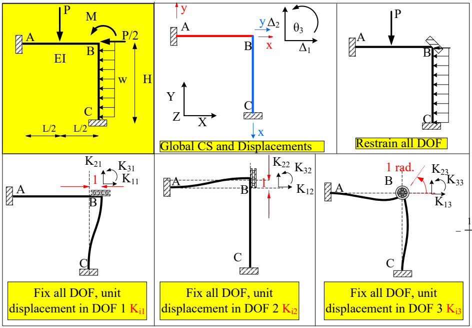
$$= \lfloor\; 0 \quad -1.5 \quad -1.5 \mid 0 \quad -1.5 \quad 1.5 \;\rfloor$$

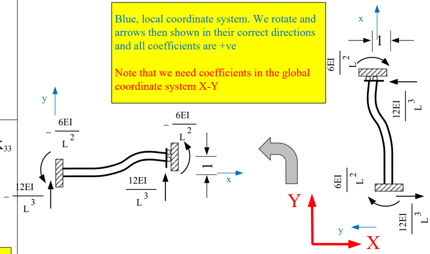
and the nodal equivalent forces at node B would have to be summed.

4. Apply a unit displacement in each of the 3 global degrees of freedom, to determine the structure global stiffness matrix. Each entry K_{ij} of the global stiffness matrix will correspond to the internal force in degree of freedom i, due to a unit displacement in degree of freedom j.

5. Recalling the force displacement relations derived earlier, we can assemble the global stiffness matrix in terms of contributions from both AB and BC:

- Need to complete the following table where columns correspond to imposed displacements on dof j, and rows correspond to the corresponding induced internal forces in each of the elements in dof i. Both are in the global coordinate system.
- $K_{1,2}$ is zero because an imposed displacement along dof 2 (horizontal), while locking all other displacements, does not induce an internal force in any of the two elements.
- K_{31} are the internal forces (moments in here) resulting from an imposed unit displacement in dof 1 (horizontal). This will not “mobilize” AB, but will activate flexure for BC. For BC from the following figure (already shown above





		K_{i1} Δ_1	K_{i2} Δ_2	K_{i3} θ_3
K_{1j} (F_X)	AB	$\frac{EA}{L}$	0	0
	BC	$\frac{12EI^c}{H^3}$	0	$\frac{6EI^c}{H^2}$
K_{2j} (F_Y)	AB	0	$\frac{12EI^b}{L^3}$	$-\frac{6EI^b}{L^2}$
	BC	0	$\frac{EA}{H}$	0
K_{3j} (M_Z)	AB	0	$-\frac{6EI^b}{L^2}$	$\frac{4EI^b}{L}$
	BC	$\frac{6EI^c}{H^2}$	0	$\frac{4EI^c}{H}$

- Note that all diagonal terms are +ve, and that the table is symmetric.

6. Summing up, the structure global stiffness matrix $[\mathrm{K}]$ is:

$$
[\mathrm{K}] = \begin{array}{c} \\ P_1 \\ P_2 \\ M_3 \end{array}
\begin{array}{c}
\begin{array}{ccc} \quad\Delta_1 \quad & \quad \Delta_2 \quad & \quad \theta_3 \quad \end{array} \\
\left[\begin{array}{ccc}
k_{44}^{AB} + k_{22}^{BC} & k_{45}^{AB} + k_{21}^{BC} & k_{46}^{AB} + k_{23}^{BC} \\
k^{AB} + k^{BC} & k_{55}^{AB} + k_{11}^{BC} & k_{56}^{AB} + k_{13}^{BC} \\
k_{64}^{AB} + k_{32}^{BC} & k_{65}^{AB} + k_{31}^{BC} & k_{66}^{AB} + k_{33}^{BC}
\end{array} \right]
\end{array}
$$

$$
= \begin{array}{c} \\ P_1 \\ P_2 \\ M_3 \end{array}
\begin{array}{c}
\begin{array}{ccc} \quad\Delta_1 \quad & \quad \Delta_2 \quad & \quad \theta_3 \quad \end{array} \\
\left[\begin{array}{ccc}
\frac{EA}{L} + \frac{12EI^c}{H^3} & 0 & \frac{6EI^c}{H^2} \\
0 & \frac{12EI^b}{L^3} + \frac{EA}{H} & -\frac{6EI^b}{L^2} \\
\frac{6EI^c}{H^2} & -\frac{6EI^b}{L^2} & \frac{4EI^b}{L} + \frac{4EI^c}{H}
\end{array} \right]
\end{array}
$$

Substituting

$$
[\mathrm{K}] = 10^6 \begin{bmatrix} 4.1139 & 0 & 0.0417 \\ 0 & 4.1139 & -0.0417 \\ 0.0417 & -0.0417 & 0.3333 \end{bmatrix}
$$

Note that the axial stiffness (EA/L) is 4.1×10^6, while the flexural one ($12EI/H^3$) is 0.0071×10^6. Axial stiffness is always much higher than flexural stiffness.

7. We need to have P_{ext} in global coordinate system. From Eq. 17 and 18 we had

$$\underbrace{\lfloor\; p_1^A \quad p_2^A \quad p_3^A \;|\; p_4^B \quad p_5^B \quad p_6^B \;\rfloor}_{AB} = \lfloor\; 0 \quad -\frac{P}{2} \quad -\frac{PL}{8} \;|\; 0 \quad -\frac{P}{2} \quad \frac{PL}{8} \;\rfloor \tag{19}$$

$$\underbrace{\lfloor\; p_1^B \quad p_2^B \quad p_3^B \;|\; p_4^C \quad p_5^C \quad p_6^C \;\rfloor}_{BC} = \lfloor\; 0 \quad -\frac{wH}{2} \quad -\frac{wH^2}{12} \;|\; 0 \quad -\frac{wH}{2} \quad \frac{wH^2}{12} \;\rfloor \tag{20}$$

8. Cast in the global coordinate system, that will be

$$\underbrace{\lfloor\; P_1^A \quad P_2^A \quad P_3^A \;|\; P_4^B \quad P_5^B \quad P_6^B \;\rfloor}_{AB} = \lfloor\; 0 \quad -\frac{P}{2} \quad -\frac{PL}{8} \;|\; 0 \quad -\frac{P}{2} \quad \frac{PL}{8} \;\rfloor \tag{21}$$

$$\underbrace{\lfloor\; P_1^B \quad P_2^B \quad P_3^B \;|\; P_4^C \quad P_5^C \quad P_6^C \;\rfloor}_{BC} = \lfloor\; -\frac{wH}{2} \quad 0 \quad -\frac{wH^2}{12} \;|\; -\frac{wH}{2} \quad 0 \quad \frac{wH^2}{12} \;\rfloor \tag{22}$$

9. The global equation of equilibrium can now be written (note that for illustrative purposes, we kept w and and a moment M at node B).

$$\underbrace{\left\{\begin{array}{c} -\frac{P}{2} \\ 0 \\ M \end{array}\right\} + \underbrace{\left\{\begin{array}{c} -\frac{wH}{2} \\ -\frac{P}{2} \\ \frac{PL}{8} - \frac{wH^2}{12} \end{array}\right\}}_{NEL}}_{P_{ext}} - \underbrace{\underbrace{\left[\begin{array}{ccc} \frac{EA}{L} + \frac{12EI^c}{H^3} & 0 & \frac{6EI^c}{H^2} \\ 0 & \frac{12EI^b}{L^3} + \frac{EA}{H} & -\frac{6EI^b}{L^2} \\ \frac{6EI^c}{H^2} & -\frac{6EI^b}{L^2} & \frac{4EI^b}{L} + \frac{4EI^c}{H} \end{array}\right]}_{[\mathrm{K}]} \left\{\begin{array}{c} \Delta_1 \\ \Delta_2 \\ \theta_3 \end{array}\right\}}_{P_{int}}$$

Substituting:

$$\left\{\begin{array}{c} -0.5 \\ 0 \\ 5 \end{array}\right\} + \underbrace{\left\{\begin{array}{c} -1.5 \\ -0.5 \\ -0.75 \end{array}\right\}}_{NEL} = 10^6 \underbrace{\left[\begin{array}{ccc} 4.1139 & 0 & 0.0417 \\ 0 & 4.1139 & -0.0417 \\ 0.0417 & -0.0417 & 0.3333 \end{array}\right]}_{[\mathrm{K}]} \left\{\begin{array}{c} \Delta_1 \\ \Delta_2 \\ \theta_3 \end{array}\right\}$$

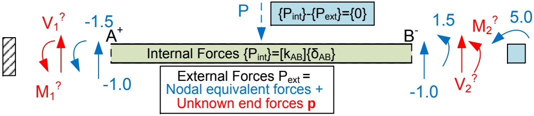
10 Solve for the displacements

$$\left\{\begin{array}{c}\Delta_1\\ \Delta_2\\ \theta_3\end{array}\right\} = 10^6 \left[\begin{array}{ccc}4.1139 & 0 & 0.0417\\ 0 & 4.1139 & -0.0417\\ 0.0417 & -0.0417 & 0.3333\end{array}\right]^{-1} \left\{\begin{array}{c}-2\\ -0.5\\ 4.25\end{array}\right\}$$

or

$$\left\{\begin{array}{c}\Delta_1\\ \Delta_2\\ \theta_3\end{array}\right\} = 10^{-6} \left\{\begin{array}{c}-0.61 \text{ m}\\ 0.0084 \text{ m}\\ 12.82 \text{ radian}\end{array}\right\}$$

11 To obtain the element internal forces, multiply each element stiffness matrix by the local displacements. For element AB, the local and global coordinates match, thus



$$\underbrace{\left\{\begin{array}{c} p_1^? \\ p_2^? \\ p_3^? \\ \hline p_4^? \\ p_5^? \\ p_6^? \end{array}\right\} + \left\{\begin{array}{c} 0 \\ -\frac{P}{2} \\ -\frac{PL}{8} \\ 0 \\ -\frac{P}{2} \\ \frac{PL}{8} \end{array}\right\}}_{P_{ext}} - \underbrace{\left[\begin{array}{ccc|ccc} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & 0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ 0 & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} & 0 & -\frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ \hline -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & 0 & \frac{12EI_y}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} & 0 & -\frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline \delta_1 \\ \delta_2 \\ \theta_3 \end{array}\right\}}_{P_{int}} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{c} p_1^? \\ p_2^? \\ p_3^? \\ \hline p_4^? \\ p_5^? \\ p_6^? \end{array}\right\} = \underbrace{10^6 \left[\begin{array}{ccc|ccc} - & - & - & -4.1 \times 10^6 & 0 & 0 \\ - & - & - & 0 & -13,889. & 41,667. \\ - & - & - & 0 & -41,667. & 83,333. \\ - & - & - & 4.1 \times 10^6 & 0 & 0 \\ - & - & - & 0 & 13,889. & -41,667 \\ - & - & -. & 0 & -41,667 & 166,667. \end{array}\right]}_{\mathtt{k}^{AB}}$$

$$\underbrace{\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ -0.61 \\ 0.0084 \\ 12.82 \end{array}\right\}}_{\delta^{AB}} - \underbrace{\left\{\begin{array}{c} 0 \\ -0.5 \\ -0.75 \\ 0 \\ -0.5 \\ 0.75 \end{array}\right\}}_{NEL^{AB}}$$

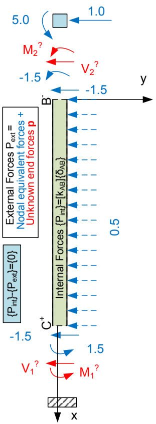
or

$$\left\{\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \hline p_4 \\ p_5 \\ p_6 \end{array}\right\} = \left\{\begin{array}{c} N_1 \\ V_1 \\ M_1 \\ \hline N_2 \\ V_2 \\ M_2 \end{array}\right\} = \left\{\begin{array}{c} 2.52 \text{ kN} \\ 1.03 \text{ kN} \\ 1.82 \text{ kN.m.} \\ -2.52 \text{ kN} \\ -0.034 \text{ kN} \\ 1.39 \text{ kN.m} \end{array}\right\}$$

12 For element BC, the local and global coordinates do not match, hence we will need to transform the displacements from their global to their local coordinate components. By inspection

Local	x	y	z
Global	$-Y$	$+X$	$+Z$

Note that there are no local or global displacements in dof 1-3, hence



$$\left\{\begin{array}{c} p_1^? \\ p_2^? \\ p_3^? \\ \hline p_4^? \\ p_5^? \\ p_6^? \end{array}\right\} = \left[\begin{array}{ccc|ccc} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & 0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ 0 & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} & 0 & -\frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ \hline -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & 0 & \frac{12EI_y}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} & 0 & -\frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{array}\right] \left\{\begin{array}{c} \delta_4 \\ \delta_5 \\ \theta_6 \\ \hline 0 \\ 0 \\ 0 \end{array}\right\}$$

$$= \underbrace{10^6 \left[\begin{array}{ccc|ccc} 4.1\times 10^6 & 0 & 0 & - & - & - \\ 0 & 13,888.9 & 41,666.7 & - & - & - \\ 0 & 41,666.7 & 16,6667. & - & - & - \\ -4.1\times 10^6 & 0 & 0 & - & - & - \\ 0 & -13,888.9 & -41,666.7 & - & - & - \\ 0 & 41,666.7 & 83,333.3 & - & - & - \end{array}\right]}_{\mathbf{k}^{BC}}$$

$$\underbrace{\left\{\begin{array}{c} -0.61 \\ 0.0084 \\ 12.82 \\ 0 \\ 0 \\ 0 \end{array}\right\}}_{\delta^{BC}} - \left\{\begin{array}{c} 0 \\ 1.5 \\ -1.5 \\ 0 \\ -1.5 \\ 1.5 \end{array}\right\} = \left\{\begin{array}{c} N_1 \\ V_1 \\ M_1 \\ \hline N_2 \\ V_2 \\ M_2 \end{array}\right\} = \left\{\begin{array}{c} -0.034 \text{ kN} \\ 2.026 \text{ kN} \\ 3.612 \text{ kN.m} \\ 0.0344 \text{ kN} \\ 0.974 \text{ kN} \\ -0.456 \text{ kN.m} \end{array}\right\}$$

```
%% Stiffness Method Frame Example 09/18
% courtesy of Xiao Fu
clear all
clc

%% Elements properties
L_elem = [6; 6]; % m
A_elem = [0.123; 0.123]; % m^2
E_elem = [200E6; 200E6]; % kN/m^2
I_elem = [1250E-6; 1250E-6]; % m^4

%% Loads
P = 1;
M = 5;
w = 0.5;

%% Structure Displacements in GCS
% Assemble global stiffness matrix
K = [A_elem(1)*E_elem(1)/L_elem(1)+12*E_elem(2)*I_elem(2)/L_elem(2)^3, 0,...
6*E_elem(2)*I_elem(2)/L_elem(2)^2;
0, A_elem(2)*E_elem(2)/L_elem(2)+12*E_elem(1)*I_elem(1)/L_elem(1)^3,...
-6*E_elem(1)*I_elem(1)/L_elem(1)^2;
6*E_elem(2)*I_elem(2)/L_elem(2)^2, -6*E_elem(1)*I_elem(1)/L_elem(1)^2, ...
4*E_elem(1)*I_elem(1)/L_elem(1)+4*E_elem(2)*I_elem(2)/L_elem(2)]

% Determine vector of external forces
NEL = [-w*L_elem(2)/2; -P/2; P*L_elem(1)/8-w*L_elem(2)^2/12]; % Nodal Equivalent Load at DOFs
F = [-P/2; 0 ; M];  % Externally applied forces
F_ext = NEL + F;    % Total External Force

% Solve for Displacement
```

```
Disp = K\F_ext

%% Internal Forces

% Element-AB
i = 1;
k_AB = stiff(E_elem(i), I_elem(i), L_elem(i), A_elem(i) ); % Element stiffness matrix in LCS
NEL_elem_AB = [0; -P/2; -P*L_elem(i)/8; 0; -P/2; P*L_elem(i)/8]; % nodal element forces in
    LCS
disp_elem_AB = [0; 0; 0; Disp(1); Disp(2); Disp(3)]; % global nodal displ. of AB in LCS
Force_elem_AB = k_AB*disp_elem_AB - NEL_elem_AB      % Internal forces of AB in LCS

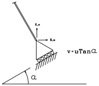
% Element-BC
i = 2;
k_BC = stiff(E_elem(2), I_elem(2), L_elem(2), A_elem(2) );
NEL_elem_BC = [0; -w*L_elem(i)/2; -w*L_elem(i)^2/12; 0; -w*L_elem(i)/2; w*L_elem(i)^2/12];
disp_elem_BC = [-Disp(2); Disp(1); Disp(3);0;0;0];
Force_elem_BC = k_BC*disp_elem_BC - NEL_elem_BC
```

```
function [k]=stiff(E,I,L,A)
EA=E*A; EI=E*I;
k=[
EA/L,        0,          0,    -EA/L,        0,          0;
0,     12*EI/L^3,  6*EI/L^2,      0,   -12*EI/L^3,  6*EI/L^2;
0,      6*EI/L^2,    4*EI/L,      0,    -6*EI/L^2,    2*EI/L;
-EA/L,       0,          0,    EA/L,         0,          0;
0,    -12*EI/L^3, -6*EI/L^2,      0,    12*EI/L^3, -6*EI/L^2;
0,      6*EI/L^2,    2*EI/L,      0,    -6*EI/L^2,    4*EI/L];
```

Cover subsequent topic only time permitting

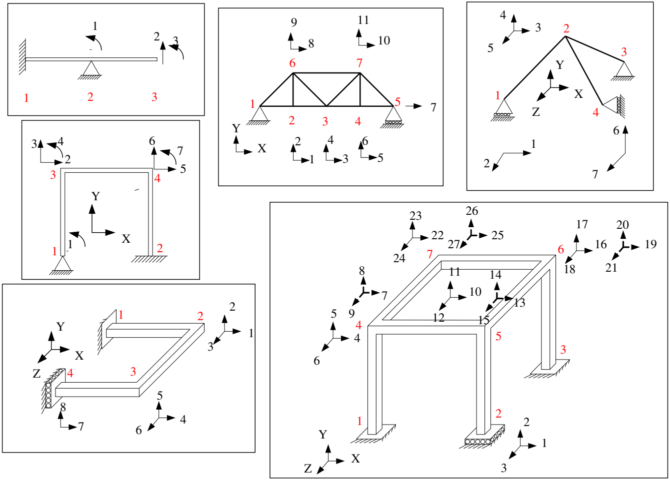
Degree of Freedom

A degree of freedom (d.o.f.) is an independent generalized nodal displacement (translation or rotation) at a node.
The displacements must be linearly independent (of coordinate system) and thus not related to each other.



Element vs Structure

An element dof is defined wrt its own local coordinate system. A structural dof is defined wrt a global coordinate system.



Structural Discretization

Numerical modeling of a structure requires that we can mathematically describe it (geometry, boundary conditions, geometry and properties of elements, and loads).

The node is characterized by its nodal id (node number), coordinates, boundary conditions, and load (this one is often defined separately)

Node No.	Coor.		B. C.		
	X	Y	X	Y	Z
1	$x(1)$	$y(1)$	$ID(1,1)$	$ID(1,2)$	$ID(1,3)$
2	$x(2)$	$y(2)$	$ID(2,1)$	$ID(2,2)$	$ID(2,3)$
3	$x(3)$	$y(3)$	$ID(3,1)$	$ID(3,2)$	$ID(3,3)$
4	$x(4)$	$y(4)$	$ID(4,1)$	$ID(4,2)$	$ID(4,3)$

0 and 1 correspond to free or fixed degree of freedom (alternatively to a 1 corresponds a reaction).
Known displacements can be zero (restrained) or non-zero.

The element is characterized by the nodes which it connects, and its group number,

Element No.	From Node	To Node	Group Number
1	1	2	1
2	3	2	2
3	3	4	2

Group number will then define both element type, and elastic/geometric properties. The last one is a pointer to a separate array,

Group No.	Element Type	Material Group
1	1	1
2	2	1
3	1	2

In this example element 1 has element id 1 (such as beam element), while element 2 has a id 2 (such as a truss element). Material group 1 would have different elastic/geometric properties than material group 2.

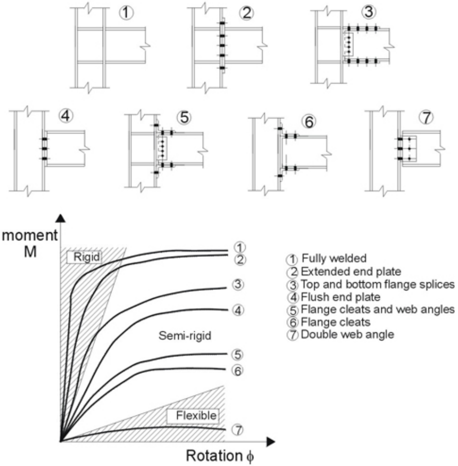
Structural idealization is as much an art as a science.

1. 2D vs 3D
2. Frame or truss
3. Rigid or semi-rigid connections
4. Effect of Relative Stiffnesses
5. Cross-Section
6. Elastic supports
7. Include or not secondary members
8. Include or not axial deformation
9. Linear or nonlinear analysis (linear analysis can not predict the peak or failure load, and will underestimate the deformations).
10. Small or large deformations
11. Time dependent effects
12. Partial collapse or local yielding
13. Static or dynamic
14. Wind load
15. Thermal load
16. Secondary stresses

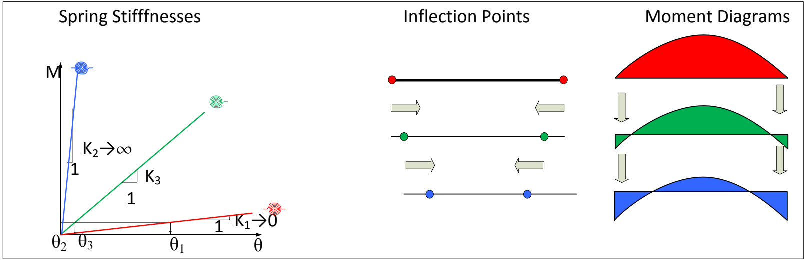
We shall review most of them separately

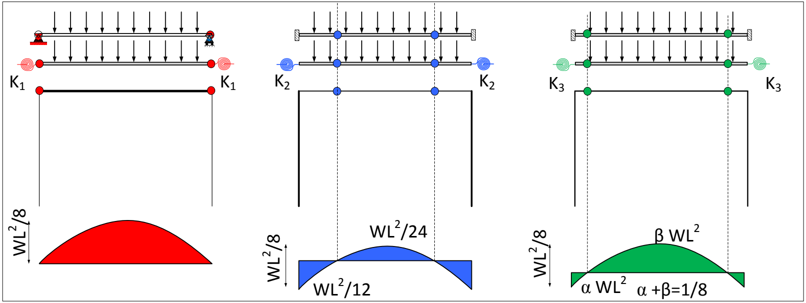
3D or simplified 2D

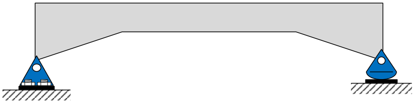
Is it a truss or a girder?

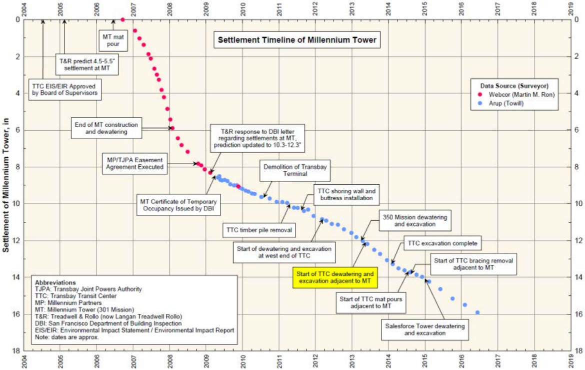


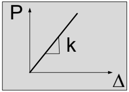
Hinge $<$ Semi-rigid connections $<$ Rigid

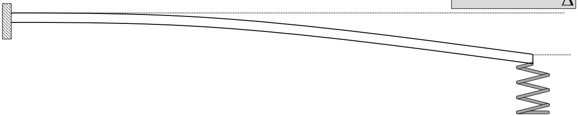

Spring Stifffnesses
Inflection Points
Moment Diagrams
M
K2→∞
1
K3
1
1
K1→0
θ2
θ3
θ1
θ

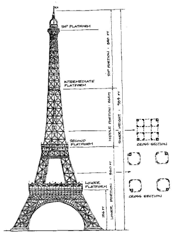

K1
K1
K2
K2
K3
K3
WL2/8
WL2/8
WL2/24
WL2/12
WL2/8
β WL2
α WL2
α +β=1/8

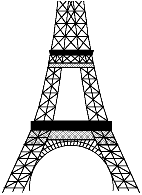



Settlement Timeline of Millennium Tower
Settlement of Millennium Tower, in
Data Source (Surveyor)
Webcor (Martin M. Ron)
Arup (Towill)
MT mat pour
T&R predict 4.5-5.5" settlement at MT
TTC EIS/EIR Approved by Board of Supervisors
End of MT construction and dewatering
T&R response to DBI letter regarding settlements at MT, prediction updated to 10.3-12.3"
MP/TJPA Easement Agreement Executed
Demolition of Transbay Terminal
MT Certificate of Temporary Occupancy Issued by DBI
TTC shoring wall and buttress installation
TTC timber pile removal
350 Mission dewatering and excavation
Start of dewatering and excavation at west end of TTC
TTC excavation complete
Start of TTC bracing removal adjacent to MT
Start of TTC dewatering and excavation adjacent to MT
Start of TTC mat pours adjacent to MT
Salesforce Tower dewatering and excavation
Abbreviations
TJPA: Transbay Joint Powers Authority
TTC: Transbay Transit Center
MP: Millennium Partners
MT: Millennium Tower (301 Mission)
T&R: Treadwell & Rollo (now Langan Treadwell Rollo)
DBI: San Francisco Department of Building Inspection
EIS/EIR: Environmental Impact Statement / Environmental Impact Report
Note: dates are approx.

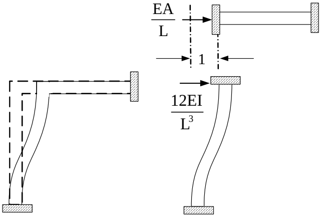

P
k
Δ





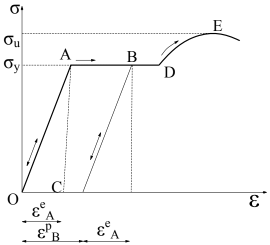


May ignore secondary members

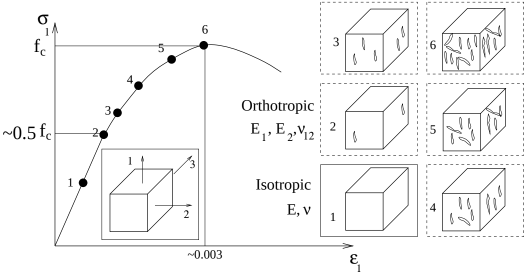


- Ratio of axial to flexural stiffness is:
  $\alpha = \frac{k_a}{k_f} = \frac{\frac{EA}{L}}{\frac{12EI}{L^3}} = \frac{AL^2}{12I}$.
- For a $b \times h$ rectangular section, with $b = h/2$, and $L = 10h$, $\Rightarrow \alpha = 100$
- For a W section
  $Z \approx \frac{wd}{9}$, $\frac{Z}{S} = \xi = 1.1$, $S = \frac{I}{\frac{d}{2}}$, $w = (490)\ \mathrm{lbs/ft}^3 A$, or
  $I \approx 0.208Ad^2$, and $\alpha = \frac{\frac{EA}{L}}{\frac{12EI}{L^3}} = \frac{\frac{EA}{L}}{\frac{12E(0.208)Ad^2}{L^3}} = 0.4\left(\frac{L}{d}\right)^2$
- For steel structure, we can assume
  $L = 20d$, $\Rightarrow \alpha = 160$ Axial stiffness is much higher than flexural stiffness. Note: we may have negligible axial deformations, however axial force is not negligible.

Non-Linear stress-strain



Steel



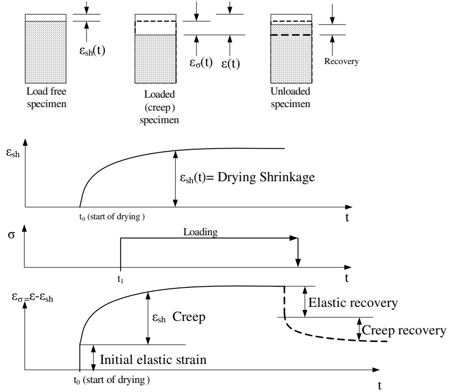
Concrete

Large Deformation

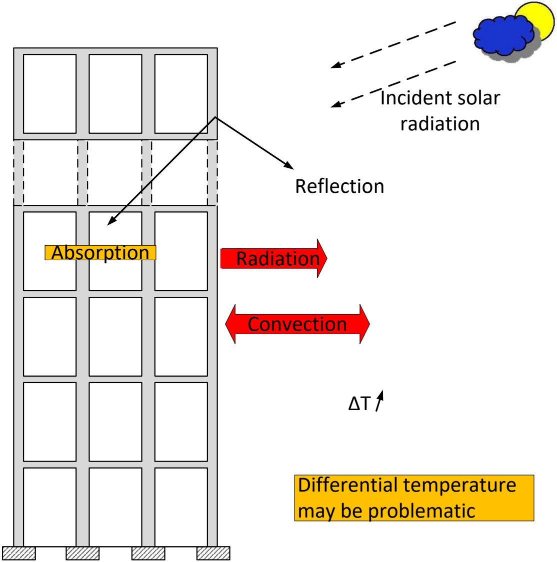
$$\varepsilon_{xx} = \underbrace{\frac{\partial u}{\partial x}}_{\text{small deformation}} + \underbrace{\frac{1}{2}\left(\frac{\partial u}{\partial x}^2 + \frac{\partial v}{\partial x}^2 + \frac{\partial w}{\partial x}^2\right)}_{\text{large deformation}}$$

u and v are the axial and transversal displacements respectively.

Time Dependent

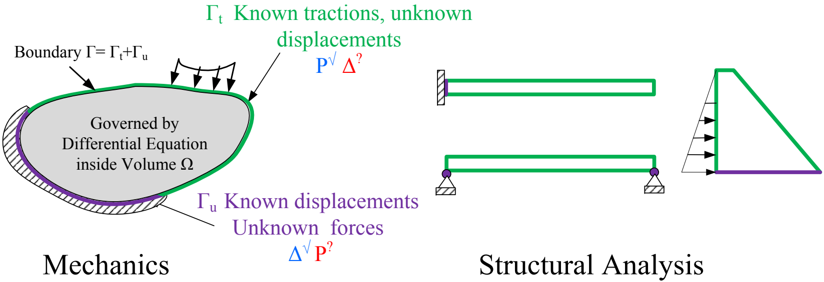


Dynamic When the frequency of the applied load (excitation) of a structure is less than about a third of its lowest natural frequency of vibration, then we can neglect inertia effects and treat the problem as a quasi-static one, otherwise a dynamic analysis must be performed.
For a very flexible structure, even a slowly applied load may necessitate a dynamic analysis.


Incident solar radiation
Reflection
Absorption
Radiation
Convection
ΔT
Differential temperature may be problematic

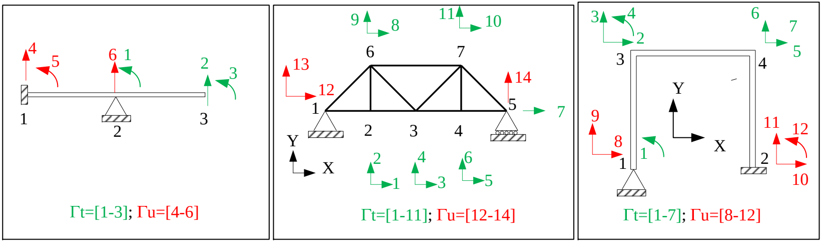
- Analysis of a structure is essentially solving a boundary value problem (governed by a differential equation over the volume Ω, and subjected to space/temporal boundary conditions along the boundary Γ).
- In our case we are discretizing our structure, and the governing differential equation (equilibrium) is embedded in $\mathrm{K}\Delta = \mathrm{P}$.
- $\Gamma = \Gamma_t \bigcup \Gamma_u$

Γ	Traction	Displ.	Math.	Struct.	DOF
Γ_t	$\mathrm{P}_t^{\surd}$	$\Delta_t^{?}$	Neuman	Essential	Free
Γ_u	$\mathrm{R}_u^{?}$	$\Delta_u^{\surd}$	Dirichlet	Natural	Fixed/Constrained

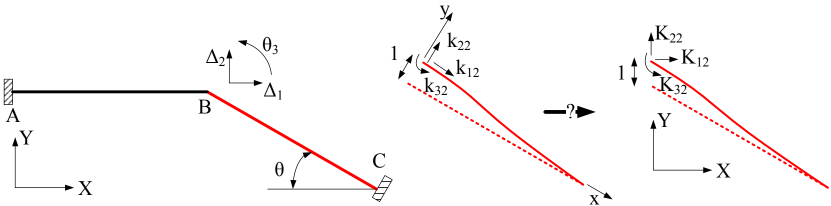


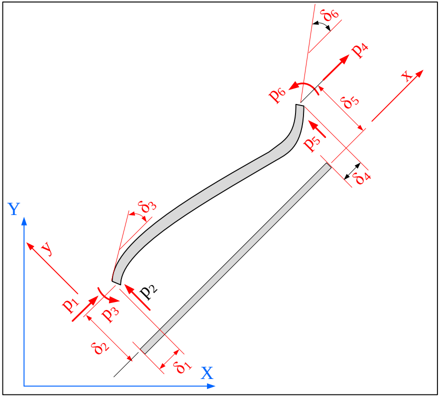
For the beam and the dam, we need to determine the displacements along Γ_t and the forces (reactions) along Γ_u.

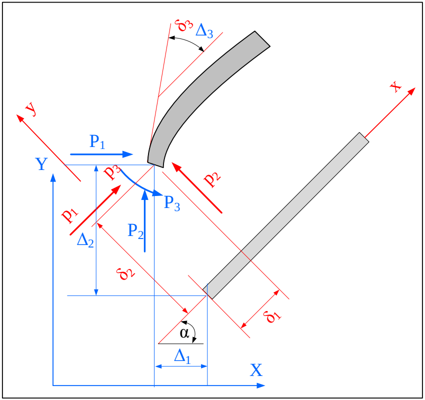
- We have labeled the global dof associated with the unconstrained dof (Γ_t), where we solve for the displacements.
- We will need to label the global dof associated with the constrained dof (Γ_u) where we will solve the reactions.
- We will label the dof along Γ_t first, and then those along Γ_u next.
- We have so far considered the stiffness matrix associated with Γ_t only.
- We will need to assemble the augmented stiffness matrix associated with $\Gamma = \Gamma_t \bigcup \Gamma_u$

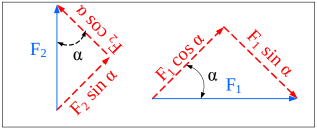


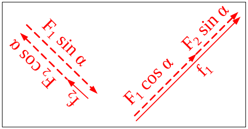
- Assembly of structure stiffness matrix is in global coordinate system, element stiffness matrix is first computed in local coordinate system.
- Need to transform $\mathtt{k}$ into $\mathtt{K}$ and δ into Δ for arbitrary structures.











$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\gamma} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$\{\mathrm{p}^e\} = [\mathrm{k}^e]\{\delta^e\} \text{ and } \{\mathrm{P}^e\} = [\mathrm{K}^e]\{\Delta^e\} \tag{23}$$

Let us define a vector transformation matrix $[\Gamma^e]$ such that:

$$\{\delta^e\} \stackrel{\text{def}}{=} [\Gamma^e]\{\Delta^e\} \text{ and } \{\mathrm{p}^e\} \stackrel{\text{def}}{=} [\Gamma^e]\{\mathrm{P}^e\} \tag{24}$$

Substituting we obtain $\{\mathrm{p}^e\} = [\Gamma^e]\{\mathrm{P}^e\} = [\mathrm{k}^e][\Gamma^e]\{\Delta^e\}$ premultiplying by $[\Gamma^e]^{-1}$: $\{\mathrm{P}^e\} = [\Gamma^e]^{-1}[\mathrm{k}^e][\Gamma^e]\{\Delta^e\}$ But since the rotation matrix is orthogonal, we have $[\Gamma^e]^{-1} = [\Gamma^e]^T$ (and $\{\Delta^e\} = [\Gamma^e]^T\{\delta^e\}$)

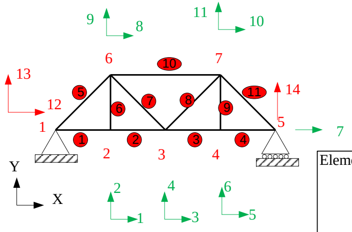
$$\{\mathrm{P}^e\} = \underbrace{[\Gamma^e]^T[\mathrm{k}^e][\Gamma^e]}_{[\mathrm{K}^e]}\{\Delta^e\}$$

$$[\mathrm{K}^e] = [\Gamma^e]^T[\mathrm{k}^e][\Gamma^e] \tag{25}$$

which is the general relationship between element stiffness matrix in local and global coordinates.

$$
\begin{aligned}
\mathrm{K}^e &= \Gamma^T \mathrm{k}^e \Gamma && (26)\\
\mathrm{K}^S &= \sum_{e=1}^{e=nelem} \mathrm{K}^e && (27)\\
\left\{ \frac{\mathrm{P}_t^{\surd}}{\mathrm{R}_u^{?}} \right\} &= \underbrace{\left[\begin{array}{c|c} \mathrm{K}_{tt} & \mathrm{K}_{tu} \\ \hline \mathrm{K}_{ut} & \mathrm{K}_{uu} \end{array} \right]}_{\text{Augmented Stiffness Matrix}} \left\{ \frac{\Delta_t^{?}}{\Delta_u^{\surd}} \right\} && (28)\\
\mathrm{K}_{tt} & \quad \mathrm{f}^{-1};\ \text{Reduced Stiffness Matrix} && (29)\\
\Delta_t &= \mathrm{K}_{tt}^{-1} (\mathrm{P}_t - \mathrm{K}_{tu}\Delta_u) && (30)\\
\mathrm{R}_u &= \mathrm{K}_{ut}\Delta_t + \mathrm{K}_{uu}\Delta_u && (31)\\
\delta^{(e)} &= \Gamma^{(e)}\Delta^{(e)} && (32)\\
p_{int}^{(e)} &= \mathrm{k}^{(e)}\delta^{(e)} && (33)
\end{aligned}
$$

LM is a mapping of element local to global dof



$$
\begin{array}{c} \text{DOF} \end{array}\;\;
[ID]=\begin{array}{cc} 1 & 2 \\ \end{array}
\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\Rightarrow
\begin{array}{cc}
\left[\begin{array}{cc} 12 & 13 \\ 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 14 \\ 8 & 9 \\ 10 & 11 \end{array}\right]
&
\begin{array}{l} \text{Node 1} \\ \text{Node 2} \\ \text{Node 3} \\ \text{Node 4} \\ \text{Node 5} \\ \text{Node 6} \\ \text{Node 7} \end{array}
\end{array}
$$

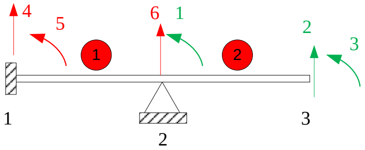
$$
[LNODS]=
\begin{array}{c}
\begin{array}{cc} \text{Nodes}\;\; i & j \end{array}\\
\left[\begin{array}{cc} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \\ 1 & 6 \\ 2 & 6 \\ 3 & 6 \\ 3 & 7 \\ 7 & 4 \\ 6 & 7 \\ 7 & 5 \end{array}\right]
\end{array}
\begin{array}{l} \\ \text{Element 1} \\ \text{Element 2} \\ \text{Element 3} \\ \text{Element 4} \\ \text{Element 5} \\ \text{Element 6} \\ \text{Element 7} \\ \text{Element 8} \\ \text{Element 9} \\ \text{Element 10} \\ \text{Element 11} \end{array}
$$

Element global dof

$$
[LM]=
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}\\
\left[\begin{array}{cc|cc} 12 & 13 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 14 \\ 12 & 13 & 8 & 9 \\ 1 & 2 & 8 & 9 \\ 3 & 4 & 8 & 9 \\ 3 & 4 & 10 & 11 \\ 10 & 11 & 5 & 6 \\ 8 & 9 & 10 & 11 \\ 10 & 11 & 7 & 14 \end{array}\right]
\end{array}
\begin{array}{l} \\ \text{Element 1} \\ \text{Element 2} \\ \text{Element 3} \\ \text{Element 4} \\ \text{Element 5} \\ \text{Element 6} \\ \text{Element 7} \\ \text{Element 8} \\ \text{Element 9} \\ \text{Element 10} \\ \text{Element 11} \end{array}
$$

LM is a mapping between element and structure global dof

[ ] User defined
[ ] Computed by program



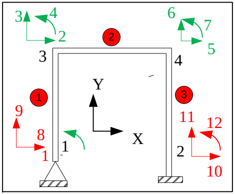
$$\begin{array}{cc} & \begin{array}{cc} \text{i} & \text{j} \end{array} \\ \text{Node} & \\ [LNODS] = & \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{array}{l} \text{Element 1} \\ \text{Element 2} \end{array} \end{array}$$

Global DOF

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ [ID] = & \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 5 \\ 6 & 1 \\ 2 & 3 \end{bmatrix} \begin{array}{l} \text{Node 1} \\ \text{Node 2} \\ \text{Node 3} \end{array} \end{array}$$

Element global dof

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ [LM] = & \left[\begin{array}{cc|cc} 4 & 5 & 6 & 1 \\ 6 & 1 & 2 & 3 \end{array}\right] \begin{array}{l} \text{Element 1} \\ \text{Element 2} \end{array} \end{array}$$



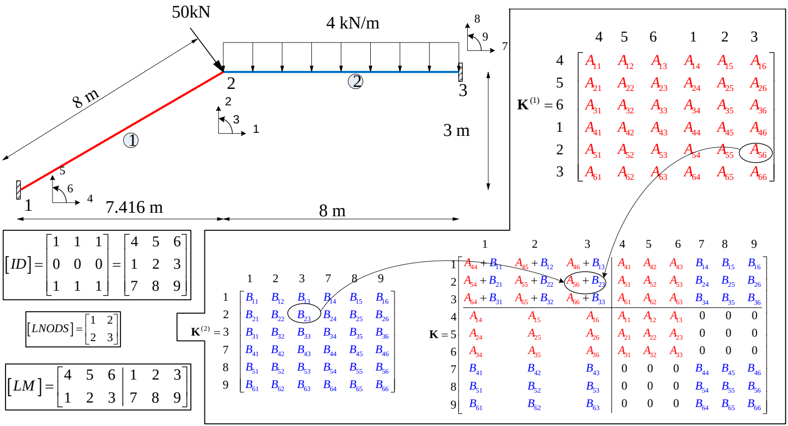
Global DOF

$$[ID] = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \Rightarrow \begin{array}{c} \begin{bmatrix} 8 & 9 & 1 \\ 10 & 11 & 12 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \end{array} \begin{array}{l} \text{Node 1} \\ \text{Node 2} \\ \text{Node 3} \\ \text{Node 4} \end{array}$$

$$[LNODS] = \begin{array}{c} \begin{array}{cc} \text{Node i} & \text{j} \end{array} \\ \begin{bmatrix} 1 & 3 \\ 3 & 4 \\ 4 & 2 \end{bmatrix} \end{array} \begin{array}{l} \text{Element 1} \\ \text{Element 2} \\ \text{Element 3} \end{array}$$

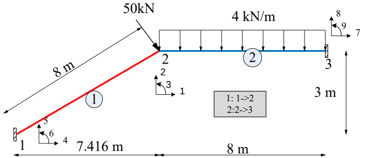
Element Global DOF

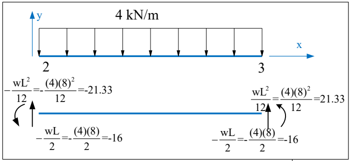
$$[LM] = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccc|ccc} 8 & 9 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 10 & 11 & 12 \end{array}\right] \end{array} \begin{array}{l} \text{Element 1} \\ \text{Element 2} \\ \text{Element 3} \end{array}$$

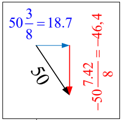


$$K_{ij}^{(e)} \rightarrow K_{st}^{(S)} \text{ and } \left\{ \begin{array}{ccc} s & = & LM(i) \\ t & = & LM(j) \end{array} \right.$$

[LM] is a mapping between the element global dof and the structure's (global) dof.

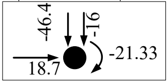






$$\left\{p_{El}^{(2)}\right\}=\begin{Bmatrix}0.\\-16.0\\-21.33\\0.0\\-16.0\\21.33\end{Bmatrix};\left\{P_{El}^{(2)}\right\}=\underbrace{\begin{bmatrix}\text{Cos }\alpha & \text{Sin }\alpha & 0 & 0 & 0 & 0\\ \text{-Sin }\alpha & \text{Cos }\alpha & 0 & & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \text{Cos }\alpha & \text{Sin }\alpha & 0\\ 0 & 0 & 0 & \text{-Sin }\alpha & \text{Cos }\alpha & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}}_{\Gamma^{(2)}}\underbrace{\begin{Bmatrix}0.\\-16.0\\-21.33\\0.0\\-16.0\\21.33\end{Bmatrix}}_{p_{El}^{(2)}}=\begin{Bmatrix}0.\\-16.0\\-21.33\\0.0\\-16.0\\21.33\end{Bmatrix}$$

$$[LM]=\left[\begin{array}{ccc|ccc}4 & 5 & 6 & 1 & 2 & 3\\ 1 & 2 & 3 & 7 & 8 & 9\end{array}\right]$$



$\Rightarrow P_t(LM^{(e)}(i))=P_{NEF}^{(e)}(i)+P_t(LM^{(e)}(i));\quad \forall\ LM^{(e)}(i)\leq size(K_{tt})$

$size(K_{tt})=3$; Corresponds to number of unconstrained dof

$$\begin{array}{llll}
LM(2,1)=1\leq 3\Rightarrow & P_t(LM(2,1))=P_{El}^{(2)}(1)+P_{nod}(LM(2,1))\ \Rightarrow & P_t(1)=0+P_{nod}(1) & =0+18.7 \quad =18.7\\
LM(2,2)=2\leq 3\Rightarrow & P_t(LM(2,2))=P_{El}^{(2)}(2)+P_{nod}(LM(2,2))\Rightarrow & P_t(2)=-16+P_{nod}(2) & =-16-46.4 \quad =-62.4\\
LM(2,3)=3\leq 3\Rightarrow & P_t(LM(2,3))=P_{El}^{(2)}(3)+P_{nod}(LM(2,3))\ \Rightarrow & P_t(3)=-21.33+P_{nod}(3)=0-21.33 & =-21.33
\end{array}$$

1. Preliminaries
   1. Read the structure mathematical model (type, coordinates, connectivity, cross-sectional and material properties, loads)
   2. Determine the number of nodes (`nnode`), number of element (`nelem`), maximum number dof/node (`ndofpn`), size of K_{tt} (`sizet`), total number of dof (`ndoft`), update ID and determine LM matrices
2. Analysis:
   1. For each element, determine
      1. Vector LM mapping local element to global structure degrees of freedoms.
      2. Element stiffness matrix $[\mathbf{k}^{(e)}]$
      3. Transformation matrix $[\Gamma^{(e)}]$
      4. Element stiffness matrix in global coordinates $[\mathrm{K}^{(e)}] = [\Gamma^{(e)}]^T[\mathbf{k}^{(e)}][\Gamma^{(e)}]$
   2. Assemble the augmented stiffness matrix $[\mathrm{K}^{(S)}]$ of unconstrained and constrained degree of freedom's.
   3. Extract $[\mathrm{K}_{tt}]$ from $[\mathrm{K}^{(S)}]$ and invert (actually decompose).
   4. Load Vector
      1. Compute nodal equivalent forces vectors for each element in local coordinate system $\mathrm{p}_{NEF}^{(e)}$ and in global coordinate system $\mathrm{P}_{NEF}^{(e)} = \Gamma^{(e)^T}\mathrm{p}_{NEF}^{(e)}$

2. Assemble the nodal load vector to include nodal loads and nodal equivalent forces (note P is for the structure).
   $P_t(LM^{(e)}(i)) = P_{NEF}^{(e)} + P_t(LM^{(e)}(i)); \forall LM^{(e)} \leq \text{size}(K_{tt})$

3. Backsubstitute and obtain nodal displacements global coordinate system, $\Delta = \mathrm{K}_{tt}^{-1}\mathrm{P}_t$
4. Extract K_{ut}
5. Solve for the reactions, $\mathrm{R}_u = \mathrm{K}_{ut}\Delta_t + \mathrm{K}_{uu}\Delta_u - \mathrm{P}(\mathtt{sizet} : \mathtt{ndof})$
6. Internal forces, for each element
   1. Determine the element nodal displacements in global coordinate system from the global nodal displacements
   2. Transform its nodal displacement from global to local coordinates $\delta^{(e)} = [\Gamma^{(e)}]\Delta^{(e)}$.
   3. Determine the internal forces $\mathrm{p}^{(e)} = \mathrm{k}^{(e)}\delta^{(e)} - p_{NEF}^{(e)}$.

```
clear all
clc
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Program based on the direct stiffness method to analyse 2D frames
% Limitaitons: all section propeties are identical; no initial displacement
%                    CVEN4525/5525 Univ. of Colorado, Boulder
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Input data
% Structural properties units: mm^2, mm^4, and MPa(10^6 N/m)
% Note this could be generalized to assign properties for individual
% element properties
A=6000;II=200*10^6;EE=200000;
% Convert units to meter and kN
A=A/10^6;II=II/10^12;EE=EE*1000;
%coordinates each ow one node
COORD=[0 0;
7.416 3;
15.416 3];
% Define ID matrix each ow one node
ID_original=[1 1 1;
0 0 0;
1 1 1];
% Connectivity matrix, each row one element
LNODS=[1 2;
2 3];
% Nodal Load each row corresponds to a node
nodal_load=[0 0 0;
50*3/8 -50*7.416/8 0;
0 0 0];
% element load (consider uniformly distributed load only)
loaded_elem=[0;-4];
```

```
%============== End of user input data ================================
%% get number of elements, nodes, and degrees of freedom per node
[nelem col]=xxx; %total number of elements
[nnodes ndofpn]=xxx; % total number of nodes and number of dof per node
ndofpe=xxx;          % number of degrees of freedom per element
ndoft=xxx;     % Size of K augmented
%% update the ID matrix
n=0;
for l=1:xxx
for k=1:xxx
if ID_original(l,k)xxx
xxx
end
end
end
size_t=xxx; % size of the unconstrained dof
for l=1:xxx
for k=1:xxx
xx
end
end
%% Compute the LM vector one row for each element
for elem=1:xxx
for nod=1:xxx
node=xxx
for dof=1:xxx
n=(nod-1)*ndofpn+dof;
LM(elem,n)=xxx
end
end
end
```

```
%% for each element compute k, K, and gamma
%% zero the matrices
k=zeros(ndofpe,ndofpe,nelem); K=zeros(ndofpe,ndofpe,nelem); Gamma=zeros(ndofpe,ndofpe,nelem);
for elem=1:nelem
%    determine coordinates of each node
nod1=xxx;nod2=xxx;
xy_1=xxx xxx]; xy_2=[xxx xxx];
[k(:,:,elem),K(:,:,elem),Gamma(:,:,elem)]=stiff(EE,II,A,xy_1,xy_2);
end
%% Assemble augmented stiffness matrix
Kaug=zeros(nnodes*ndofpn);
for elem=1:xxx
for l=1:xxx
lr=xxx;
for c=1:xxx
lc=xxx;
Kaug(lr,lc)=xxx;
end
end
end
%% Handle the load
% Initialize to zero
P=zeros(ndoft,1); %initialize the vecotr of load size=ndoft
% loop on each loaded node
for l=1:nnodes % Loop on each loaded node
for c=1:ndofpn
if ID(l,c)<=xxx
P(ID(l,c))=xxx
end
end
end
```

```
% loop on each loaded element
 for elem=1:nelem
 for c=1:xxx % loop on each dof of the element
 NEF_local(elem,c)=0; %initialize to zero
 end
 w=loaded_elem(elem);
 if w xxx%only if element is loaded compute non zero NEF
 xxx
 L=sqrt((xy_1(1)-xy_2(1))^2+(xy_1(2)-xy_2(2))^2);
 NEF_local(elem,:)=[0; xxx; xxx; xxx; xxx; xxx];
 end
 NEF_global(elem,:)=xxx
 for c=1:ndofpe % add to the P vector terms associated with constrained dof
 global_dof=xxx
 P(global_dof)=xxx
 end
 end
%% Solve FO the displacements
% Extract Ptt
 Ptt=P(1:xxx);
% Extract the unconstrained structures Stiffness Matrix
 Ktt=xxx;
% Solve for the Displacements inverse of Ktt times load vector
 Displacements=Ktt\Ptt
%% Solve for the reactions
% Extract Kut
 Kut=xxx;
% Compute the Reactions and do not forget to add fixed end actions
 Reactions=xxx;
%% Solve for the internal foces
% Assign the vector of global displacements for the element
```

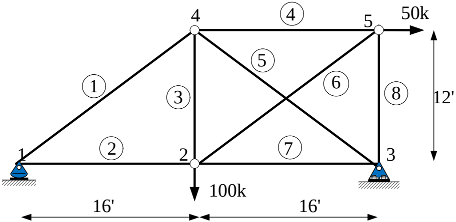
```
for elem=1:nelem
for c=1:ndofpe
global_dof=xxx;
xxx
end
end
% get the element internal foces
for elem=1:nelem
dis_local=xxx;
int_forces=xxx;
end
```

```
function [k,K,Gamma]= stiff(EE,II,A,xy_1,xy_2)
% Determine the length
L=xxx
% Compute the angle theta (careful with vertical members!)
if(xy_2(1)-xy_1(1))~=0
alpha=xxx
else
alpha=xxx
end
% form rotation matrix Gamma
Gamma=[
cos(alpha)   sin(alpha)   0      0           0           0;
xxx
];
% form element stiffness matrix in local coordinate system
EI=EE*II; EA=EE*A;
k=[
EA/L,          0,            0,    -EA/L,         0,          0;
```

```
xxx];
% Element stiffness matrix in global coordinate system
K=xxx
```

This will generate the following results:

```
Displacements =     Reactions        int_forces =   int_forces =
0.0010            130.4973
-0.0050             55.6766         141.8530       149.2473
-0.0005             13.3742           2.6758         9.3266
-149.2473         13.3742          -8.0315
22.6734       -141.8530       -149.2473
-45.3557          -2.6758          22.6734
8.0315       -45.3557
```



$$ID = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 2 & 3 \\ 9 & 10 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}; \qquad [LM] = \begin{bmatrix} 1 & 8 & 4 & 5 \\ 1 & 8 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 4 & 5 \\ 2 & 3 & 6 & 7 \\ 2 & 3 & 9 & 10 \\ 9 & 10 & 6 & 7 \end{bmatrix}$$

$$[K^{(e)}] = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$c = \cos\alpha = \frac{x_2 - x_1}{L}$; $s = \sin\alpha = \frac{Y_2 - Y_1}{L}$

Element 1: $L = 20^{'}$, $c = \frac{16-0}{20} = 0.8$, $s = \frac{12-0}{20} = 0.6$,

$\frac{EA}{L} = \frac{(30{,}000ksi)(10in^2)}{20^{'}} = 15{,}000\ k/ft.$

$$
[K_1] = \begin{array}{c} \\ 1 \\ 8 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 8 & 4 & 5 \end{array} \\
\begin{bmatrix}
9{,}600 & 7200 & -9{,}600 & -7{,}200 \\
7{,}200 & 5{,}400 & -7{,}200 & -5{,}400 \\
-9{,}600 & -7{,}200 & 9{,}600 & 7{,}200 \\
-7{,}200 & -5{,}400 & 7{,}200 & 5{,}400
\end{bmatrix}
\end{array}
$$

Element 2: $L = 16^{'}$, $c = 1$, $s = 0$, $\frac{EA}{L} = 18{,}750$ k/ft.

$$
[K_2] = \begin{array}{c} \\ 1 \\ 8 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 8 & 2 & 3 \end{array} \\
\begin{bmatrix}
18{,}750 & 0 & -18{,}750 & 0 \\
0 & 0 & 0 & 0 \\
-18{,}750 & 0 & 18{,}750 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

Element 3 $L = 12^{'}$, $c = 0$, $s = 1$, $\frac{EA}{L} = 25{,}000\ k/ft$

$$[K_3] = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 25,000 & 0 & -25,000 \\ 0 & 0 & 0 & 0 \\ 0 & -25,000 & 0 & 25,000 \end{bmatrix} \end{array}$$

Element 8 $L = 12^{'}$, $c = 0$, $s = 1$, $\frac{EA}{L} = 25,000\ k/ft$

$$[K_8] = \begin{array}{c} \\ 9 \\ 10 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 9 & 10 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 25,000 & 0 & -25,000 \\ 0 & 0 & 0 & 0 \\ 0 & -25,000 & 0 & 25,000 \end{bmatrix} \end{array}$$

Assemble the global stiffness matrix in k/ft Note that we are not assembling the augmented stiffness matrix, but rather its submatrix $[\mathrm{K}_{tt}]$.

Convert to k/in and simplify

$$\underbrace{\left\{\begin{array}{c} 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ 50 \\ 0 \end{array}\right\}}_{\mathtt{P}_t} = \underbrace{\left[\begin{array}{ccccccc} 2,362.5 & -1,562.5 & 0.00 & -800 & -600 & 0 & 0 \\ & 3,925.0 & 600 & 0 & 0 & -800 & -600 \\ & & 2,533.33 & 0.00 & -2,083.33 & -600 & -450 \\ & & & 3,162.5 & 0 & -1,562.5 & 0 \\ & \mathrm{SYMMETRIC} & & & 2,983.33 & 0 & 0 \\ & & & & & 2,362.5 & 600 \\ & & & & & & 2,533.33 \end{array}\right]}_{\mathtt{K}_{tt}} \underbrace{\left\{\begin{array}{c} U_1 \\ U_2 \\ V_3 \\ U_4 \\ V_5 \\ U_6 \\ V_7 \end{array}\right\}}_{\mathtt{u}_t}$$

Invert stiffness matrix and solve for displacements

$$\left\{\begin{array}{c} U_1 \\ U_2 \\ V_3 \\ U_4 \\ V_5 \\ U_6 \\ V_7 \end{array}\right\} = \left\{\begin{array}{c} -0.0223 \text{ in.} \\ 0.00433 \text{ in.} \\ -0.116 \text{ in.} \\ -0.0102 \text{ in.} \\ -0.0856 \text{ in.} \\ -0.00919 \text{ in.} \\ -0.0174 \text{ in.} \end{array}\right\}$$

Solve for member internal forces (in this case axial forces) in local coordinate systems

$$\left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\} = \underbrace{\frac{AE}{L}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right]}_{\mathbf{k}} \underbrace{\underbrace{\left[\begin{array}{cccc} C & S & 0 & 0 \\ 0 & 0 & C & S \end{array}\right]}_{\Gamma} \underbrace{\left\{\begin{array}{c} U_1 \\ V_1 \\ U_2 \\ V_2 \end{array}\right\}}_{\Delta}}_{\delta}$$

$$= \frac{AE}{L}\left[\begin{array}{cccc} c & s & -c & -s \\ -c & -s & c & s \end{array}\right]\left\{\begin{array}{c} U_1 \\ V_1 \\ U_2 \\ V_2 \end{array}\right\}$$

Element 1:

$$\left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\}^1 = (15,000 \text{ kpf})\left(\frac{1}{12}\frac{\text{ft.}}{\text{in.}}\right)\left[\begin{array}{cccc} 0.8 & 0.6 & -0.8 & -0.6 \\ -0.8 & -0.6 & 0.8 & 0.6 \end{array}\right]\left\{\begin{array}{c} -0.0223 \\ 0.00 \\ -0.0102 \\ -0.0856 \end{array}\right\}$$

$$= \left\{\begin{array}{c} 52.1 \text{ kip} \\ -52.1 \text{ kip} \end{array}\right\} \text{ Compression}$$

Element 2:

$$\left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\}^2 = 18,750 \text{ kpf} \left(\frac{1}{12}\frac{\text{ft}}{\text{in.}}\right) \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \left\{\begin{array}{c} -0.0233 \\ 0.00 \\ 0.00433 \\ -0.116 \end{array}\right\}$$

$$= \left\{\begin{array}{c} -43.2 \text{ kip} \\ 43.2 \text{ kip} \end{array}\right\} \text{Tension}$$

Element 3:

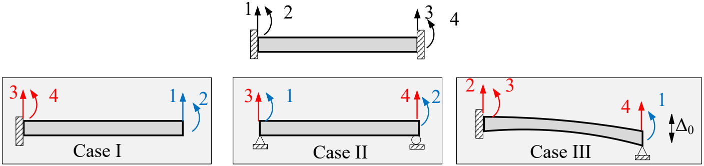
$$\left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\}^3 = 25,000 \text{ kpf} \left(\frac{1}{12}\frac{\text{ft.}}{\text{in.}}\right) \left[\begin{array}{cccc} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{array}\right] \left\{\begin{array}{c} 0.00433 \\ -0.116 \\ -0.0102 \\ -0.0856 \end{array}\right\}$$

$$= \left\{\begin{array}{c} -63.3 \text{ kip} \\ 63.3 \text{ kip} \end{array}\right\} \text{Tension}$$

Element 4:

$$\left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\}^4 = 18,750 \text{ kpf} \left(\frac{1}{12}\frac{\text{ft.}}{\text{in.}}\right) \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \left\{\begin{array}{c} -0.0102 \\ -0.0856 \\ -0.00919 \\ -0.0174 \end{array}\right\}$$

$$= \left\{\begin{array}{c} -1.58 \text{ kip} \\ 1.58 \text{ kip} \end{array}\right\} \text{Tension}$$



We consider the third case, a cantilevered Beam with initial Displacement and no other load.

1. The *element* stiffness matrix is

$$\mathbf{k}^{(e)} = \begin{array}{c c} & \begin{array}{cccc} 2 & 3 & 4 & 1 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \\ 1 \end{array} & \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & 6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$

2. The augmented *structure* stiffness matrix is assembled

$$\mathrm{K}^{(S)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} \\ \begin{bmatrix} 4EI/L & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{bmatrix} \end{array}$$

3. The global augmented matrix can be decomposed as

$$\left\{ \begin{array}{c} M_1(=0)\checkmark \\ \hline R_2\,? \\ R_3\,? \\ R_4\,? \end{array} \right\} = \left[\begin{array}{c|ccc} 4EI/L & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array} \right] \left\{ \begin{array}{c} \theta_1\,? \\ \hline \Delta_2\checkmark \\ \theta_3\checkmark \\ \Delta_4\checkmark \end{array} \right\}$$

4. K_{tt} is inverted (or actually decomposed) and stored in the same global matrix storage location

$$\left[\begin{array}{c|ccc} \boxed{L/4EI} & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array} \right]$$

5. Next we compute the equivalent load, $\mathrm{P}_t' = \mathrm{P}_t - \mathrm{K}_{tu}\Delta_u$, and overwrite P_t by P_t' (Note that we are boxing terms of interest only).

$$\mathrm{P}_t - \mathrm{K}_{tu}\Delta_u = \left\{\begin{array}{c}\boxed{M_1 = 0}\\ \hline R_2?\\ R_3?\\ R_4?\end{array}\right\} - \left[\begin{array}{c|ccc} L/4EI & \boxed{6EI/L^2} & \boxed{2EI/L} & \boxed{-6EI/L^2}\\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3\\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2\\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3\end{array}\right]\left\{\begin{array}{c}\theta_1\\ \hline \boxed{0}\\ \boxed{0}\\ \boxed{\Delta^0}\end{array}\right\}$$

$$= \left\{\begin{array}{c}\boxed{6EI\Delta^0/L^2}\\ \hline R_2?\\ R_3?\\ R_4?\end{array}\right\}$$

6. Solve for the displacements from $\Delta_t = \mathrm{K}_{tt}^{-1}(\mathrm{P}_t - \mathrm{K}_{tu}\Delta_u)$ and overwrite P_t by Δ_t

$$\left\{\begin{array}{c}\boxed{\theta_1}\\ \hline 0\\ 0\\ \Delta^0\end{array}\right\} = \left[\begin{array}{c|ccc} \boxed{L/4EI} & 6EI/L^2 & 2EI/L & -6EI/L^2\\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3\\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2\\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3\end{array}\right]\left\{\begin{array}{c}\boxed{6EI\Delta^0/L^2}\\ \hline R_2?\\ R_3?\\ R_4?\end{array}\right\}$$

$$= \left\{\begin{array}{c}3\Delta^0/2L\\ \hline 0\\ 0\\ 0\end{array}\right\}$$

7. Finally, we solve for the reactions, $\mathrm{R}_u = \mathrm{K}_{ut}\Delta_{tt} + \mathrm{K}_{uu}\Delta_u$, and overwrite Δ_u by R_u

$$
\left\{\begin{array}{c} M_1 \\ \hline \boxed{R_2} \\ \boxed{R_3} \\ \boxed{R_4} \end{array}\right\}
=
\left[\begin{array}{c|ccc}
L/4EI & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline
\boxed{6EI/L^2} & \boxed{12EI/L^3} & \boxed{6EI/L^2} & \boxed{-12EI/L^3} \\
\boxed{2EI/L} & \boxed{6EI/L^2} & \boxed{4EI/L} & \boxed{-6EI/L^2} \\
\boxed{-6EI/L^2} & \boxed{-12EI/L^3} & \boxed{-6EI/L^2} & \boxed{12EI/L^3}
\end{array}\right]
\left\{\begin{array}{c} \boxed{3\Delta^0/2L} \\ \hline \boxed{0} \\ \boxed{0} \\ \boxed{\Delta^0} \end{array}\right\}
$$

$$
= \left\{\begin{array}{c} -6EI\Delta^0/L^2 \\ \hline \boxed{-3EI\Delta^0/L^3} \\ \boxed{-3EI\Delta^0/L^2} \\ \boxed{3EI\Delta^0/L^3} \end{array}\right\}
$$

Structural Design

Virtual Work

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Spring 2022

Table of Contents I

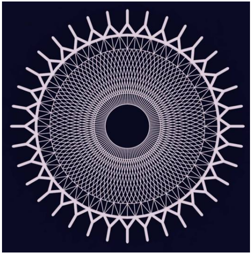
- Architects design structural envelope; structural engineer, analyzes and dimensions it (no change in form).
- "Large" structures are designed by structural engineers who must obey the Vitruvian virtues (or the Vitruvian Triad)

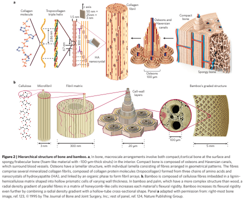
  Firmitas i.e. solid (Strength, Stiffness, Stability in modern parlance).
  Utilitas i.e useful (not an issue anymore in modern times).
  Venustas i.e beautiful (often forgotten).

- Motivation for this chapter:

  Vitruvius: *architecture is an imitation of nature*, indeed there is nowadays attempts to have bioinspired structural materials (e.g. bones and bamboos).
  Sullivan: *Form ever follows function*

- ⇒ shape optimization: standard deviation of the stress distribution should be nearly zero.





This chapter will address how a structural engineer can also be structural designer (Maillart, Gaudi, Nervi, Frei, Calatrava, and many others).

1 Le temple

2 Théâtre / Hôtel de ville / Bibliothèque

3 Hôpital / Sanatorium

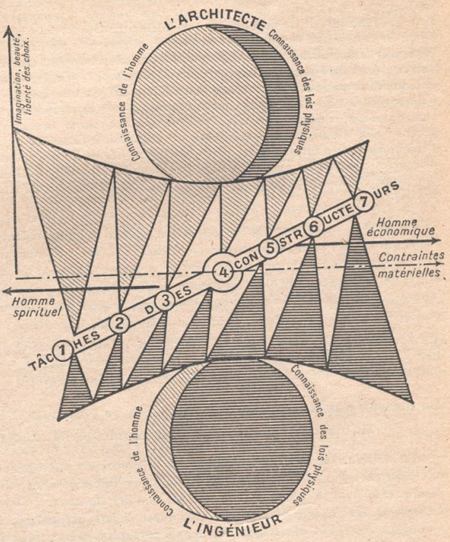
4 Le Logis

5 Le Bureau / Atelier

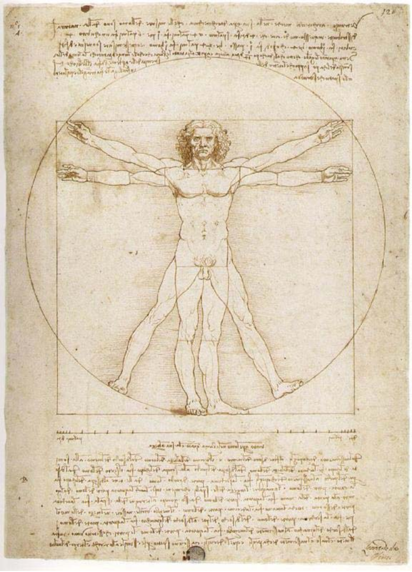
6 Grande industrie

7 Le pont / Le barrage / La route

(Tableau extrait de A. Boll. *Habitation moderne et Urbanisme* et de F. de Pierrefeu et Le Corbusier, *La Maison des hommes*).



Proportions and Dimensions (Vitruvian Man and Modulor)



Aristotle Mentioned *virtual velocities*: *heavy bodies located at the end of a lever are equilibrated when, in their possible motion, velocities are in the inverse ratio to the weights*.

Newton: *If I have seen further, it is by standing on the shoulders of giants*.

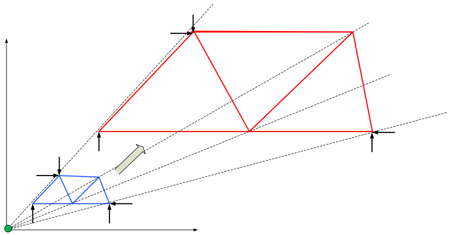
Maxwell greatest physicist between Newton and Einstein. His fame in physics overshadowed his pioneering work in the theory of structures: analysis of trusses (applying equations of equilibrium at each joint); b) foundation of virtual work; c) flexibility method.

Mohr rediscovered the work of Maxwell and formalized the modern principle of virtual work.

Einstein: (asked if he stood on Newton's shoulders): *No, on the shoulders of Maxwell*.

- Maxwell wrote:
    1. *In any system of points in equilibrium in a plane under the action of repulsions and attractions, the sum of the products of each attraction multiplied by the distance of the points between which it acts, is equal to the sum of products of the repulsions multiplied each by the distance of the points between which it acts.*
    2. *Multiply each load by the height of the point at which it acts, and each tension by the length of the piece on which it acts, and add all these products together.*
    3. *Then multiply the vertical pressures on the supports of the frame each by the height at which it acts, and each pressure by the length of the piece on which it acts, and add the products together. This sum will be equal to the former sum.*
- Simply put: *the sum of a structure's tension load path minus the sum of the compression paths is equal to a value related to the applied external forces*:

$$\Sigma F_T L_T - \Sigma F_C L_c = \Sigma \vec{P}_i \vec{\Delta}_i$$



Purely geometrical proof

- A truss with externally forces is in equilibrium.
- Dilate the space from an arbitrary point, tension members do positive work equal to the tensile force times the member change in length. Compression members will do negative work.
- Energy conservation the internal work equals external work (dot product on the right hand side).

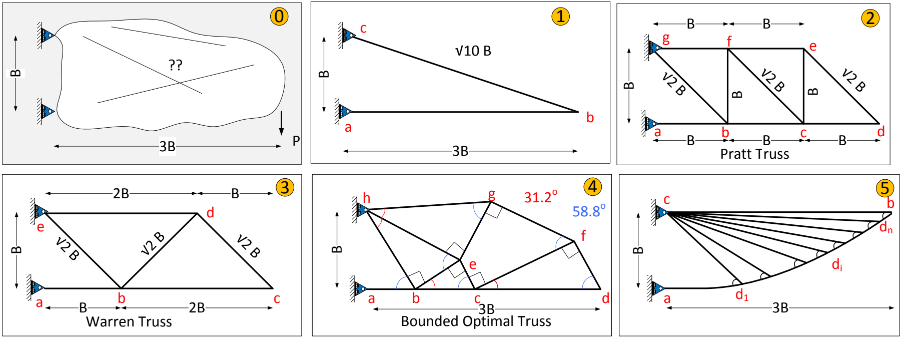
Implication for Design

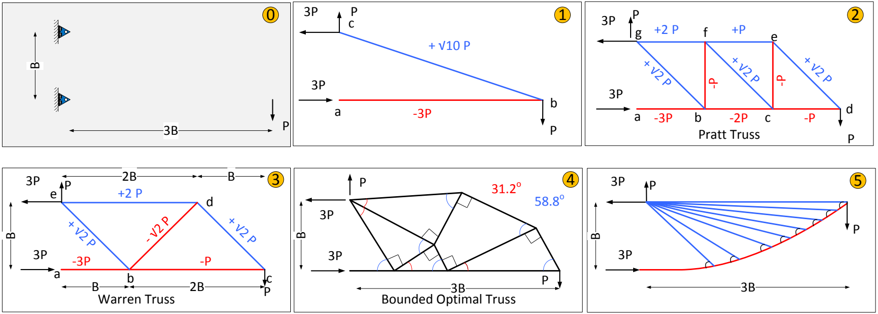
- Load paths must be equal.
- If a tension (or compression) load path is "too long", the truss will be penalized twice: once in tension and once in compression.
- Seek a configuration that minimizes tension load path (compression load path will automatically be minimized).

- Maxwell did not explicitly mention *Virtual Force*, however many credit him (Mohr) for the laying down the foundations for that theorem.
- A closer look to what he wrote can be rephrased as follows:
  1. Replace *attraction* by *compression* and *repulsion* by *tension*.
  2. Divide the sum $\sum P_i^{(e)} L_i$ by area and Young's modulus.
  3. replace height by displacement Δ.
- This is clearly the principle of virtual force,

$$\Sigma_{i=1}^{n}(\Delta_i)\delta\overline{P}_i = \Sigma_1^n \delta\overline{P}^{(i)} \frac{P^{(i)} L_i}{A_i E_i}$$

- Maxwell goes on saying: *The importance of this theorem to the engineer arises from the circumstance that the strength of a piece is in general proportional to its section, so that if the strength of each piece is proportional to the stress which it has to bear, its weight will be proportional to the product of the stress multiplied by the length of the piece. Hence these sums of products give an estimate of the total quantity of material which must be used in sustaining tension and pressure respectively*.
- Hence, PVW can be used to design a structure with minimum weight.
- Because *form should follow function*, the structure with minimum weight (yet meeting requirements) would be the most "elegant".
- Hence, design must consider structural topology optimization.


0
B
??
P
3B
1
c
√10 B
B
a
b
3B
2
B
B
g
f
e
√2 B
B
√2 B
B
√2 B
B
a
b
c
d
B
B
B
Pratt Truss
3
2B
B
e
d
√2 B
√2 B
√2 B
B
a
b
c
B
2B
Warren Truss
4
h
g
31.2°
58.8°
f
e
B
a
b
c
d
3B
Bounded Optimal Truss
5
c
b
d_n
B
d_i
a
d_1
3B



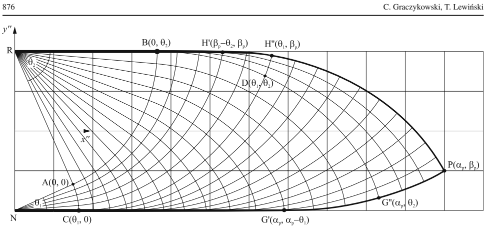
Shape Optimization

A	B	C	D	E	F	G	H	I	J
		Length L/B	Force F/P		FL/BP		A-B	A+B	Deflection
			+ve	-ve	A +ve	B -ve			Δ/(σB/E)
Two bar Truss									
1	a-b	3.000		3.000		9.0	-9.0	9.0	
2	b-c	3.162	3.162		10.0		10.0	10.0	
Sum		**6.162**			**10.0**	**9.0**	**1.0**	**19.0**	**19.0**
Warren Truss									
1	a-b	1.000		3.000		3.0	-3.0	3.0	
2	b-c	2.000		1.000		2.0	-2.0	2.0	
3	c-d	1.414	1.414		2.0		2.0	2.0	
4	d-e	2.000	2.000		4.0	0.0	4.0	4.0	
5	b-e	1.414	1.414		2.0	0.0	2.0	2.0	
6	b-d	1.414		1.414		2.0	-2.0	2.0	
Sum		**9.243**			**8.0**	**7.0**	**1.0**	**15.0**	**15.0**

A	B	C	D	E	F	G	H	I	J
		Length L/B	Force F/P		FL/BP		A-B	A+B	Deflection
			+ve	-ve	A +ve	B -ve			Δ/(σB/E)
Pratt Truss									
1	a-b	1.000		3.000		3.0	-3.0	3.0	
2	b-c	1.000		2.000		2.0	-2.0	2.0	
3	c-d	1.000		1.000		1.0	-1.0	1.0	
4	d-e	1.414	1.414		2.0		2.0	2.0	
5	e-f	1.000	1.000		1.0		1.0	1.0	
6	f-g	1.000	2.000		2.0		2.0	2.0	
7	b-g	1.414	1.414		2.0		2.0	2.0	
8	c-f	1.414	1.414		2.0		2.0	2.0	
9	b-f	1.000		1.000		1.0	-1.0	1.0	
10	c-e	1.000		1.000		1.0	-1.0	1.0	
Sum		**11.243**			**9.0**	**8.0**	**1.0**	**17.0**	**17.0**

- Self weight ignored.
- Column I is proportional to the total volume of material (assuming same allowable stress in tension and compression).
- Deflection (J) is proportional to the volume of material I)
- Exercise: Repeat analysis for: a) Howe and K trusses; b) 4:1 cantilevered truss; c) Any other truss.

	Load Paths				Deflection
	Tensile Load $A=\frac{\sum F_T L_T}{PB}$	Compressive Load $B=\frac{\sum F_C L_C}{PB}$	$A-B$	$A+B$	$\frac{\Delta}{\frac{\sigma B}{E}}$
①	10	9	1.	19	19
②	9↘	8↘	1. →	17↘	17↘
③	8↘	7↘	1. →	15↘	15↘
④	7.7↘	6.7↘	1. →	14.47↘	14.47↘
⑤	8.52	7.52	1.	16.04	16.04


876
C. Graczykowski, T. Lewiński
y''
R
B(0, θ₂)
H'(β_p−θ₂, β_p)
H''(θ₁, β_p)
θ₂
D(θ₁, θ₂)
x''
P(α_p, β_p)
A(0, 0)
θ₁
G''(α_p, θ₂)
N
C(θ₁, 0)
G'(α_p, α_p−θ₁)

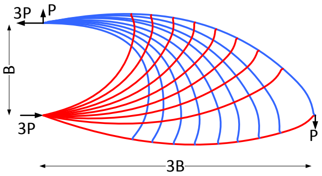
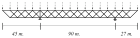
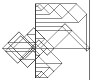
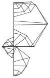
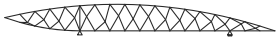
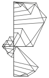
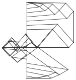
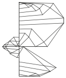
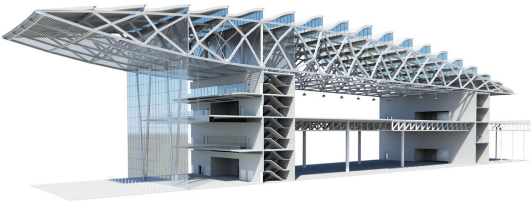
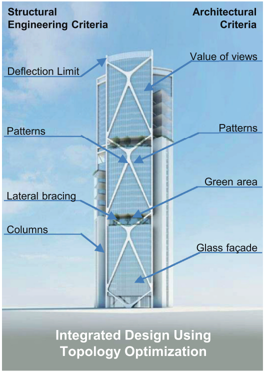

3P
P
B
3P
P
3B

Table 2 Optimization of the roof truss

Description	Form Diagram	Force Diagram	Normalized Volume
(a) Initial truss connectivity			1.000
(b) Benchmark truss; top chord, cantilevers and web members unconstrained			0.552
(c) Chord profiles constrained for architectural and functional reasons			0.629
(d) Depth constrained, straight web members (X-diagonals)			0.852
(e) Depth constrained			0.669



Structural optimization using graphic statics, Structural and Multidisciplinary Optimization, 2013,

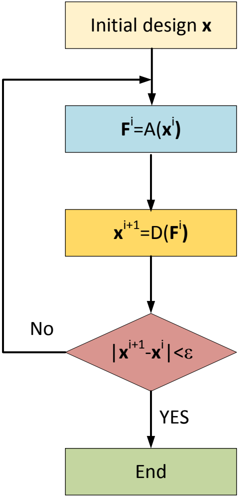

Structural Engineering Criteria
Architectural Criteria
Deflection Limit
Value of views
Patterns
Patterns
Green area
Lateral bracing
Columns
Glass façade
Integrated Design Using Topology Optimization

- The structural engineer usually assumes (based on experience, tables) initial dimensions for members, such as A and I for each truss members or beam element.
- An analysis is performed. Most structures are statically indeterminate. Hence results depend on A and I.
- Following the analysis, we have the truss axial forces, or beam moment diagrams.
- We must then check our design.

$$\begin{aligned} \text{Truss} \quad & \sigma_i = \frac{P_i}{A_i} \\ \text{Beam} \quad & \sigma = \frac{Mc}{I} \end{aligned}$$

  and compare with allowable stresses σ_{all}.
- if $\sigma > \sigma_{all}$, then we need to re-dimension the element, and re-analyze.

For small structures, an experienced structural engineer may not need more than 2-3 iterations. For large structures, the process is automated in many commercial codes.



Architect's initial dimensioning (r/c); or Engineer's based on experience. This is the first initial best estimate.
x denotes a vector (thus it is bold faced) of structural dimensions. Size of of x is equal to the number of design parameters (such number of truss elements to be dimensioned).

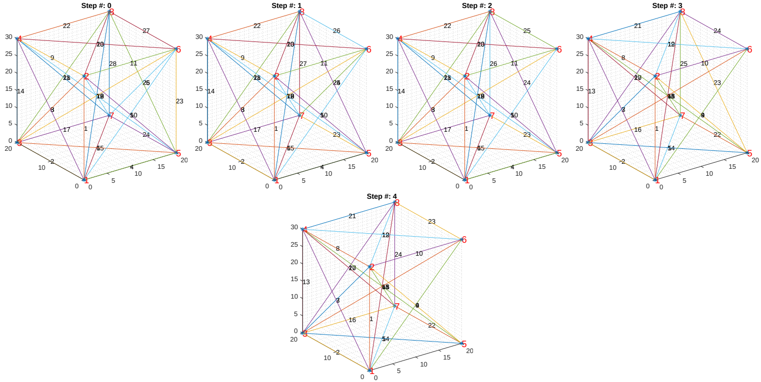
$A(\mathbf{x}^i)$ is an Operator with input $\mathbf{x}^i$ and output element internal forces $\mathbf{F}^i$ (such as axial force, shear force, moment). Thus $A(\mathbf{x}^i)$ is analysis. It could be your hand calculations, or a computer program.
Recall that for statically indeterminate structures internal forces depend on relative dimensions/stiffnesses ($M_a = K_a / \Sigma K_i$). If you change a dimension, you change K, and the corresponding moment.

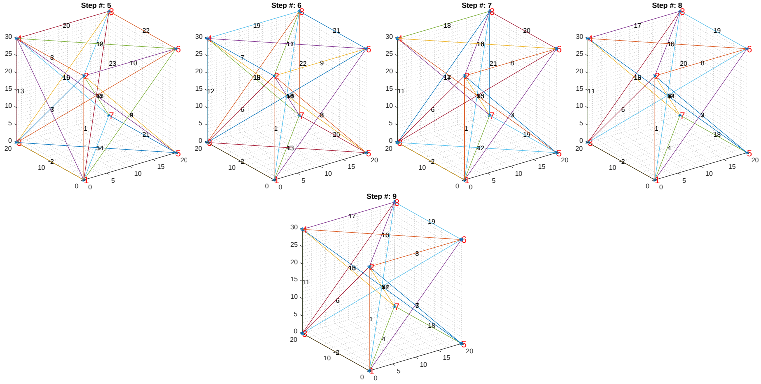
$D(\mathbf{F}^i)$ is another operator with input $\mathbf{F}^i$ and output dimensions $\mathbf{x}^{i+1}$. Thus $D(\mathbf{F}^i)$ is design. This can be hand based or computer based. Again, we need the internal force diagrams in order to design (such as A-F/σ_{all}).

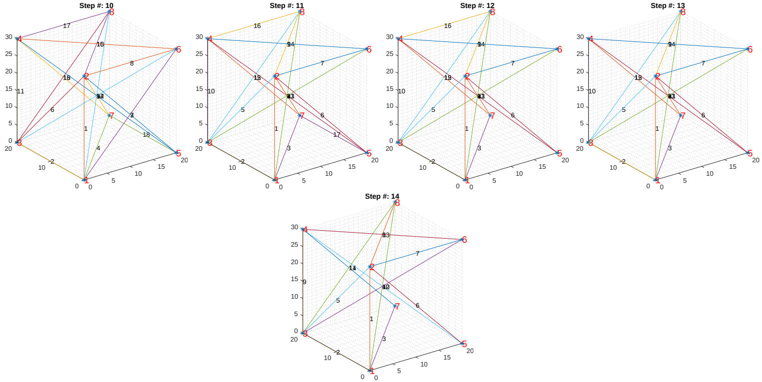
This is a check for convergence. If our last set of dimensions is close enough to the previous one (within a certain tolerance), then that is good enough.
Careful, in practice we often have few sections (steel) to use or formworks for concrete elements. This is to simplify the construction, minimize risk of error, and reduce the cost.

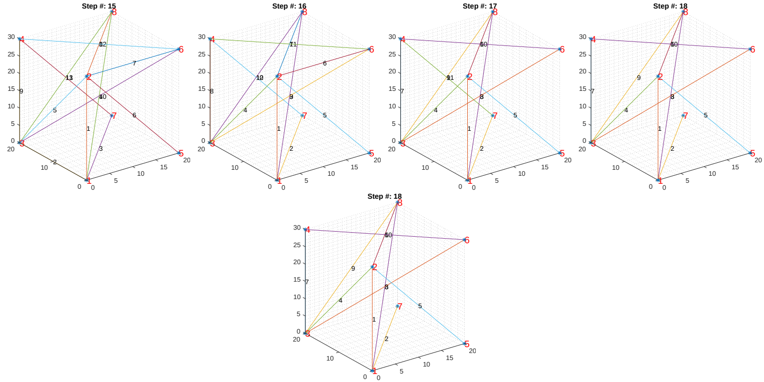
- Lattice models have their roots in Physics, and have been extensively used in fracture modeling of cementitious materials.
- Simple concept:
  1. Start with a densely packed mesh. Elements are typically Bernoulli's beam columns or truss elements.
  2. Perform a finite element analysis.
  3. Identify elements whose stress exceed a failure criterion.
  4. Remove those elements and reanalyze.
- Similar concept can be used for design.

  *By slowly removing inefficient material from a structure, the shape of the structure evolves towards an optimum. This is the simple concept of evolutionary structural optimization (ESO). [**?**]*


Step #: 0
Step #: 1
Step #: 2
Step #: 3
Step #: 4


Step #: 5
Step #: 6
Step #: 7
Step #: 8
Step #: 9


Step #: 10
Step #: 11
Step #: 12
Step #: 13
Step #: 14



Click to activate video

Listing: Main

```
clear; close all;clc
%===============================================================
% Generate Nodes
Random=1;MaxSteps=30;
[x,y,z,N] = Node_Generation(Random);
%% Generate members
[r,c,v] = find(hankel(2:N));           \%\# Create unique combinations of indices
index = [v c].';                       \%'\# Reshape the indices
nnodes=N;
nelem=size(index,2);
Inods=index';
step=0;OK=0;
%====================
% Video for fun
v=VideoWriter('Design_Opti.avi');
v.FrameRate = 1; %set to 25 frames per second
open(v)
%============================
PlotMesh(x,y,z,Inods,nelem,step)
frame=getframe(gcf);writeVideo(v,frame);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
while OK==0 \&\& step<MaxSteps
    % Pass the data to analyze with casap and retrieve element(s) to be removed
    % remove one element at a time, and regenerate the mesh
    ElemOut = randi([1 nelem],1,1); % generate a random number for testing
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Highlight element
    X=[x(Inods(ElemOut,1)) x(Inods(ElemOut,2))];Xc=mean(X);
    Y=[y(Inods(ElemOut,1)) y(Inods(ElemOut,2))];Yc=mean(Y);
    Z=[z(Inods(ElemOut,1)) z(Inods(ElemOut,2))];Zc=mean(Z);
    plot3(X,Y,Z,'LineWidth',3,'Color','r');
    frame=getframe(gcf);writeVideo(v,frame);
    hold on
```

```
     NodesOut=lnods(ElemOut,1:2);
     % compact lnods
     for i=ElemOut:nelem-1
         lnods(i,1:2)=lnods(i+1,1:2);
     end
     nelem=nelem-1;step=step+1;
     PlotMesh(x,y,z,lnods,nelem,step)
     % check how many other elements are connected to each of the two nodes
     % Minimum acceptable \# of elements per node min_elem
     min_elem=2;
     Del_Nodes=Check_Lonely(NodesOut,lnods,min_elem,v);
     frame=getframe(gcf);writeVideo(v,frame);
end
```

Listing: Node Generation

```
function [x,y,z,N] = Node_Generation(Random)
if Random == 0
    N = 8;                          %# Number of points
    x = rand(1,N);                  %# A set of random x values
    y = rand(1,N);                  %# A set of random y values
    z = rand(1,N);                  %# A set of random z values
else
    Rx=20;Ry=20;RZ=60;
    Deltax=20;Deltay=20;Deltaz=30;
    xmin=0.;ymin=0.;zmin=0.;
    nx=round(Rx/Deltax)+1;ny=round(Rx/Deltay)+1;nz=round(Rx/Deltaz)+1;
    N=0;
    for ix=1:nx
        for iy=1:ny
            for iz=1:nz
                N=N+1;
                x(N)=xmin+(ix-1)*Deltax;
```

```
                y(N)=ymin+(iy-1)*Deltay;
                z(N)=zmin+(iz-1)*Deltaz;
            end
        end
    end
end
end
```

Listing: Plot Mesh

```
function PlotMesh(x,y,z,lnods,nelem,step)
figure
plot3(x,y,z,'*'); % May omit
hold on
N=size(x,2);
for i=1:N
    text(x(i),y(i),z(i),num2str(i),'Color','red','FontSize',14);
end
hold on
%
for ie=1:nelem
    X=[x(lnods(ie,1)) x(lnods(ie,2))];Xc=mean(X);
    Y=[y(lnods(ie,1)) y(lnods(ie,2))];Yc=mean(Y);
    Z=[z(lnods(ie,1)) z(lnods(ie,2))];Zc=mean(Z);
    plot3(X,Y,Z);
    text(Xc,Yc,Zc,num2str(ie),'FontSize',10)
    hold on
end
title(['Step #: ' num2str(step)]);
pbaspect([1 1 1]);grid minor;view(-36,24);
%%%%%%%%%%%%%%%%%%%%%%%%
%=======================
%% Save plot
```

```
% ====================
GS = 'c:/Program Files/gs/gs9.10/bin/gswin64.exe';
set(gcf,'PaperPositionMode','auto');
FileName=['./Figs/Mesh-No-' num2str(step) '.eps'];
print(FileName,'-depsc');
eps2pdf(FileName,GS,0);
end
```

Listing: Delete Nodes?

```
function Del_Nodes= Check_Lonely(NodesOut,Inods,min_elem,v)
% for each of the two nodes connected to the deleted element, find out how
% many remaining elements are still connected to them
Del_Nodes=0;
for i=1:2
    x=sum(Inods(:,:)==NodesOut(i));
    n(i)=x(1)+x(2);
     if n(i)<min_elem
        strg=['Number of elements connected to node ' num2str(NodesOut(i)) ...
            ' dropped below minimum (' num2str(min_elem) ')'];
        errordlg(strg,'END EXECUTION');
%           frame=getframe(gcf);writeVideo(v,frame);
%           close(v);
        stop
        Del_Nodes=n(i);
    end
end
```

Structural Analysis

Influence Lines

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Spring 2022

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- So far, load was fix, and we made no distinction between fixed (dead) load, and variable load (live).
- Since a variable/Live load can move, a key question is how would a reaction or an internal force at a given point be affected by the positioning of the live load.
- Hence, we introduce the concept of Influence line

  > *An influence line is a diagram whose ordinates, which is plotted as a function of distance along the span, give the same internal force, a reaction, or a displacement at a particular point in a structure as unit load moves across the structure.*

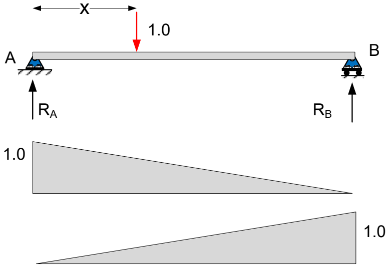
- This will facilitate placement of load to maximize an internal force (shear, or moment).
- Mathematically, an influence line can be described as IL_{ij} where IL is the quantity of interest (again, reaction, shear or moment) at degree of freedom i due to a unit load at degree of freedom j.

- For statically determinate structures, IL will consist of only straight line segments between critical ordinate values.
- IL for a shear force at a given location will contain a translational discontinuity at this location. The summation of the positive and negative shear forces at this location is equal to unity.
- Except at an internal hinge location, the slope to the shear force IL will be the same on each side of the critical section since the bending moment is continuous at the critical section.
- Likewise, IL for a bending moment will contain a unit rotational discontinuity at the point where the bending moment is being evaluated.
- Two methods:

  Equilibrium: Write an equation for the function being determined, e.g., the equation for the shear, moment, or axial force induced at a point due to the application of a unit load at any other location on the structure.

  Müller Breslau Principle to draw qualitative influence lines, which are directly proportional to the actual influence line.

A downward concentrated load of magnitude 1 unit moves from A to B across the simply supported beam AB as shown below. Draw influence lines for reactions at *A* and *B*, and for shear and moment at *C*.



Equilibrium

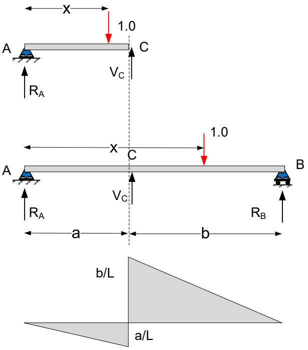
$$
\begin{aligned}
\Sigma M_B &= 0 \\
R_A L &= 1.(L - x) \\
\Rightarrow R_{Ax} &= 1 - \frac{x}{L}
\end{aligned}
$$

Linear equation in x for the reaction.
Likewise

$$
\begin{aligned}
\Sigma M_A &= 0 \\
R_B L &= 1.0x \\
\Rightarrow R_{Bx} &= \frac{x}{L}
\end{aligned}
$$

Influence Line for Shear at C

Segment AC

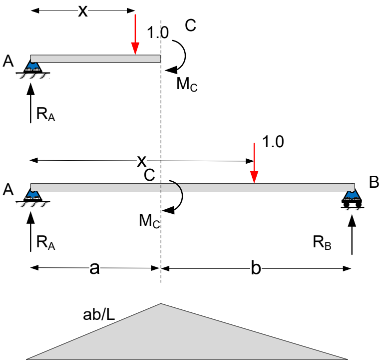


$$
\begin{aligned}
\circlearrowleft \sum M_B &= 0 \\
\Rightarrow R_A(L) - (L-x)(1) &= 0 \\
\Rightarrow R_A &= 1 - \frac{x}{L} \\
V \text{ at C due to unit load at x (left of C)} &= V_{Cx}^{-} \\
&= R_A - 1.0 \\
&= \left(1 - \frac{x}{L}\right) - 1.0 \\
&= -\frac{x}{L}
\end{aligned}
$$

When $x = 0$, $V_{CA} = 0$, and when $x = a$ (just before point C), $V_{CC} = -a/L$ Unit load is beyond the segment AC

$$
\begin{aligned}
V_{Cx}^{+} &= R_A \\
&= 1 - \frac{x}{L}
\end{aligned}
$$

When $x = a$ (just after point C), $V_{CC} = 1 - a/L = (L-a)/L = b/L$ and when $x = L$, $V_{CB} = 1 - L/L = 0$



segment AC

$$
\begin{aligned}
M_{Cx}^{-} &= aR_A - 1.0(a-x) \\
&= a\left(1-\frac{x}{L}\right) - (a-x) \\
&= a - \frac{ax}{L} - a + x = -\frac{ax}{L} + x
\end{aligned}
$$

When $x = 0$, $M_{Cx} = 0$, and when $x = a$ (just just before point C),

$$
\begin{aligned}
M_{CC} &= -a^2/L + a \\
&= (-a^2 + aL)/L \\
&= a(L-a)/L \\
&= ab/L
\end{aligned}
$$

Unit load is beyond the segment AC

$$M_{Cx}^{+} = aR_A = a(1-\frac{x}{L}) = a - \frac{ax}{L}$$

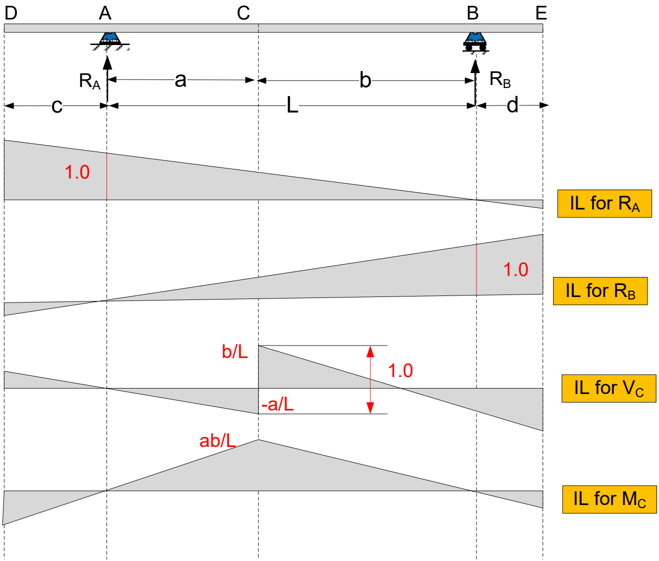
When $x = a$ (just after point C),

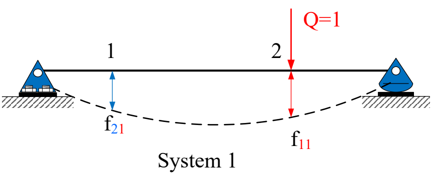
$$
\begin{aligned}
M_{CC} &= a - a^2/L \\
&= (aL - a^2)/L \\
&= a(L-a)/L \\
&= ab/L
\end{aligned}
$$

and when $x = L$,

$$
\begin{aligned}
M_{CB} &= a - aL/L \\
&= a - a \\
&= 0
\end{aligned}
$$

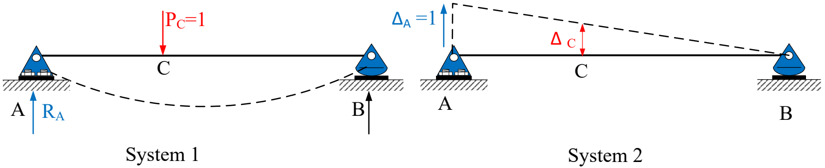
Example: SSB with overhangs





- Recall that f_{ij}, i.e.displacement at i due to a unit force at j: $1.\Delta_i = \int \delta\overline{M}_i \frac{M_i}{EI} dx$
- Displacement at dof i due to a unit force at j is: $f_{ij} = \int \delta\overline{M}_i \frac{M_j}{EI} dx$
- Displacement at dof j due to a unit force at i: $f_{ji} = \int \delta\overline{M}_j \frac{M_i}{EI} dx$
- Both virtual loads and real loads are unit: $f_{ij} = f_{ji}$
- Which is Maxwell-Betti's reciprocal theorem *for a linear elastic structure subject to two sets of forces* P *and* Q *the work done by the set* P *through the displacements produced by the set* Q *is equal to the work done by the set* Q *through the displacements produced by the set* P.

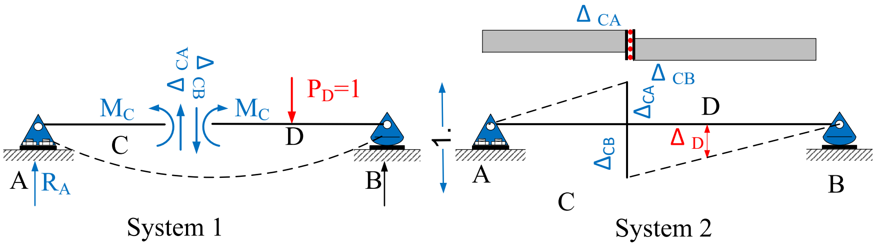
- Revisit the MBRT *the virtual work done by the forces in System 1 going through the corresponding displacements in System 2 should be equal to the virtual work done by the forces in System 2 going through the corresponding displacements in System 1.*
- "Trick" the problem, and consider the following two systems:



where the unit force P_C may be "traveling" between A and B (i.e analogous to the moving unit load which will generate the corresponding reaction at A), and apply the MBRT

$$\underbrace{P_C}_{1} \Delta_C = R_A \underbrace{\Delta_A}_{1} \Rightarrow \Delta_C = R_A$$

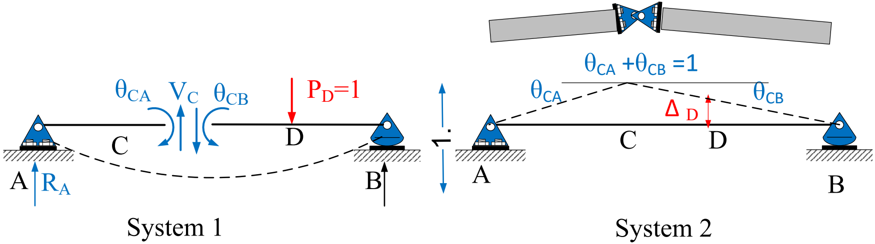
Hence, the displacement at C is equal to the reaction at A. This is the Müller-Breslau theorem: *The influence line for any reaction or internal force corresponds to the deflected shape of the structure produced by removing the capacity of the structure to carry that force and then introducing into the modified (or released) structure a unit deformation corresponding to the restrained removed.*



- Apply a shear release (but not moment) at C
- From Maxwell-Betti (as before)

$$\underbrace{(P_D)}_{1}\Delta_D = V_C\underbrace{(\Delta_{CA}+\Delta_{CB})}_{1} \Rightarrow \Delta_D = V_C$$

- Thus, the deflected shape in System 2 represents the influence line for shear force V_C.



- Apply a moment (but not shear) release at C
- From Maxwell-Betti (as before)

$$\underbrace{(P_D)}_{1}\Delta_D = M_C\underbrace{(\theta_{CA}+\theta_{CB})}_{1} \Rightarrow \Delta_D = M_C$$

- Thus, the deflected shape in System 2 represents the influence line for moment M_C.

Structural Analysis

Safety and Probability

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Spring 2019

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- C: Capacity; and D Demand.
- In the Allowable Stress Design (ASD) method, we simply impose

$$D < \frac{C}{SF}$$

where SF is a safety factor ($\sim 1.5 - 2$)

- In this approach only capacity is reduced (because of uncertainties), we are implicitly assuming that demand is purely deterministic.

- Both Capacity and Demand in the Load and Resistance Factor Design are considered to be random variables with their own probability distribution functions.
- There is a probability of failure.
- Load will be multiplied by a factor α, (ASCE-7-10) and we shall consider the ultimate resistance (reduced by Φ)
- We will assign α and Φ such that the probability of failure does not exceed a certain value.
- LRFD is generally expressed as

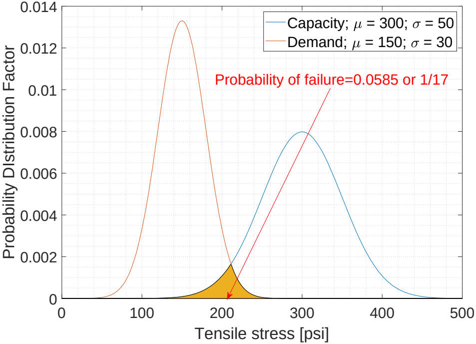
$$\Phi C_n \geq \Sigma \alpha_i D_i \tag{1}$$

where C_n and D are the nominal capacity and demands (or nominal resistance and load).

- Limit state is generally determined from Plastic capacity without a nonlinear analysis.

- LRFD seeks to have a Reliability Index such that $\beta > \sim 3.5$. The Reliability Index is a “universal” indicator on the adequacy of a structure, and can be used as a metric to 1) assess the health of a structure, and 2) compare different structures targeted for possible remediation.

- Capacity C and demand D are both random variables (usually assumed to be normal, though a log-normal may be prefereable in some instances).

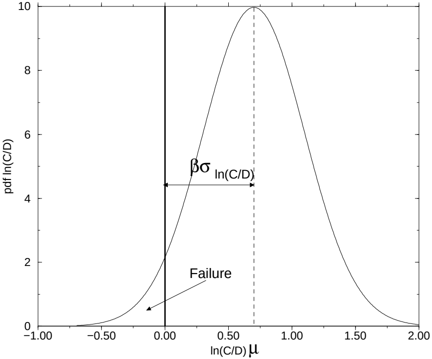


- Note area under each curve is 1.;
- In the shaded areea, $C < D$.

- Two approaches to determine β depending on how is the safety margin computed.

$$
\begin{aligned}
M &= C - D \\
\mu_M &= \mu_C - \mu_D \\
\sigma_M &= \sqrt{\sigma_C^2 + \sigma_D^2} \\
\beta &= \frac{\mu_M}{\sigma_M} \\
&= \frac{\mu_C - \mu_D}{\sqrt{\sigma_C^2 + \sigma_D^2}}
\end{aligned}
$$

$$
\begin{aligned}
M &= \ln C - \ln D \\
\mu_M &= \mu_C - \mu_D \quad \text{First order} \\
\sigma_M &= \sqrt{\frac{\sigma_C^2}{\mu_C^2} + \frac{\sigma_D^2}{\mu_D^2}} = \sqrt{V_C^2 + V_D^2} \\
\beta &= \frac{\mu_M}{\sigma_M} = \frac{\ln \mu_C - \ln \mu_D}{\sqrt{V_C^2 + V_D^2}} \\
&= \frac{\ln \mu_C/\mu_D}{\sqrt{V_C^2 + V_D^2}}
\end{aligned}
$$


10
8
6
4
2
0
pdf ln(C/D)
βσ ln(C/D)
Failure
−1.00
−0.50
0.00
0.50
1.00
1.50
2.00
ln(C/D) μ

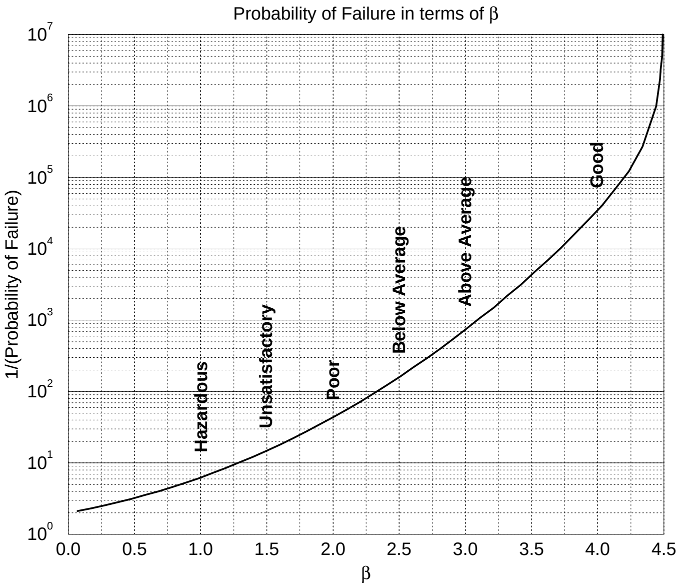
- β is selected to reflect failure consequences

Type of Load/Member	β
AISC	
DL + LL; Members	3.0
DL + LL; Connections	4.5
DL + LL + WL; Members	3.5
DL + LL +EL; Members	1.75
ACI	
Ductile Failure	3-3.5
Brittle Failures	3.5-4

The probability of failure P_f is equal to the ratio of the shaded area to the total area under the curve and is given by $\Phi(-\beta)$ where Φ is the standard normal cumulative probability function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = \frac{1}{2}\left[1 + erf\left(\frac{x}{\sqrt{2}}\right)\right] \quad (2)$$

Target values for β


Probability of Failure in terms of β
1/(Probability of Failure)
β
Hazardous
Unsatisfactory
Poor
Below Average
Above Average
Good
10^0
10^1
10^2
10^3
10^4
10^5
10^6
10^7
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5

Intermediary Structural Analysis

Introduction

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Fall 2021

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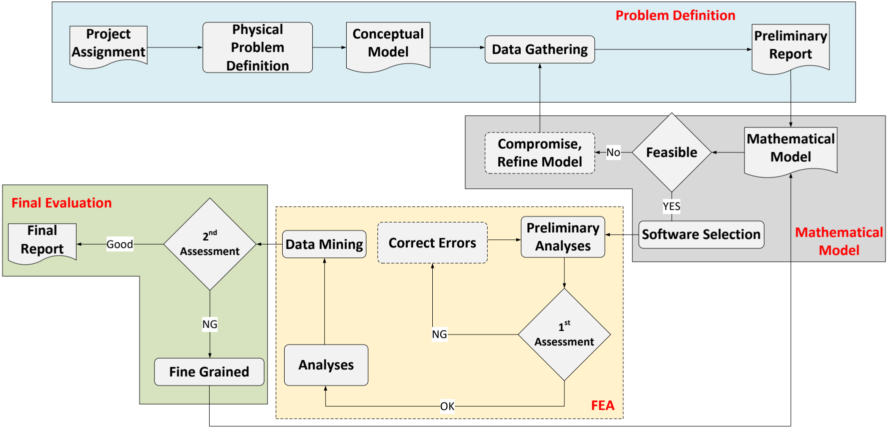
Different titles could be given to the course

- Matrix Structural Analysis.
- Analysis of Framed Structures.
- Finite Element I.
- Intermediary Structural Analysis.

Objectives

- Consolidate basic understanding of structural analysis/behaviour and introduce analysis techniques which will be used professionally.
- Examine interaction between analysis and design.
- Good understanding of the underpinning of the finite element method as applied to framed structures. Brief exposure to dynamic and nonlinear analysis.

- Early constructions, rules of thumbs, Vitruvius, Gothic cathedrals.
- Father of experimental mechanics Galileo.
- Mathematics $\rightarrow$ Mechanics (18th century, mostly French) $\rightarrow$ Structural analysis (19th century, mostly German and American) $\rightarrow$ finite element or computer based analysis (20th century, American).
- Slide rule + Moment distribution led to the design of many structures (skyscrapers in NY).
- Advent of stiffness method in late 60s (coming from aerospace) and rise of the finite element method.
- Not much has changed since then in terms of core method for linear analysis, mostly refinements for nonlinear analysis and parallel computation.
- Great improvements in (Graphical) user interfaces: punched cards $\rightarrow$ separate tools for drawing (AutoCad) and Analysis (Sap) $\rightarrow$ integrated tools for architectural modeling and structural analysis (Revit) with realistic rendering.
- Rather than focusing on how to use these codes, the course will focus on what is inside the core of theses codes.
- Emphasis will be on theory (80%), programming (10%), and modeling (10%)



- There are three phases in computational structural analysis:
  1. Modeling
  2. Number Crunching
  3. Interpretation

- In practice *Modeling* and *interpretation* are the most important, yet this course will focus more on the *number crunching* part (with some lectures on the other two).

- Early on, it was easy to develop a feel for a structural behavior using hand calculation (such as the moment distribution).

- It has been argued that this is no longer possible with computers. This is not correct.

Structural analysis must take into consideration

1. Load (static or dynamic). When the frequency of the applied load (excitation) of a structure is less than about a third of its lowest natural frequency of vibration, then we can neglect inertia effects and treat the problem as a quasi-static one, otherwise a dynamic analysis must be performed.
2. Structure model
    1. Global geometry
        - small deformation ($\epsilon = \frac{\partial u}{\partial x}$)
        - large deformation:

          Material level: $\varepsilon_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2$

          Structural level P-Δ effects
    2. Structural elements element types:
        - 1D framework (truss, beam, columns)
        - 2D finite element (plane stress, plane strain, axisymmetric, plate or shell elements)
        - 3D finite element (solid elements)
    3. Material Properties: linear (steel), nonlinear (concrete).

4. Sectional properties: constant *v.s.* variable
5. Structural connections: rigid, semi-flexible (linear or nonlinear)
6. Structural supports: rigid, semi-rigid/spring.

Structural design must satisfy:

1. Strength ($\sigma < \sigma_f$)
2. Stiffness ("small" deformations)
3. Stability (buckling, cracking)

Structural analysis must satisfy

1. Statics (equilibrium)
2. Constitutive relation (stress-strain or force displacement relations)
3. Kinematics (compatibility of displacement or strains)

Internal Forces (for flexure)

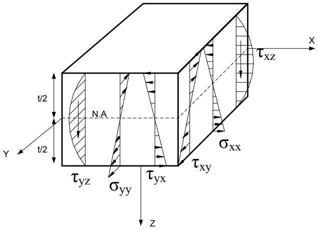
In Structural Mechanics (or Mechanics of Materials), emphasis has been on the stress and strain tensors, it is often more convenient to operate on the resultant forces in structural engineering.

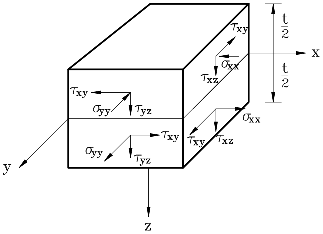
Engineering Theories

Instead of solving for the stress components throughout the body, we solve for certain stress resultants (normal, shear forces, and Moments and torsions) resulting from an integration over the body.

Internal Forces

Resultants *per unit width*





$$\mathbf{N} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \boldsymbol{\sigma} dz \tag{1}$$

$$N_{xx} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx} dz; \qquad N_{yy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{yy} dz; \qquad N_{xy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xy} dz; \tag{2}$$

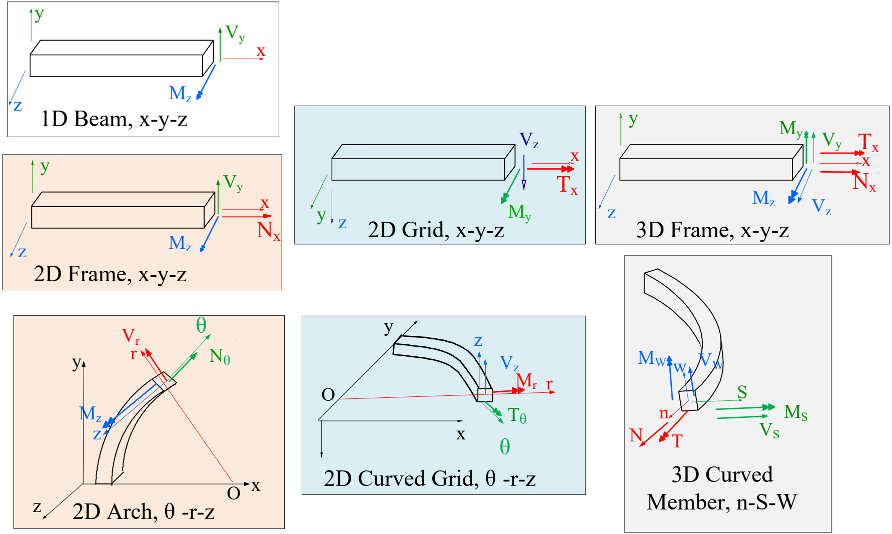
$$\mathbf{M} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \boldsymbol{\sigma} z dz; \tag{3}$$

$$M_{xx} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx} z dz; \qquad M_{yy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{yy} z dz; \qquad M_{xy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xy} z dz \tag{4}$$

$$\mathbf{V} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \boldsymbol{\tau} dz \tag{5}$$

$$V_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz} dz; \qquad V_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz} dz \tag{6}$$

In plate theory, we ignore membrane forces, those will be accounted for in shells.



1D Beam, x-y-z
y
V_y
x
M_z
z
2D Frame, x-y-z
y
V_y
x
N_x
M_z
z
2D Grid, x-y-z
V_z
x
T_x
M_y
y
z
3D Frame, x-y-z
y
M_y
V_y
T_x
x
N_x
M_z
V_z
z
2D Arch, θ -r-z
V_r
r
θ
N_θ
y
M_z
z
O
x
2D Curved Grid, θ -r-z
y
z
V_z
M_r
r
O
T_θ
x
θ
3D Curved Member, n-S-W
M_W
W
V_W
S
M_S
V_S
n
N
T

Cartesian						
	Forces			Moments		
	x	y	z	x	y	z
Beam		V_y				M_z
2D Frame	N_x	V_y				M_z
Grid			V_z	T_x	M_y	
3D Frame	N_x	V_y	V_z	T_x	M_y	M_z
Polar						
	Forces			Moments		
	r	θ	z	r	θ	z
Arch	V_r	N_θ				M_z
Curved Beam			V_z	M_r	T_θ	
Curved						
	Forces			Moments		
	n	s	w	n	s	w
Curved	N_n	V_s	V_w	T_n	M_s	M_w

Intermediary Structural Analysis

Stiffness and Transformation Matrices

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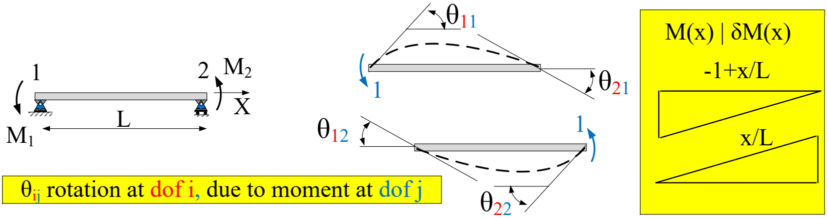
- In structural analysis an influence coefficient C_{ij} is the effect on d.o.f. i of a unit action at d.o.f. j for an individual element or a whole structure.
- It is indeed a tensor of order 2.

	Unit Action	Effect on
Influence Line	Load	Shear
Influence Line	Load	Moment
Influence Line	Load	Deflection
Stress σ_{ij}	traction t_j	*face i*
Flexibility Coefficient d_{ij}	Force j	Displacement i
Stiffness Coefficient k_{ij}	Displacement j	Force i

- We seek to determine the flexibility matrix for the following statically determinate beam.
- The flexibility matrix here would be a 2×2, and each term d_{ij} corresponds to the displacement at degree of freedom i due to a unit force at degree of freedom j.
- We have here two DOF corresponding to the rotations at each end.



- The force displacement relationship is now expressed as

$$\left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\} = \left[\begin{array}{cc} d_{11} & d_{12} \\ d_{21} & d_{22} \end{array} \right] \left\{ \begin{array}{c} M_1 \\ M_2 \end{array} \right\} \tag{1}$$

where M_i correspond to the externally applied moments, $d_{ij} = \theta_{ij}$, and θ_i to the corresponding unknown rotations at dof i due to a moment at dof j.

- Using the complementary virtual work, or more specifically, the virtual force method to analyze this problem, :

$$d_{ij} = \underbrace{\int_0^l \delta\overline{M}(x)_i \frac{M(x)_j}{EI_z} dx}_{\text{Internal}} = \underbrace{\delta\overline{P}_i \, \Delta_j + \delta\overline{M}_i \, \theta_j}_{\text{External}} \tag{2}$$

where $\delta\overline{M}(x)$, $\frac{M(x)}{EI_z}$, $\delta\overline{P}$ and Δ are the virtual internal force, real internal displacement, virtual external load, and real external displacement respectively.

- Here, both the external virtual force and moment are usually taken as unity.

- Recall of the derivation of the virtual force:

$$\left.\left.\begin{array}{rcl} \delta U & = & \int \delta\overline{\sigma}_x\, \varepsilon_x\, d\text{vol} \\ \delta\overline{\sigma}_x & = & \delta \frac{\overline{M}_x y}{I} \\ \varepsilon_x & = & \frac{\sigma_x}{E} = \frac{My}{EI} \\ \int y^2 dA & = & I \\ d\text{vol} & = & dAdx \end{array}\right\} \delta U = \int_0^L \delta\overline{M}\frac{M}{EI}dx \\ \begin{array}{l} \delta W = \delta\overline{P}\Delta \\ \delta U = \delta W \end{array} \right\} \int_0^L \delta\overline{M}\frac{M}{EI}dx = \delta\overline{P}\Delta \tag{3}$$

- Hence:

$$EI\underbrace{1}_{\delta\overline{M}}\underbrace{d_{11}}_{\Delta} = \int_0^L \underbrace{\left(-1+\frac{x}{L}\right)}_{\delta\overline{M}(x)}\underbrace{\left(-1+\frac{x}{L}\right)}_{M(x)}dx = \frac{L}{3} \tag{4}$$

- Similarly, we would obtain:

$$EId_{22} = \int_0^L \left(\frac{x}{L}\right)^2 dx = \frac{L}{3} \tag{5}$$

$$EId_{12} = \int_0^L \left(-1+\frac{x}{L}\right)\frac{x}{L}dx = -\frac{L}{6} = EId_{21} \tag{6}$$

- Those results can be summarized in a matrix form as:

$$[\mathrm{d}] = \frac{L}{6EI_z}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{7}$$

and we could then solve for the displacements (rotations) due to the external moments.

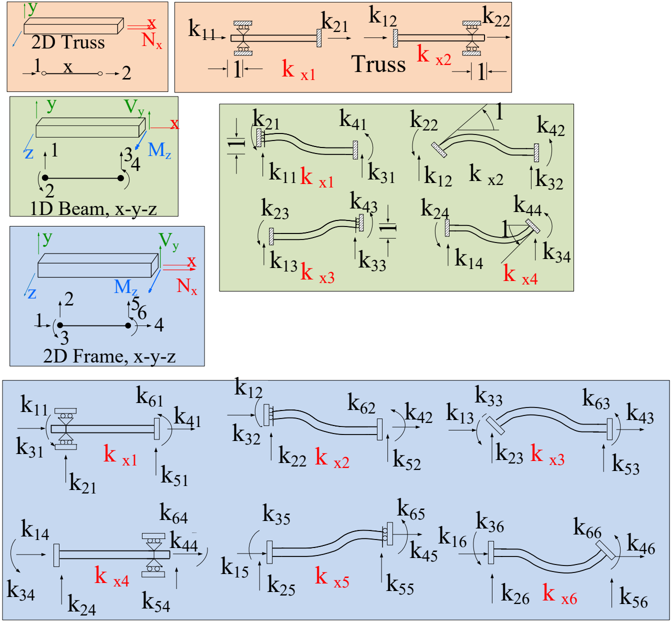
- In the flexibility method, we made a structure statically determinate.
- In the stiffness method we make the structure (element or entire structure) kinematically determinate by
  1. Constraining all the degrees of freedom

$$\{p\} = [k]\{\delta\} \tag{8}$$

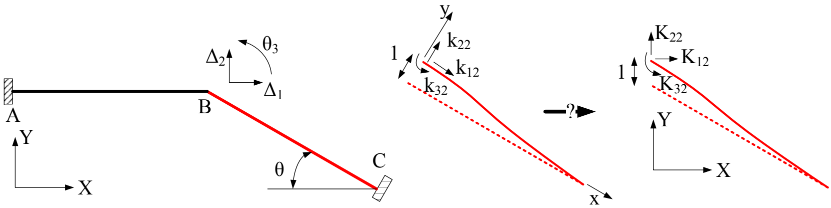
  $k_i j$will correspond to the reaction at dof i due to a unit deformation (translation or rotation) at dof j.

- Flexibility: displacements in terms of the externally applied forces ($\Delta(F)$). Derived for a structure.
- Stiffness: (internal) forces in terms of the externally imposed displacements ($F(\Delta)$). Derived first for elements, and then those are combined for a structure.

- We seek to determine forces (reactions) due an externally applied unit displacement.
- All forces and displacements are shown in the positive direction.
- Once we determine all the k_{ij} coefficients, we could then easily assemble the element stiffness matrix.


2D Truss
Truss
1D Beam, x-y-z
2D Frame, x-y-z

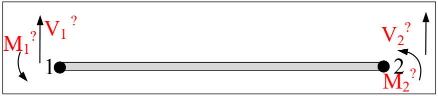
1. Currently, we are seeking to determine the element stiffness matrix of an individual element in local coordinate syatem $\mathtt{k}^e$ (x axis aligned with the member).
2. This element stiffness matrix of an element will be transformed to $\mathtt{K}^e$ through the transformation matrix $\mathtt{K}^e = \Gamma^T \mathtt{k}^e \Gamma$ from the local to the global coordinate syatem.
3. Finally, we will assemble the global stiffness matrix of a structure in the global coordinate system $\mathtt{K}^S = \sum_{e=1}^{e=nelem} \mathtt{K}^e$

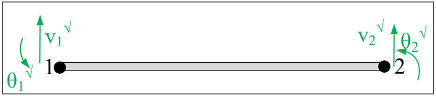


- Note local coordinate system $(x - y)$ and global coordinate system $(X - Y)$.

$$\sigma = E\epsilon \Rightarrow \underbrace{A\sigma}_{P} = \underbrace{\frac{AE}{L}}_{k^{axial}} \underbrace{\Delta}_{1} \tag{9}$$

Hence, for a unit displacement, the applied force should be equal to $\frac{AE}{L}$. From statics, the force at the other end must be equal and opposite.

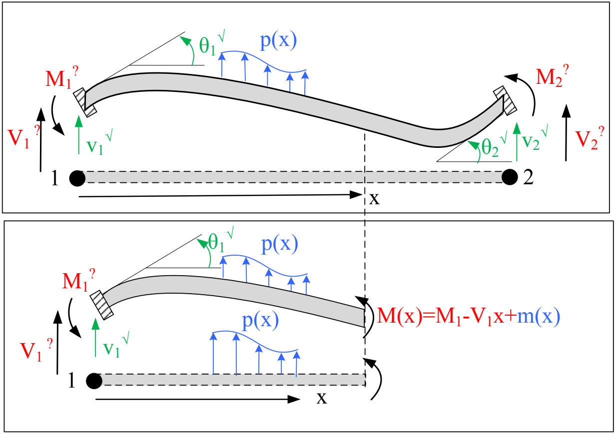




- Objective: solve for forces in terms of known displacements in a beam: Four unknowns forces ($V_1^?$, $V_2^?$, $M_1^?$ and $M_2^?$) in terms of four known displacements ($v_1^{\surd}$, $v_2^{\surd}$, $\theta_1^{\surd}$ and $\theta_2^{\surd}$)

$$\begin{array}{rclrcl} V_1^? & = & V_1^?(v_1^{\surd},\theta_1^{\surd},v_2^{\surd},\theta_2^{\surd}) & M_1^? & = & M_1^?(v_1^{\surd},\theta_1^{\surd},v_2^{\surd},\theta_2^{\surd}) \\ V_2^? & = & V_2^?(v_1^{\surd},\theta_1^{\surd},v_2^{\surd},\theta_2^{\surd}) & M_2^? & = & M_2^?(v_1^{\surd},\theta_1^{\surd},v_2^{\surd},\theta_2^{\surd}) \end{array} \tag{10}$$

- Four unknowns, need four equations. Two provided by the second order linear differential equation governing flexure, and two from the two equations of equilibrium.



- A. Differential equation

$$M = \underbrace{-EI\frac{d^2v}{dx^2}}_{\text{Diff Eq.}} = \underbrace{M_1^? - V_1^? x + m(x)}_{\text{Statics}} \tag{11}$$

- $m(x)$ moment due to applied load $q(x)$ at section x (for uniformly distributed load: $m(x) = -\frac{1}{2}wx^2$)
- Integrating twice

$$
\begin{aligned}
-EIv' &= M_1^? x - \frac{1}{2} V_1^? x^2 + f(x) + C_1 && (12)\\
-EIv &= \frac{1}{2} M_1^? x^2 - \frac{1}{6} V_1^? x^3 + g(x) + C_1 x + C_2 && (13)
\end{aligned}
$$

where $f(x) = \int m(x)dx$, and $g(x) = \int f(x)dx$.

- Boundary conditions at $x = 0$

$$
\left.\begin{aligned} v' &= \theta_1^{\surd} \\ v &= v_1^{\surd} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} C_1 &= -EI\theta_1^{\surd} \\ C_2 &= -EIv_1^{\surd} \end{aligned}\right. \qquad (14)
$$

- Boundary conditions at $x = L$ and combining with C_1 and C_2

$$
\left.\begin{aligned} v' &= \theta_2^{\surd} \\ v &= v_2^{\surd} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} -EI\theta_2^{\surd} &= M_1^? L - \tfrac{1}{2} V_1^? L^2 + f(L) - EI\theta_1^{\surd} \\ -EIv_2^{\surd} &= \tfrac{1}{2} M_1^? L^2 - \tfrac{1}{6} V_1^? L^3 + g(L) - EI\theta_1^{\surd} L - EIv_1^{\surd} \end{aligned}\right. \qquad (15)
$$

- Though we could solve for $M_1^?$ and $V_1^?$ in terms of $v_1^{\surd}$, $v_2^{\surd}$, $\theta_1^{\surd}$ and $\theta_2^{\surd}$, we proceed with
- B. Equilibrium

$$V_1^? + P + V_2^? = 0 \qquad M_1^? - V_1^? L + m(L) + M_2^? = 0 \tag{16}$$

where $P = \int_0^L p(x)dx$,

- thus

$$V_1^? = \frac{(M_1^? + M_2^?)}{L} + \frac{1}{L} m(L) \qquad V_2^? = -(V_1^? + P) \tag{17}$$

- Substituting V_1 into θ_2 and v_2 (Eq. 15)

$$\left\{ \begin{array}{rl} M_1^? - M_2^? & = \frac{2EI_z}{L}\theta_1^{\surd} + \frac{2EI_z}{L}\theta_2^{\surd} + m(L) - \frac{2}{L}f(L) \\ 2M_1^? - M_2^? & = \frac{6EI_z}{L}\theta_1^{\surd} - \frac{6EI_z}{L^2}v_1^{\surd} - \frac{6EI_z}{L^2}v_2^{\surd} + m(L) - \frac{6}{L^2}g(L) \end{array} \right. \tag{18}$$

- Solve for the moments

$$M_1 = \underbrace{\frac{2EI_z}{L}\left(2\theta_1^{\surd} + \theta_2^{\surd}\right) - \frac{6EI_z}{L^2}\left(v_2^{\surd} - v_1^{\surd}\right)}_{I} + \underbrace{M_1^F}_{II} \tag{19}$$

$$M_2 = \underbrace{\frac{2EI_z}{L}\left(\theta_1^{\surd} + 2\theta_2^{\surd}\right) - \frac{6EI_z}{L^2}\left(v_2^{\surd} - v_1^{\surd}\right)}_{I} + \underbrace{M_2^F}_{II} \tag{20}$$

where

$$M_1^F = \frac{2}{L^2}\left[Lf(L) - 3g(L)\right] \tag{21}$$

$$M_2^F = -\frac{1}{L^2}\left[L^2 m(L) - 4Lf(L) + 6g(L)\right] \tag{22}$$

- M_1^F and M_2^F are the fixed end moments for $\theta_1 = \theta_2 = 0$ and $v_1 = v_2 = 0$.
- In Eq. 19 and 20 we observe that the moments developed at the end of a member are caused by: I) end rotation and displacements; and II) fixed end members.

- We can substitute those expressions in Eq. 17 and solve for the shear forces:

$$V_1 = \underbrace{\frac{6EI_z}{L^2}\left(\theta_1^{\surd} + \theta_2^{\surd}\right) - \frac{12EI_z}{L^3}\left(v_2^{\surd} - v_1^{\surd}\right)}_{I} + \underbrace{V_1^F}_{II} \quad (23)$$

$$V_2 = \underbrace{-\frac{6EI_z}{L^2}\left(\theta_1^{\surd} + \theta_2^{\surd}\right) + \frac{12EI_z}{L^3}\left(v_2^{\surd} - v_1^{\surd}\right)}_{I} + \underbrace{V_2^F}_{II} \quad (24)$$

where

$$V_1^F = \frac{6}{L^3}\left[Lf(L) - 2g(L)\right] \quad (25)$$

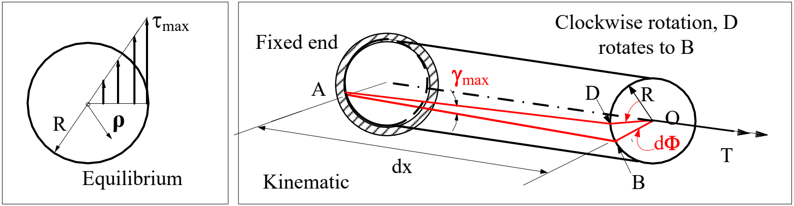
$$V_2^F = -\left[\frac{6}{L^3}\left[Lf(L) - 2g(L)\right] + q\right] \quad (26)$$

- The end shear and moments are in terms of $v_2 - v_1$ which is the “drift” sometimes denoted by Ψ.
- It is very important to note that the derived equations are based on:
  1. Equilibrium

2. Stress-strain
3. Compatibility

Torsion causes twisting and warping. Two types of Torsion:

- S^t Venant/Constant torsion If the member is allowed to warp freely, then the applied torque is resisted solely by S^t Venant shearing stresses: pure or uniform torsion.
- Non-Uniform if the member is restrained from warping freely, the applied torque is resisted by a combination of S^t Venant shearing stresses and warping torsion. All cross-sections will warp out of plane except circular ones.



- Determine torque T required to impose a unit rotation Φ

- Assuming a linear elastic material, and a linear strain (and thus stress) distribution along the radius of a circular cross section subjected to torsional load:
- From Equilibrium (Internal torsion must be equal and opposite to external torsion)

$$T_{ext} = \underbrace{\underbrace{\int_A \underbrace{\frac{\rho}{R}\tau_{max}}_{\text{stress}} \underbrace{dA}_{\text{area}}}_{\text{Shear Force}} \underbrace{\rho}_{\text{arm}}}_{T_{int}} = \frac{\tau_{max}}{R} \underbrace{\int_A \rho^2 dA}_{J} \Rightarrow \tau_{max} = \frac{TR}{J} \tag{27}$$

Note analogy with $\sigma = \frac{Mc}{I_z}$.

- $\int_A \rho^2 dA$ is the polar moment of inertia J (S^t Venant's torsion constant). For circular cross sections

$$J = \int_A \rho^2 dA = \int_0^R \rho^2 \underbrace{2\pi\rho}_{C} d\rho = \frac{\pi R^4}{2} = \frac{\pi D^4}{32} \tag{28}$$

  where C is the circumference at radius ρ.

- For rectangular sections $b \times d$, and $b < d$, an approximate expression is given by

$$J \simeq kb^3 d \tag{29}$$

$$k \simeq \frac{0.3}{1 + \left(\frac{b}{d}\right)^2} \tag{30}$$

- Kinematics: We have a relation between torsion and shear stress, we now seek a relation between torsion and torsional rotation. we consider the arc length BD

$$\left.\begin{array}{rcl} \gamma_{max} dx = d\Phi R \Rightarrow \frac{d\Phi}{dx} & = & \frac{\gamma_{max}}{R} \\ \text{Stress-strain } \gamma_{max} & = & \frac{\tau_{max}}{G} \end{array}\right\} \left.\begin{array}{rcl} \frac{d\Phi}{dx} & = & \frac{\tau_{max}}{GR} \\ \tau_{max} & = & \frac{TR}{J} \end{array}\right\} \frac{d\Phi}{dx} = \frac{T}{GJ} \tag{31}$$

CORRECT BOOK, REPLACE C BY R

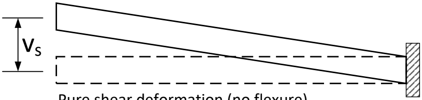
- Integrating $\int T dx = \int GJ d\Phi$ and obtain:

$$T = \frac{GJ}{L}\Phi \tag{32}$$

Note the similarity between this equation and Equation 9 ($P = \frac{AE}{L}\Delta$)

- In general, shear deformations are quite small. However, for beams with low span to depth ratio, those deformations can not be neglected.
  - Bernouilli Beam, we do not account for shear deformation, plane section remains plane.
  - Timoshenko beam accounts for shear deformation.

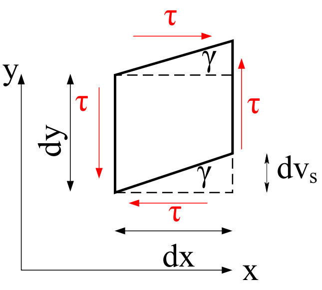
Objective: Determine shear deformation (no flexure) and its impact on stiffness coefficients.



Pure shear deformation (no flexure)

Coverage

1. Review
2. Shear coefficient
3. Example: Deflection cantilevered beam
4. Shear Factor
5. Shear deformation
   - Translation
   - Rotation



- linear elastic material, shear strain (small displacement, i.e. $\tan\gamma \approx \gamma$)

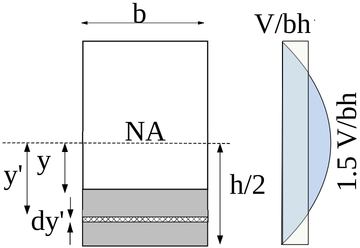
$$\tan\gamma \approx \gamma = \underbrace{\frac{dv_s}{dx}}_{\text{Kinematics}} = \underbrace{\frac{\tau}{G}}_{\text{Stress-strain}} \tag{33}$$

- $\frac{dv_s}{dx}$ slope of the beam neutral axis wrt horizontal,vertical sections remain undeformed, G shear modulus, τ shear stress, v_s shear induced displacement.

- In general (Equilibrium)

$$\tau(y) = \frac{VQ(y)}{Ib} \tag{34}$$

V shear force, Q first moment (or static moment) about neutral axis of the portion of the cross-sectional area outside of the section where the shear stress is to be determined, I moment of inertia, b width.



- Define shear coefficient α_s as the ratio of the shear stress at the neutral axis $\tau(y=0)$ to the average shear stress ($\tau = V/bh$).

$$\tau(y) = \frac{VQ(y)}{Ib} \tag{35}$$

$$= \frac{1}{Ib}\underbrace{\underbrace{\int_y^{h/2} \underbrace{V\, bdy'}_{dA}}_{dF} y'}_{dM} = \frac{V}{2I}\left(\frac{h^2}{4}-y^2\right)$$

$$= \frac{6V}{bh^3}\left(\frac{h^2}{4}-y^2\right) \tag{36}$$

- Shear stress is zero for $y = h/2$ and maximum at the neutral axis where ($y = 0$ and $\tau_{max} = 1.5\frac{V}{bh}$). $\Rightarrow \alpha_s = 1.5$

- Consider a cantilevered (rectangular *bxh*) beam subjected to a point load P at its free end.
- From the principle of complementary virtual work (and noting that $\delta M = (1)x$, $M = Px$, $\delta\tau = 1$, and τ is given by Eq. 36):

$$
\begin{aligned}
(1)\Delta &= \underbrace{\int_0^L \delta\overline{M}\frac{M}{EI}dx}_{\text{Flexure}} + \underbrace{\int_{Vol} \delta\overline{\tau}\frac{\tau}{G}dVol}_{\text{Shear}} \\
&= \int_0^L x\frac{Px}{EI}dx + \frac{1}{G}\int_{-h/2}^{h/2} \underbrace{(1)}_{\delta\tau} \underbrace{\left\{\frac{P}{2I}\left[\left(\frac{h}{2}\right)^2 - y^2\right]\right\}}_{\tau} \underbrace{Lbdy}_{dVol} \\
&= \underbrace{\frac{PL^3}{3EI}}_{\Delta_{flex}} + \underbrace{\frac{6}{5}\frac{PL}{AG}}_{v_s} \qquad (37) \\
\Delta &= \frac{PL^3}{3EI}\left(1 + \frac{3E}{10G}\left(\frac{h}{L}\right)^2\right) \qquad (38)
\end{aligned}
$$

- for $E/G = 2.5$ (typical value for steel), $\Delta = EI\left(1 + 0.75\left(\frac{h}{L}\right)^2\right)\Delta_{flex}$
- For $L = h$ total deflection is 1.75 times the one due to flexure only.
- For $L = 10h$ the deflection due to shear is less than 1% of Δ_{felx}.

- Just as we had $\sigma = P/A$, can we assume $\tau = V/A$? NO
- Normalizing the shear force V by A ($\tau = V/A$, just like $\sigma = P/A$) is incorrect since the shear stress is not uniformly distributed along the depth, hence we define $\tau \stackrel{\text{def}}{=} V/A_s$, where $A_s \stackrel{\text{def}}{=} \frac{A}{\lambda_s}$ is the effective cross section for shear. We seek to determine λ_s, the shear factor
- To determine λ_s we are going to equate the average shear strain energy U

$$U_{aver} = \int_0^L \frac{V^2}{2GA_s} dx = \lambda_s^? \int_0^L \frac{V^2}{2GA} dx \tag{39}$$

to the exact one determined from the actual shear stress distribution

$$U_{exact} = \frac{1}{2} \int_\Omega \gamma \underbrace{G\gamma}_{\tau} d\Omega = \int_\Omega \frac{\tau^2}{2G} d\Omega = \int \int \int \frac{\tau^2}{2G} dxdydz \tag{40}$$

Note that in structural mechanics Ω represents a volume, and Γ (or $\delta\Omega$) the corresponding surface.

- Starting with the exact expression of the shear stress

$$\tau(y) = \frac{VQ(y)}{Ib}; \qquad Q(y) = \int_y^{h/2} by'dy' = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right) \tag{41}$$

- Substituting into Eq. 40 to determine the exact strain energy.

$$\begin{aligned}
U_{exact} &= \int_0^L \left[\int_0^b \int_{-h/2}^{h/2} \frac{V^2}{8GI^2}\left(\frac{h^2}{4} - y^2\right)^2 dydz\right] dx && (42)\\
&= \int_0^L \frac{V^2 b}{8GI^2}\left[\int_{-h/2}^{h/2}\left(\frac{h^4}{16} - \frac{h^2y^2}{2} + y^4\right) dy\right] dx && (43)\\
&= \int_0^L \frac{V^2 b}{8GI^2}\left[\frac{h^4 y}{16} - \frac{h^2y^3}{6} + \frac{y^5}{5}\right]_{-h/2}^{h/2} dx && (44)\\
&= \int_0^L \frac{V^2bh^5}{240GI^2} dx && (45)
\end{aligned}$$

- For a rectangular section $I = bh^3/12$

$$U_{exact} = \frac{3}{5}\int_0^L \frac{V^2}{GA}dx = \underbrace{\frac{6}{5}}_{1.2}\int_0^L \frac{V^2}{2GA}dx \tag{46}$$

- Comparing with Eq. 39, we note that the shear form factor $\lambda_s = 1.2$. Thus $\tau = V/A_s$ and $A_s = A/1.2$
- For shear deformation, we thus adopt $\tau = V/A_s$ and from Eq. 33 we obtain

$$\tan\gamma \simeq \gamma = \frac{dv_s}{dx} = \frac{V}{GA_s} = \frac{\lambda_s V}{AG} \tag{47}$$

- Note analogy with $\varepsilon = \frac{du}{dx} = \frac{P}{AE}$

- The shear deformation for a beam clamped at one end subjected to a point load at the other (as in th definition of a stiffness coefficient term) will be determined next.
- From above, $\int dv_s = \int \frac{V}{GA_s}dx$. Assuming V to be constant, integrate Eq. 47

$$v_s = \frac{V}{GA_s}x + C_1 \tag{48}$$

- If the displacement v_s is zero at the opposite end of the beam, then $C_1 = -\frac{V}{GA_s}(x - L)$ and

$$v_s = \frac{V}{GA_s}(x - L) \tag{49}$$

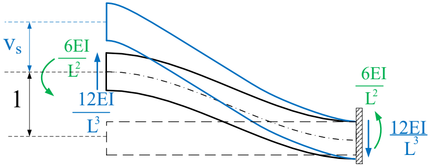
- At $x = 0$

$$v_s = \frac{V}{GA_s}L \tag{50}$$

- What is the "parasitic" displacement due to shear deformation when we applied loads meant to induce unit displacements?
- First, arbitrarily define (recall that $r = \sqrt{\frac{I}{A}}$ and $G = \frac{E}{2(1+\nu)}$)

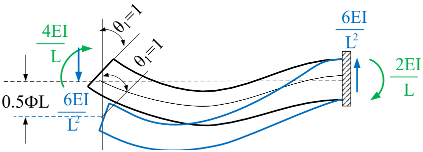
$$\Phi \stackrel{\text{def}}{=} \frac{12EI}{GA_sL^2} = 24(1+\nu)\frac{A}{A_s}\left(\frac{r}{L}\right)^2 \tag{51}$$

- It will be shown that v_s is related to Φ.
- Recall that $A_s = A/\lambda_s$, then due to a unit vertical translation, the end shear force is obtained from Eq. 23 and setting $v_1 = 1$ and $\theta_1 = \theta_2 = v_2 = 0$, or $V = \frac{12EI_z}{L^3}$. At $x = 0$ we have (Eq. 50)



$$\left.\begin{array}{lcl} v_s & = & \frac{VL}{GA_s} \\ V & = & \frac{12EI_z}{L^3} \end{array}\right\} v_s = \underbrace{\frac{12EI}{GA_sL^2}}_{\Phi} \tag{52}$$

- Shear deformation has increased the total translation from 1 to $1+\Phi$.
- Similar arguments apply to the



- Even when a rotation θ_1 is applied, an internal shear force is induced, and this in turn is going to give rise to shear deformations (translation) which must be accounted for.

- The shear force is obtained from Eq. 23 and setting $\theta_1 = 1$ and $\theta_2 = v_1 = v_2 = 0$, or $V = \frac{6EI_z}{L^2}$. At $x = 0$,

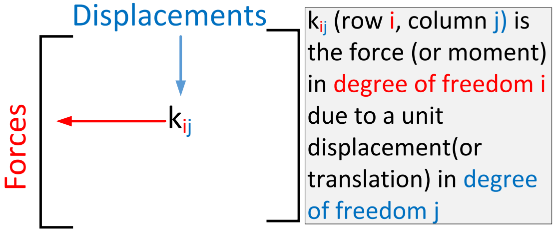
$$\left.\begin{array}{rcl} v_s & = & \frac{VL}{A_sG} \\ V & = & \frac{6EI_z}{L^2} \\ \Phi & = & \frac{12EI}{GA_sL^2} \end{array}\right\} v_s = 0.5\Phi L \tag{53}$$

- Shear deformation has moved the end of the beam (which was supposed to have zero translation) down by by $0.5\Phi L$.

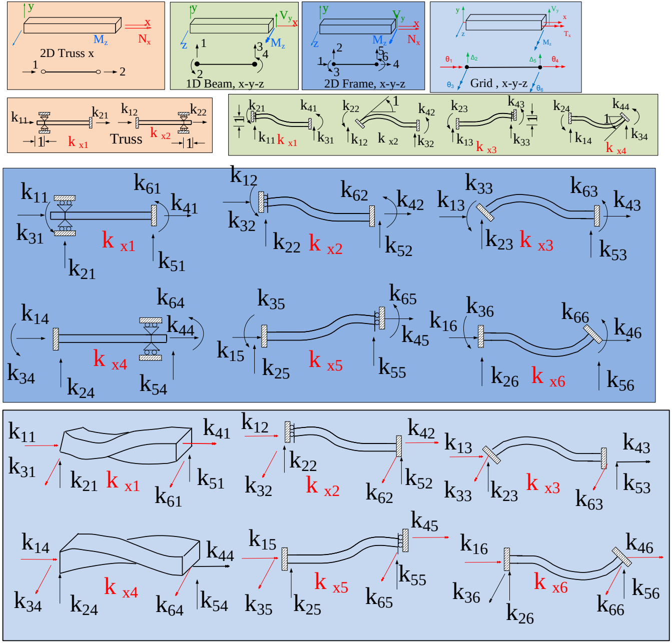
- We have now derived all the proper equations relating displacements to forces.
- Next we shall define the stiffness matrices of different types of elements based on the following coordinate system for both 2D and 3D.

	Forces			Moments		
	x	y	z	x	y	z
Beam		V_y				M_z
2D Frame	N_x	V_y				M_z
Grid		V_y		T_x		M_z
3D Frame	N_x	V_y	V_z	T_x	M_y	M_z

- Recall the definition of the stiffness matrix:



- Identify all the terms that need to be determined


2D Truss x
1D Beam, x-y-z
2D Frame, x-y-z
Grid , x-y-z
Truss

- The truss element (whether in 2D or 3D) has only one degree of freedom associated with each node. Hence, from Eq. 9, we have

$$[\mathrm{k}^t] = \frac{AE}{L} \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \tag{54}$$

- Using Equations 19, 20, 23 and 24 we can determine the forces associated with each unit displacement.

$$
[\mathbf{k}^b] = \begin{array}{c|cccc}
 & v_{1y} & \theta_{1z} & v_{2y} & \theta_{2z} \\
\hline
V_{1y} & \text{Eq. } 23(v_1 = 1) & \text{Eq. } 23(\theta_1 = 1) & \text{Eq. } 23(v_2 = 1) & \text{Eq. } 23(\theta_2 = 1) \\
M_{1z} & \text{Eq. } 19(v_1 = 1) & \text{Eq. } 19(\theta_1 = 1) & \text{Eq. } 19(v_2 = 1) & \text{Eq. } 19(\theta_2 = 1) \\
V_{2y} & \text{Eq. } 24(v_1 = 1) & \text{Eq. } 24(\theta_1 = 1) & \text{Eq. } 24(v_2 = 1) & \text{Eq. } 24(\theta_2 = 1) \\
M_{2z} & \text{Eq. } 20(v_1 = 1) & \text{Eq. } 20(\theta_1 = 1) & \text{Eq. } 20(v_2 = 1) & \text{Eq. } 20(\theta_2 = 1)
\end{array} \tag{55}
$$

- Substituting

$$
[\mathbf{k}^b] = \begin{array}{c|cccc}
 & v_{1y} & \theta_{1z} & v_{2y} & \theta_{2z} \\
\hline
V_{1y} & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\
M_{1z} & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\
V_{2y} & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\
M_{2z} & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L}
\end{array} \tag{56}
$$

- Note row i corresponds to the force in dof i, column j corresponds to the unit displacement in dof j, intersection will be k_{ij}.

- $\mathbf{k}^{2dfr} = \mathbf{k}^b \bigcup \mathbf{k}^t$, Note no coupling between the axial forces and the shear/moment.

$$[\mathbf{k}^{2dfr}] = \begin{array}{c} \\ N_{1x} \\ V_{1y} \\ M_{1z} \\ N_{2x} \\ V_{2y} \\ M_{2z} \end{array} \begin{array}{c} \begin{array}{cccccc} u_{1x} & v_{1y} & \theta_{1z} & u_{2x} & v_{2y} & \theta_{2z} \end{array} \\ \left[\begin{array}{cccccc} k_{11}^t & 0 & 0 & k_{12}^t & 0 & 0 \\ 0 & k_{11}^b & k_{12}^b & 0 & k_{13}^b & k_{14}^b \\ 0 & k_{21}^b & k_{22}^b & 0 & k_{23}^b & k_{24}^b \\ k_{21}^t & 0 & 0 & k_{22}^t & 0 & 0 \\ 0 & k_{31}^b & k_{32}^b & 0 & k_{33}^b & k_{34}^b \\ 0 & k_{41}^b & k_{42}^b & 0 & k_{43}^b & k_{44}^b \end{array} \right] \end{array} \tag{57}$$

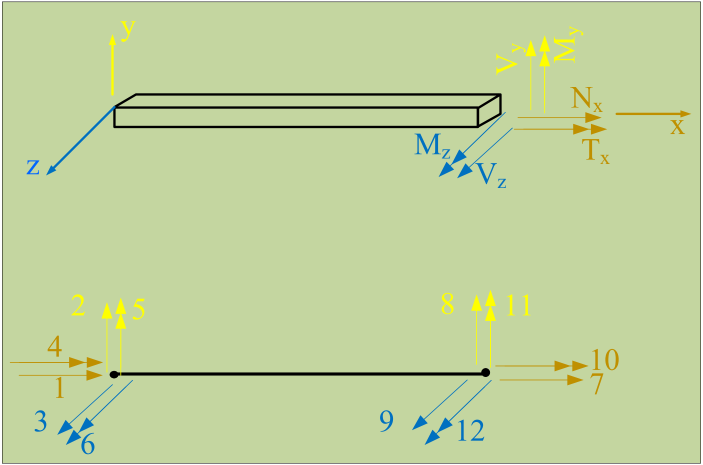
$$[\mathbf{k}^{2dfr}] = \begin{array}{c} \\ N_{1x} \\ V_{1y} \\ M_{1z} \\ N_{2x} \\ V_{2y} \\ M_{2z} \end{array} \begin{array}{c} \begin{array}{cccccc} u_{1x} & v_{1y} & \theta_{1z} & u_{2x} & v_{2y} & \theta_{2z} \end{array} \\ \left[\begin{array}{cccccc} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{array} \right] \end{array} \tag{58}$$

- Stiffness matrix of the grid element is very analogous to the one ofthe 2D frame element, except that the axial component is replaced by the torsional one.

$$[\mathbf{k}^g] = \begin{array}{c} \\ T_{1x} \\ V_{1y} \\ M_{1z} \\ T_{2x} \\ V_{2y} \\ M_{2z} \end{array} \begin{array}{c} \begin{array}{cccccc} \alpha_{1x} & v_{1y} & \beta_{1z} & \alpha_{2x} & v_{2y} & \beta_{2z} \end{array} \\ \begin{bmatrix} \text{Eq. 32} & 0 & 0 & -\text{Eq. 32} & 0 & 0 \\ 0 & k^b_{11} & k^b_{12} & 0 & k^b_{13} & k^b_{14} \\ 0 & k^b_{21} & k^b_{22} & 0 & k^b_{23} & k^b_{24} \\ -\text{Eq. 32} & 0 & 0 & \text{Eq. 32} & 0 & 0 \\ 0 & k^b_{31} & k^b_{32} & 0 & k^b_{33} & k^b_{34} \\ 0 & k^b_{41} & k^b_{42} & 0 & k^b_{43} & k^b_{44} \end{bmatrix} \end{array} \tag{59}$$

- Substituting

$$[\mathbf{k}^g] = \begin{array}{c} \\ T_{1x} \\ V_{1y} \\ M_{1z} \\ T_{2x} \\ V_{2y} \\ M_{2z} \end{array} \begin{array}{c} \begin{array}{cccccc} \alpha_{1x} & v_{1y} & \beta_{1z} & \alpha_{2x} & v_{2y} & \beta_{2z} \end{array} \\ \begin{bmatrix} \frac{GI_x}{L} & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{GI_x}{L} & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \end{array} \tag{60}$$


y
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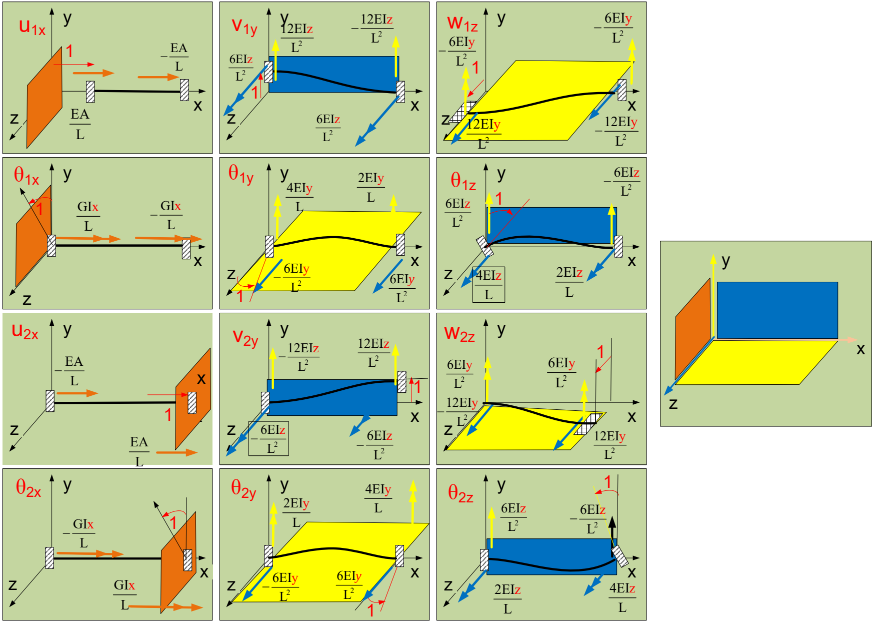
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2EIy
θ1z
4EIz
2EIz
u2x
v2y
w2z
θ2x
θ2y
θ2z

For $[\mathbf{k}_{11}^{3D}]$ and with we obtain:

$$
k^{3dfr} = \begin{array}{c} \\ N_{x1} \\ V_{y1} \\ V_{z1} \\ T_{x1} \\ M_{y1} \\ M_{z1} \\ N_{x2} \\ V_{y2} \\ V_{z2} \\ T_{x2} \\ M_{y2} \\ M_{z2} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} u_{1x} & v_{1y} & w_{1z} & \theta_{1x} & \theta_{1y} & \theta_{1z} & u_{2x} & v_{2y} & w_{2z} & \theta_{2x} & \theta_{2y} & \theta_{2z} \end{array} \\
\begin{pmatrix}
\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -6\frac{EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
-\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{pmatrix}
\end{array}
\tag{61}
$$

- If shear deformations are present, we need to alter the stiffness matrix Eq. 56
  1. Translation: divide (or normalize) coefficients of the first and third columns of the stiffness matrix by $1+\Phi$ so that the net translation at both ends is unity otherwise displacement would be $1+\Phi$ instead of 1.
  2. Due to rotation and the effect of shear deformation
     1. The forces induced at the ends due to a unit rotation at end 1 (second column) neglecting shear deformations, are

$$V_1 = -V_2 = \frac{6EI}{L^2}; \qquad M_1 = \frac{4EI}{L}; \qquad M_2 = \frac{2EI}{L} \tag{62}$$

     2. There is a net positive translation of $0.5\Phi L$ at end 1 when we applied a unit rotation, no additional forces induced.

3. For a unit rotation, all other displacements should be zero $\Rightarrow$. Hence, should counteract this parasitic shear deformation by an equal and opposite one $\Rightarrow$ apply an additional Δ vertical displacement $-0.5\Phi L$ and the additional forces induced at the ends (first column) are given by

$$\Delta V_1 = -\Delta V_2 = \underbrace{\frac{12EI}{L^3}}_{k^b_{11}} \frac{1}{1+\Phi} \underbrace{(-0.5\Phi L)}_{v_s} \tag{63}$$

$$\Delta M_1 = \Delta M_2 = \underbrace{\frac{6EI}{L^2}}_{k^b_{21}} \frac{1}{1+\Phi} \underbrace{(-0.5\Phi L)}_{v_s} \tag{64}$$

Denominators have already been divided by $1+\Phi$ in k^b.

4. Summing up all the forces, we have the forces induced as a result of a unit rotation only when the effects of both bending and shear deformations are included.

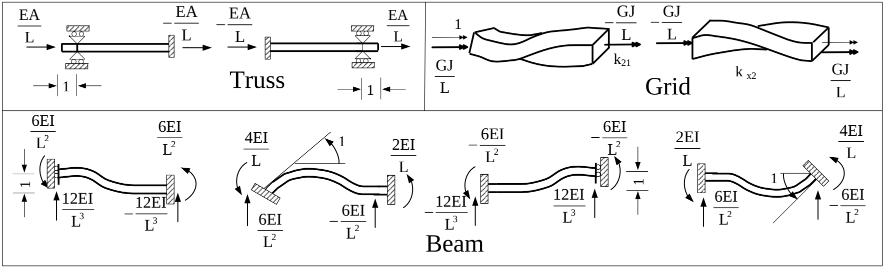
$$V_1 = -V_2 = \underbrace{\frac{6EI}{L^2}}_{\text{Due to unit rotation}} + \underbrace{\underbrace{\frac{12EI}{L^3}}_{k_{11}^t} \frac{1}{1+\Phi} \underbrace{(-0.5\Phi L)}_{v_s}}_{\text{Due to Parasitic Shear}} = -\frac{6EI}{L^2}\frac{1}{1+\Phi} \quad (65)$$

$$M_1 = \underbrace{\frac{4EI}{L}}_{\text{Due to unit rotation}} + \underbrace{\underbrace{\frac{6EI}{L^2}}_{k_{21}^t} \frac{1}{1+\Phi} \underbrace{(-0.5\Phi L)}_{v_s}}_{\text{Due to parasitic shear}} = \frac{4+\Phi}{1+\Phi}\frac{EI}{L} \quad (66)$$

$$M_2 = \underbrace{\frac{2EI}{L}}_{\text{Due to unit rotation}} + \underbrace{\underbrace{\frac{6EI}{L^2}}_{k_{21}^t} \frac{1}{1+\Phi} \underbrace{(-0.5\Phi L)}_{v_s}}_{\text{Due to parasitic shear}} = \frac{2-\Phi}{1+\Phi}\frac{EI}{L} \quad (67)$$

- Element stiffness matrix given in Eq. 56 becomes

$$
[\mathrm{k}^{bV}] = \begin{array}{c|cccc}
 & v_{1y} & \theta_{1z} & v_{2y} & \theta_{2z} \\
\hline
V_{1y} & \frac{12EI_z}{L^3(1+\Phi_y)} & \frac{6EI_z}{L^2(1+\Phi_y)} & -\frac{12EI_z}{L^3(1+\Phi_y)} & \frac{6EI_z}{L^2(1+\Phi_y)} \\
M_{1z} & \frac{6EI_z}{L^2(1+\Phi_y)} & \frac{(4+\Phi_y)EI_z}{(1+\Phi_y)L} & -\frac{6EI_z}{L^2(1+\Phi_y)} & \frac{(2-\Phi_y)EI_z}{L(1+\Phi_y)} \\
V_{2y} & -\frac{12EI_z}{L^3(1+\Phi_y)} & -\frac{6EI_z}{L^2(1+\Phi_y)} & \frac{12EI_z}{L^3(1+\Phi_y)} & -\frac{6EI_z}{L^2(1+\Phi_y)} \\
M_{2z} & \frac{6EI_z}{L^2(1+\Phi_y)} & \frac{(2-\Phi_y)EI_z}{L(1+\Phi_y)} & -\frac{6EI_z}{L^2(1+\Phi_y)} & \frac{(4+\Phi_y)EI_z}{L(1+\Phi_y)}
\end{array} \tag{68}
$$



1. Singularity All the derived stiffness matrices are singular, that is there is at least one row and one column which is a linear combination of others. For example in the beam-column element, row 4 = −row 1; and L times row 2 is equal to the sum of row 3 and 6. This singularity (not present in the flexibility matrix) is caused by the linear relations introduced by the equilibrium equations which are embedded in the formulation.
2. Symmetry All matrices are symmetric due to Maxwell-Betti's reciprocal theorem, and the stiffness flexibility relation.

More about the stiffness matrix properties later.

- In the presence of thermal load (or initial strains), nodal equivalent forces $\mathtt{P}_{el}$ can be readily determined as follows:
- Trusss

$$F_1^T = -AE\alpha\Delta T \qquad F_2^T = AE\alpha\Delta T \tag{69}$$

- Beam

$$\begin{aligned} F_1^T &= -AE\alpha\Delta T^{avg} & F_2^T &= AE\alpha\Delta T^{avg} \\ M_1^T &= \frac{EI\alpha(\Delta T^{top}-\Delta T^{bot})}{h} & M_2^T &= -\frac{EI\alpha(\Delta T^{top}-\Delta T^{bot})}{h} \end{aligned} \tag{70}$$

  where α is the coefficient of thermal expansion, $\Delta T^{avg} = \frac{\Delta T^{top}+\Delta T^{bot}}{2}$.
- For initial forces (such as prestressed members) one needs to simply specify $\alpha\Delta T$ for the initial strain induced by prestressing
- In the load input data file one simply needs to specify $\alpha\Delta T$ for the thermally loaded truss, and $\alpha(\Delta T^{top} - \Delta T^{bot})$ and h for beams.

- Nodal equivalent forces P_{el} should not be confused with fixed end actions (they are equal but with opposite signs).
- From above

$$
\begin{aligned}
m(x) &= \text{moment due to the applied loads at section } x && (71)\\
f(x) &= \int m(x)dx; \quad g(x) = \int f(x)dx; \quad q(x) = \int p(x)dx && (72)\\
& \quad \text{total load on the span} && (73)
\end{aligned}
$$

- and

$$
\begin{aligned}
M_1^F &= \frac{2}{L^2}\left[Lf(L) - 3g(L)\right] && (74)\\
M_2^F &= -\frac{1}{L^2}\left[L^2 m(L) - 4Lf(L) + 6g(L)\right] && (75)\\
V_1^F &= \frac{6}{L^3}\left[Lf(L) - 2g(L)\right] && (76)\\
V_2^F &= -\frac{6}{L^3}\left[Lf(L) - 2g(L)\right] - q && (77)
\end{aligned}
$$

- For a uniformly distributed load w over the entire span,

$$m(x) = -\frac{1}{2}wx^2; \quad f(x) = -\frac{1}{6}wx^3; \quad g(x) = -\frac{1}{24}wx^4; \quad q = wL \tag{78}$$

- Substituting

$$M_1^F = \frac{2}{L^2}\left[L\left(-\frac{1}{6}wL^3\right) - 3\left(-\frac{1}{24}wL^4\right)\right] = \frac{wL^2}{12} \tag{79}$$

$$M_2^F = -\frac{1}{L^2}\left[L^2\left(-\frac{1}{2}wL^2\right) - 4L\left(-\frac{1}{6}wL^3\right) + 6\left(-\frac{1}{24}wL^4\right)\right] = \frac{wL^2}{12} \tag{80}$$

$$V_1^F = \frac{6}{L^3}\left[L\left(-\frac{1}{6}wL^3\right) - 2\left(-\frac{1}{24}wL^4\right)\right] = \frac{wL}{2} \tag{81}$$

$$V_2^F = -\frac{6}{L^3}\left[L\left(-\frac{1}{6}wL^3\right) - 2\left(-\frac{1}{24}wL^4\right)\right] - wL = \frac{wL}{2} \tag{82}$$

- Use the unit step function to find $m(x)$. For a concentrated load P acting at a from the left-hand end with $b = L - a$,

$$\begin{array}{rclcrcl} m(x) & = & -P(x-a)H_a & \text{gives} & m(L) & = & -Pb \\ f(x) & = & -\frac{1}{2}P(x-a)^2H_a & & f(L) & = & -\frac{1}{2}Pb^2 \\ g(x) & = & -\frac{1}{6}P(x-a)^3H_a & & g(L) & = & -\frac{1}{6}Pb^3 \end{array} \tag{83}$$

- where we define $H_a = 0$ if $x < a$, and $H_a = 1$ if $x \geq a$, and

$$q = P \tag{84}$$

$$M_1^F = \frac{2}{L^2}\left[L\left(-\frac{1}{2}Pb^2\right) - 3\left(-\frac{1}{6}Pb^3\right)\right] == \frac{Pb^2a}{L^2} \tag{85}$$

$$M_2^F = -\frac{1}{L^2}\left[L^2(-Pb) - 4L\left(-\frac{1}{2}Pb^2\right) + 6\left(-\frac{1}{6}Pb^3\right)\right] = \frac{Pb}{1^2}\left(L^2 - 2Lb + b^2\right) \tag{86}$$

$$= \frac{Pba^2}{L^2} \tag{87}$$

$$V_1^F = \frac{6}{L^3}\left[L\left(-\frac{1}{2}Pb^2\right) - 2\left(-\frac{1}{6}Pb^3\right)\right] = -\frac{Pb^2}{L^3}(3L - 2b) = \frac{Pb^2}{L^3}(3a + b) \tag{88}$$

$$V_2^F = -\left(\frac{6}{L^3}\left[L\left(-\frac{1}{2}Pb^2\right) - 2\left(-\frac{1}{6}Pb^3\right)\right] + P\right) = \frac{Pa^2}{L^3}(a + 3b) \tag{89}$$

- If the load is applied at midspan ($a = B = L/2$), then the previous equationreduces to

$$M_1^F = -\frac{PL}{8} \tag{90}$$
$$M_2^F = \frac{PL}{8} \tag{91}$$
$$V_1^F = -\frac{P}{2} \tag{92}$$
$$V_2^F = -\frac{P}{2} \tag{93}$$

Intermediary Structural Analysis

Stiffness Method; I Orthogonal Structures

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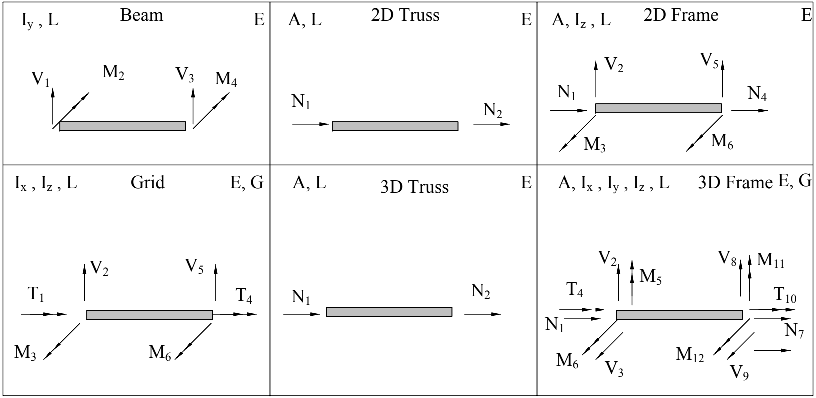
Fall 2021

Table of Contents

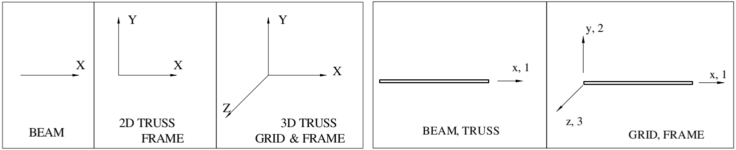
- A structure is a system composed of individual components (elements).
- Structure must be discretized
  - Revisit local and global d.o.f/coordinates.
  - Element internal forces.
  - Element stiffness matrices.
- We now seek to analyze a structure (system).
- For convenience, we will start with orthogonal 2D structures.
- We will assemble the structure stiffness in terms of unrestrained d.o.f.

- Completely different than in structural analysis/design (where we focused mostly on flexure and defined a positive moment as one causing “tension below”. This would be awkward to program!).
- Consistent with the prevailing coordinate system (i.e. a positive moment as one which is counter-clockwise


I_y , L
Beam
E
V_1
M_2
V_3
M_4
A, L
2D Truss
E
N_1
N_2
A, I_z , L
2D Frame
E
V_2
V_5
N_1
N_4
M_3
M_6
I_x , I_z , L
Grid
E, G
V_2
V_5
T_1
T_4
M_3
M_6
A, L
3D Truss
E
N_1
N_2
A, I_x , I_y , I_z , L
3D Frame
E, G
V_2
M_5
V_8
M_{11}
T_4
T_{10}
N_1
N_7
M_6
V_3
M_{12}
V_9

Two coordinate systems:

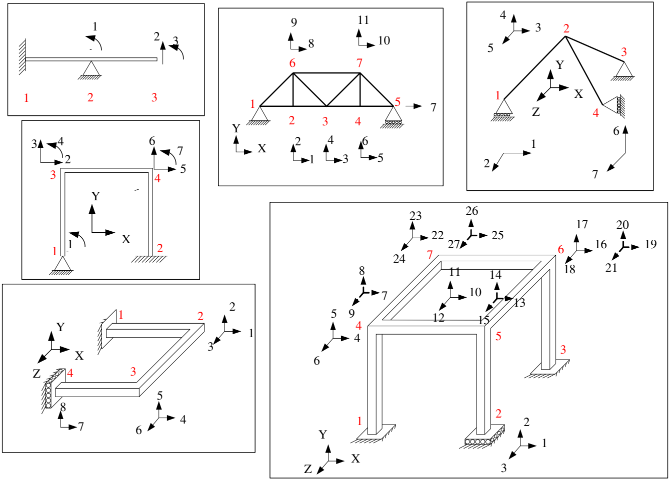
1. Global: to describe the structure nodal coordinates.
   - Arbitrarily selected provided it is a Right Hand Side (RHS) one
   - Upper case axis labels, X, Y, Z, or 1,2,3 (running indices within a computer program).



2. Local: system is associated with each element
   - Describe the element internal forces.
   - lower case axis labels, x, y, z (or 1,2,3). The x-axis is assumed to be along the member, and the direction.
   - Selected such that it points from the 1st node to the 2nd node.

- A degree of freedom (d.o.f.) is an independent generalized nodal displacement (translation or rotation) at a node.
- The displacements must be linearly independent (of coordinate system) and thus not related to each other.
- An element dof is defined wrt its own local coordinate system. A structural dof is defined wrt a global coordinate system.

Type		Node 1	Node 2	$[\mathtt{k}^{(e)}]$ (Local)	$[\mathtt{K}^{(e)}]$ (Global)
1 Dimensional					
Beam	$\{\mathtt{p}\}$	F_{y1}, M_{z2}	F_{y3}, M_{z4}	4×4	4×4
	$\{\delta\}$	v_1, θ_2	v_3, θ_4		
2 Dimensional					
Truss	$\{\mathtt{p}\}$	F_{x1}	F_{x2}	2×2	4×4
	$\{\delta\}$	u_1	u_2		
Frame	$\{\mathtt{p}\}$	F_{x1}, F_{y2}, M_{z3}	F_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\delta\}$	u_1, v_2, θ_3	u_4, v_5, θ_6		
Grid	$\{\mathtt{p}\}$	T_{x1}, F_{y2}, M_{z3}	T_{x4}, F_{y5}, M_{z6}	6×6	6×6
	$\{\delta\}$	θ_1, v_2, θ_3	θ_4, v_5, θ_6		
3 Dimensional					
Truss	$\{\mathtt{p}\}$	F_{x1},	F_{x2}	2×2	6×6
	$\{\delta\}$	u_1,	u_2		
Frame	$\{\mathtt{p}\}$	F_{x1}, F_{y2}, F_{y3}, T_{x4} M_{y5}, M_{z6}	F_{x7}, F_{y8}, F_{y9}, T_{x10} M_{y11}, M_{z12}	12×12	12×12
	$\{\delta\}$	u_1, v_2, w_3, θ_4, θ_5 θ_6	u_7, v_8, w_9, θ_{10}, θ_{11} θ_{12}		



1. Determine degree of static indeterminacy, n.
2. Define a primary structure which statically determinate by removing n arbitrarily reactions to have a statically determinate (and stable) structure.
3. Analyse the primary structure, subjected to the actual load, and solve for the n displacements corresponding to the n reactions removed, Δ_j
4. Apply a unit load at point at each of the d.o.f. corresponding to the redundant forces, and solve for deflections f_{ij} at node i due to a unit force at node j.
5. Write the compatibility of displacement equation $f_{ij}R_j - \Delta_i = 0$ For $n = 2$, this corresponds to:

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \left\{ \begin{array}{c} R_1 \\ R_2 \end{array} \right\} - \left\{ \begin{array}{c} \Delta_1 \\ \Delta_2 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}$$

6. Invert the matrix, and solve for the reactions.

Note that Reactions are the primary unknowns, subsequently from statics one can determine the internal forces, and finally the displacements.

1. Determine the degree of kinematic indeterminacy.
2. Fix all displacements, the structure is now kinematically determinate (all displacements are known and are equal to zero).
3. Determine end nodal forces for each loaded element, sum up.
4. Apply a unit displacement (rotation or displacement) at each free/unrestrained degree of freedom j at a time, and in each case we shall determine the internal reaction forces at degrees of freedom j, K_{ij}.
5. Assemble the reduced structure stiffness matrix in global coordinate system in terms of the individual element stiffness matrices transformed to the global one. This will result in an equation of equilibrium at each node:

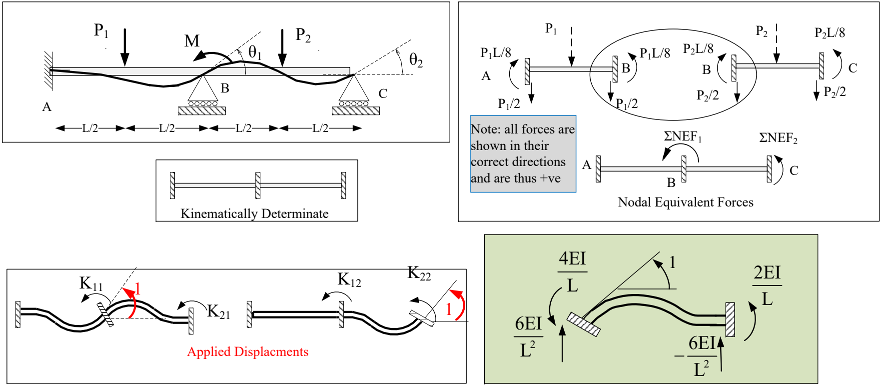
$$\underbrace{\mathrm{K}\Delta}_{\mathrm{P}_{int}} - \mathrm{P}_{ext} = 0. \tag{1}$$

   Where P_{ext} includes nodal forces and nodal equivalent loads.
6. Reduced because we are not considering the restrained degrees of freedom.

- Note analogy with moment distribution method.

- Displacements are the primary unknowns, subsequently from the displacement force relations (again element stiffness matrix) we solve for both internal forces and reactions.

- Flexibility: What are the forces (reactions) that will ensure compatibility (of displacements at released dof)?
- Stiffness: What are the displacements that will ensure equilibrium?



$P_1 = 2P$, $M = PL$, and $P_2 = P$. Solve for the displacements.

1. Degree of kinematic indeterminacy is 2.

2. Using the previously defined sign convention, determine thenodal equivalent load (to the load applied along the member)

$$\Sigma P_{el,1} = \underbrace{\frac{P_1 L}{8}}_{BA} - \underbrace{\frac{P_2 L}{8}}_{BC} = \frac{2PL}{8} - \frac{PL}{8} = \frac{PL}{8}$$

$$\Sigma P_{el,2} = \underbrace{\frac{PL}{8}}_{CB}$$

3. If it takes $\frac{4EI}{L}$ (k_{44}^{BA}) to rotate BA and $\frac{4EI}{L}$ (k_{22}^{BC}) to rotate BC, it will take a total force of $\frac{8EI}{L}$ to simultaneously rotate BA and BC.

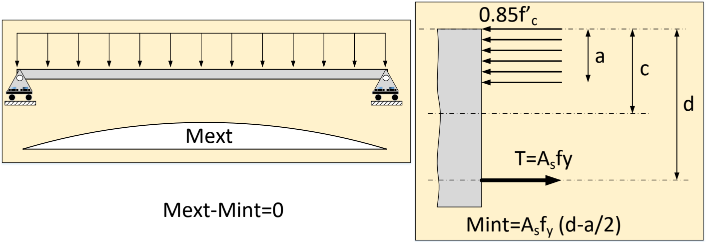
4. The sum of the rotational stiffnesses at global d.o.f. 1 is $K_{11} = \frac{8EI}{L}$; similarly, $K_{21} = \frac{2EI}{L}$ (k_{42}^{BC}).

5. If we now rotate d.o.f. 2 by a unit angle, we will have $K_{22} = \frac{4EI}{L}$ (k_{22}^{BC}) and $K_{12} = \frac{2EI}{L}$ (k_{42}^{BC}).

6. Equation of equilibrium:

$$\underbrace{\underbrace{\left\{ \begin{array}{c} PL \\ 0 \end{array} \right\}}_{\mathrm{p}_{nodes}} + \underbrace{\left\{ \begin{array}{c} \frac{PL}{8} \\ \frac{PL}{8} \end{array} \right\}}_{\mathrm{P}_{el}}}_{\mathrm{P}_{ext}} - \underbrace{\underbrace{\left[\begin{array}{cc} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array} \right]}_{\mathrm{K}} \underbrace{\left\{ \begin{array}{c} \theta_1^? \\ \theta_2^? \end{array} \right\}}_{\Delta}}_{\mathrm{P}_{int}} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}$$

7. Note that we have $\mathrm{P}_{ext} - \mathrm{P}_{int} = 0$ and not $\mathrm{P}_{ext} + \mathrm{P}_{int} = 0$ because the external forces must be resisted by the internal ones in an equal and opposite direction. By analogy



8. Simplifying

$$\left\{ \begin{array}{c} PL + \frac{PL}{8} \\ +\frac{PL}{8} \end{array} \right\} = \left[\begin{array}{cc} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array} \right] \left\{ \begin{array}{c} \theta_1^? \\ \theta_2^? \end{array} \right\}$$

Note that we will always write the equilibrium relationship as $\mathrm{P}_{ext} - \mathrm{P}_{int} = 0$

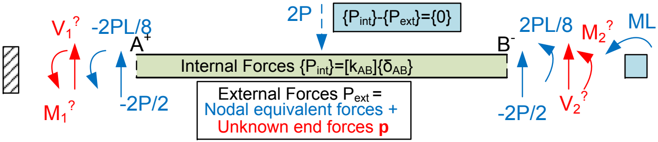
9. Invert the two by two matrix

$$\left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\} = \left[\begin{array}{cc} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{array} \right]^{-1} \left\{ \begin{array}{c} PL + \frac{PL}{8} \\ +\frac{PL}{8} \end{array} \right\} = \left\{ \begin{array}{c} \frac{17}{112}\frac{PL^2}{EI} \\ -\frac{5}{112}\frac{PL^2}{EI} \end{array} \right\}$$

10. Recall that for each element $\{\mathrm{p}\} = [\mathrm{k}]\{\delta\}$, and in this case $\{\mathrm{p}\} = \{\mathrm{P}\}$ and $\{\delta\} = \{\Delta\}$ for element *AB*. The element stiffness matrix has been previously derived, and in this case the global and local d.o.f. are the same.

11. Next, we need to compute the element internal forces.

12. Equilibrium equation for element AB, at the element level, can be written as (note that we must include the nodal equivalent loads to maintain equilibrium):



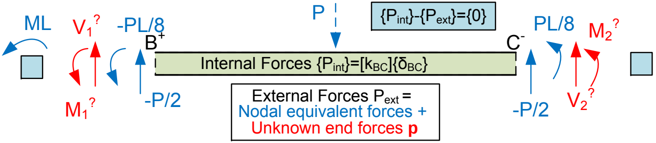
$$\underbrace{\underbrace{\left\{\begin{array}{c} V_1^? \\ M_1^? \\ V_2^? \\ M_2^? \end{array}\right\}}_{\left\{\mathrm{p}^{AB}_{int-for}\right\}} + \underbrace{\left\{\begin{array}{c} -\frac{2P}{2} \\ -\frac{2PL}{8} \\ -\frac{2P}{2} \\ \frac{2PL}{8} \end{array}\right\}}_{\mathrm{p}^{AB}_{el}}}_{\mathrm{p}^{AB}_{ext}} - \underbrace{\underbrace{\left[\begin{array}{cccc} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{array}\right]}_{[\mathrm{k}^{AB}]} \underbrace{\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \frac{17}{112}\frac{PL^2}{EI} \end{array}\right\}}_{\left\{\delta^{AB}\right\}}}_{\mathrm{p}^{AB}_{int}} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right\}$$

Note: This step is called Force recovery, i.e. we determine the internal forces from the nodal displacements. It is in terms of local forces p and not the global ones P.

Solving

$$\lfloor\ V_1 \quad M_1 \quad V_2 \quad M_2\ \rfloor = \lfloor\ \frac{107}{56}P \quad \frac{31}{56}PL \quad \frac{5}{56}P \quad \frac{5}{14}PL\ \rfloor$$

13 Similarly, for element *BC*:



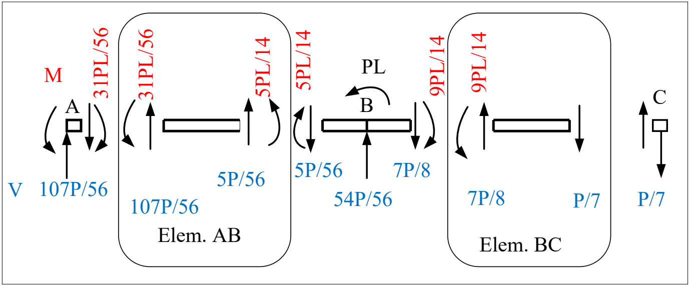
$$\underbrace{\underbrace{\begin{Bmatrix} V_1^? \\ M_1^? \\ V_2^? \\ M_2^? \end{Bmatrix}}_{\{\mathbf{p}^{BC}_{int-for}\}} + \underbrace{\begin{Bmatrix} -\frac{P}{2} \\ -\frac{PL}{8} \\ -\frac{P}{2} \\ \frac{PL}{8} \end{Bmatrix}}_{\mathbf{p}^{BC}_{el}}}_{\mathbf{p}^{BC}_{ext}} - \underbrace{\underbrace{\begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}}_{[\mathbf{k}^{BC}]} \underbrace{\begin{Bmatrix} 0 \\ \frac{17}{112}\frac{PL^2}{EI} \\ 0 \\ -\frac{5}{112}\frac{PL^2}{EI} \end{Bmatrix}}_{\{\delta^{BC}\}}}_{\mathbf{p}^{BC}_{int}} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

or

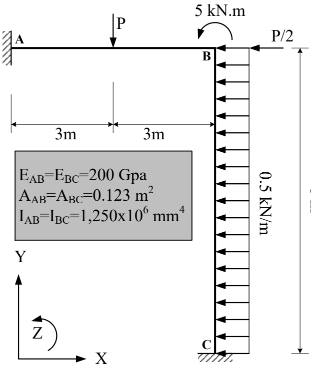
$$\lfloor\; V_1 \quad M_1 \quad V_2 \quad M_2 \;\rfloor = \lfloor\; \frac{7}{8}P \quad \frac{9}{14}PL \quad -\frac{P}{7} \quad 0 \;\rfloor$$

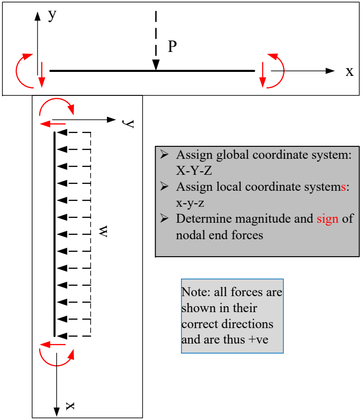
14 This simple example calls for the following observations:

1. Node A has contributions from element *AB* only, while node B has contributions from both *AB* and *BC*.
2. We observe that $p_3^{AB} \neq p_1^{BC}$ even though they both correspond to a shear force at node B, the difference between them is equal to the reaction at B. Similarly, $p_4^{AB} \neq p_2^{BC}$ due to the externally applied moment at node B.
3. Must conclude with free body, shear and moment diagrams.



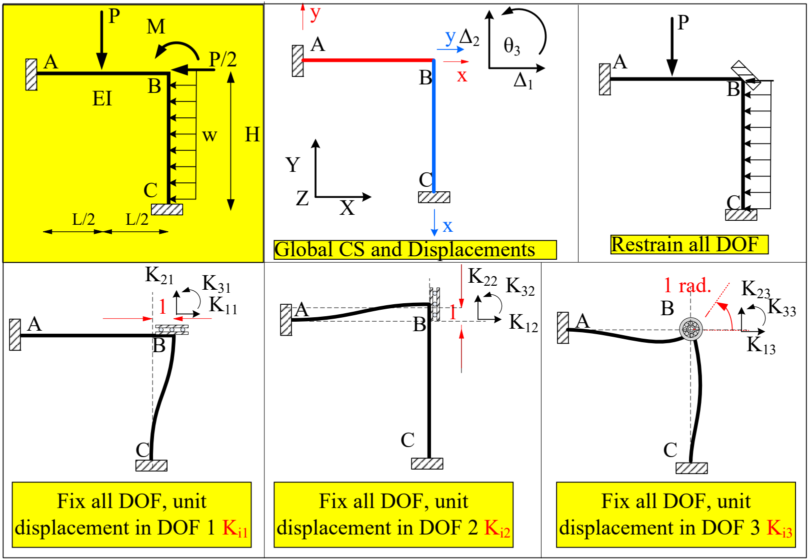
Analise the following frame for $P = 2$ kN, $L = H = 6$ m, $M = 5$ kN.m, $w = 0.5$ kN/m, $E = 2 \times 10^8$ kPa, $A = 0.123$ m^2, and $I^b = I^c = 0.00125$ m^4





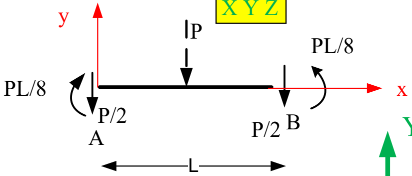
- Assign global coordinate system: X-Y-Z
- Assign local coordinate systems: x-y-z
- Determine magnitude and sign of nodal end forces

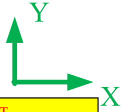
Note: all forces are shown in their correct directions and are thus +ve


P
M
P/2
A
B
EI
w
H
C
L/2
L/2
y
A
y Δ2
θ3
B
x
Δ1
Y
Z
X
C
x
Global CS and Displacements
P
A
B
C
Restrain all DOF
K21
K31
1
K11
A
B
C
Fix all DOF, unit displacement in DOF 1 Ki1
K22
K32
A
1
B
K12
C
Fix all DOF, unit displacement in DOF 2 Ki2
1 rad.
K23
B
K33
A
K13
C
Fix all DOF, unit displacement in DOF 3 Ki3

1. Assuming axial deformations, we do have three global degrees of freedom, Δ_1, Δ_2, and θ_3.
2. Constrain all the degrees of freedom, and thus make the structure kinematically determinate.
3. Determine the nodal equivalent loads for each element in local coordinate system in its own local coordinate system (element 1 is assumed to be defined from A to B, and element 2 from B to C):

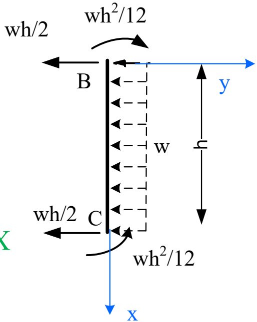
x	y	z
X	Y	Z





$$p_{AB} = \lfloor 0 \;\; -P/2 \;\; -PL/8 \;\; 0 \;\; -P/2 \;\; PL/8 \rfloor^T$$
$$P_{AB} = \lfloor 0 \;\; -P/2 \;\; -PL/8 \;\; \boxed{0 \;\; -P/2 \;\; PL/8} \rfloor^T$$

$$p_{BC} = \lfloor 0 \;\; -wh/2 \;\; -wh^2/12 \;\; 0 \;\; -wh/2 \;\; wh^2/12 \rfloor^T$$
$$P_{BC} = \lfloor \boxed{-wh/2 \;\; 0 \;\; -wh^2/12} \;\; -wh/2 \;\; 0 \;\; wh^2/12 \rfloor^T$$



x	y	z
-Y	X	Z

$$\underbrace{\lfloor\; p_1^A \quad p_2^A \quad p_3^A \mid p_4^B \quad p_5^B \quad p_6^B \;\rfloor}_{AB} = \lfloor\; 0 \quad -\frac{P}{2} \quad -\frac{PL}{8} \mid 0 \quad -\frac{P}{2} \quad \frac{PL}{8} \;\rfloor \tag{2}$$

$$= \lfloor\; 0 \quad -\frac{2}{2} \quad -\frac{(2)(6)}{8} \mid 0 \quad -\frac{2}{2} \quad \frac{(2)(6)}{8} \;\rfloor$$

$$= \lfloor\; 0 \quad -1.0 \quad -1.5 \mid 0 \quad -1.0 \quad 1.5 \;\rfloor$$

$$\underbrace{\lfloor\; p_1^B \quad p_2^B \quad p_3^B \mid p_4^C \quad p_5^C \quad p_6^C \;\rfloor}_{BC} = \lfloor\; 0 \quad -\frac{wH}{2} \quad -\frac{wH^2}{12} \mid 0 \quad -\frac{wH}{2} \quad \frac{wH^2}{12} \;\rfloor \tag{3}$$

$$= \lfloor 0 - \frac{(0.5)(6)}{2} \;\; -\frac{(0.5)(6)^2}{12} \Big| 0 - \frac{(0.5)(6)}{2} \;\; \frac{(0.5)(6)^2}{12} \rfloor$$

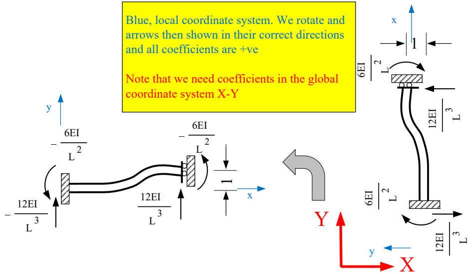
$$= \lfloor\; 0 \quad -1.5 \quad -1.5 \mid 0 \quad -1.5 \quad 1.5 \;\rfloor$$

and the nodal equivalent forces at node B would have to be summed.

4. Apply a unit displacement in each of the 3 global degrees of freedom, to determine the structure global stiffness matrix. Each entry K_{ij} of the global stiffness matrix will correspond to the internal force in degree of freedom i, due to a unit displacement in degree of freedom j.

5. Recalling the force displacement relations derived earlier, we can assemble the global stiffness matrix in terms of contributions from both AB and BC:

- Need to complete the following table where columns correspond to imposed displacements on dof j, and rows correspond to the corresponding induced internal forces in each of the elements in dof i. Both are in the global coordinate system.
- $K_{1,2}$ is zero because an imposed displacement along dof 2 (horizontal), while locking all other displacements, does not induce an internal force in any of the two elements.
- K_{31} are the internal forces (moments in here) resulting from an imposed unit displacement in dof 1 (horizontal). This will not “mobilize” AB, but will activate flexure for BC. For BC from the following figure (already shown above



		K_{i1} Δ_1	K_{i2} Δ_2	K_{i3} θ_3
K_{1j} (F_X)	AB	$\frac{EA}{L}$	0	0
	BC	$\frac{12EI^c}{H^3}$	0	$\frac{6EI^c}{H^2}$
K_{2j} (F_Y)	AB	0	$\frac{12EI^b}{L^3}$	$-\frac{6EI^b}{L^2}$
	BC	0	$\frac{EA}{H}$	0
K_{3j} (M_Z)	AB	0	$-\frac{6EI^b}{L^2}$	$\frac{4EI^b}{L}$
	BC	$\frac{6EI^c}{H^2}$	0	$\frac{4EI^c}{H}$

- Note that all diagonal terms are +ve, and that the table is symmetric.

6. Summing up, the structure global stiffness matrix $[\mathrm{K}]$ is:

$$[\mathrm{K}] = \begin{array}{c} \\ P_1 \\ P_2 \\ M_3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad\Delta_1 \quad & \quad\Delta_2\quad & \quad\theta_3\quad \end{array} \\ \begin{bmatrix} k_{44}^{AB} + k_{22}^{BC} & k_{45}^{AB} + k_{21}^{BC} & k_{46}^{AB} + k_{23}^{BC} \\ k^{AB} + k^{BC} & k_{55}^{AB} + k_{11}^{BC} & k_{56}^{AB} + k_{13}^{BC} \\ k_{64}^{AB} + k_{32}^{BC} & k_{65}^{AB} + k_{31}^{BC} & k_{66}^{AB} + k_{33}^{BC} \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ P_1 \\ P_2 \\ M_3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad\Delta_1 \quad & \quad\Delta_2\quad & \quad\theta_3\quad \end{array} \\ \begin{bmatrix} \frac{EA}{L} + \frac{12EI^c}{H^3} & 0 & \frac{6EI^c}{H^2} \\ 0 & \frac{12EI^b}{L^3} + \frac{EA}{H} & -\frac{6EI^b}{L^2} \\ \frac{6EI^c}{H^2} & -\frac{6EI^b}{L^2} & \frac{4EI^b}{L} + \frac{4EI^c}{H} \end{bmatrix} \end{array}$$

Substituting

$$[\mathrm{K}] = 10^6 \begin{bmatrix} 4.1139 & 0 & 0.0417 \\ 0 & 4.1139 & -0.0417 \\ 0.0417 & -0.0417 & 0.3333 \end{bmatrix}$$

Note that the axial stiffness (EA/L) is 4.1×10^6, while the flexural one ($12EI/H^3$) is 0.0071×10^6. Axial stiffness is always much higher than flexural stiffness.

7. We need to have $\mathbf{P}_{ext}$ in global coordinate system. From Eq. 2 and 3 we had

$$\underbrace{\lfloor\; p_1^A \quad p_2^A \quad p_3^A \mid p_4^B \quad p_5^B \quad p_6^B \;\rfloor}_{AB} = \lfloor\; 0 \quad -\frac{P}{2} \quad -\frac{PL}{8} \mid 0 \quad -\frac{P}{2} \quad \frac{PL}{8} \;\rfloor \tag{4}$$

$$\underbrace{\lfloor\; p_1^B \quad p_2^B \quad p_3^B \mid p_4^C \quad p_5^C \quad p_6^C \;\rfloor}_{BC} = \lfloor\; 0 \quad -\frac{wH}{2} \quad -\frac{wH^2}{12} \mid 0 \quad -\frac{wH}{2} \quad \frac{wH^2}{12} \;\rfloor \tag{5}$$

8. Cast in the global coordinate system, that will be

$$\underbrace{\lfloor\; P_1^A \quad P_2^A \quad P_3^A \mid P_4^B \quad P_5^B \quad P_6^B \;\rfloor}_{AB} = \lfloor\; 0 \quad -\frac{P}{2} \quad -\frac{PL}{8} \mid 0 \quad -\frac{P}{2} \quad \frac{PL}{8} \;\rfloor \tag{6}$$

$$\underbrace{\lfloor\; P_1^B \quad P_2^B \quad P_3^B \mid P_4^C \quad P_5^C \quad P_6^C \;\rfloor}_{BC} = \lfloor\; -\frac{wH}{2} \quad 0 \quad -\frac{wH^2}{12} \mid -\frac{wH}{2} \quad 0 \quad \frac{wH^2}{12} \;\rfloor \tag{7}$$

9. The global equation of equilibrium can now be written (note that for illustrative purposes, we kept w and and a moment M at node B).

$$\underbrace{\underbrace{\left\{\begin{array}{c} -\frac{P}{2} \\ 0 \\ M \end{array}\right\}}_{\text{P}_{nodes}} + \underbrace{\left\{\begin{array}{c} -\frac{wH}{2} \\ -\frac{P}{2} \\ \frac{PL}{8} - \frac{wH^2}{12} \end{array}\right\}}_{\text{P}_{el}}}_{\text{P}_{ext}} - \underbrace{\underbrace{\left[\begin{array}{ccc} \frac{EA}{L} + \frac{12EI^c}{H^3} & 0 & \frac{6EI^c}{H^2} \\ 0 & \frac{12EI^b}{L^3} + \frac{EA}{H} & -\frac{6EI^b}{L^2} \\ \frac{6EI^c}{H^2} & -\frac{6EI^b}{L^2} & \frac{4EI^b}{L} + \frac{4EI^c}{H} \end{array}\right]}_{[\text{K}]} \left\{\begin{array}{c} \Delta_1 \\ \Delta_2 \\ \theta_3 \end{array}\right\}}_{\text{P}_{int}} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right\}$$

Substituting:

$$\left\{\begin{array}{c} -0.5 \\ 0 \\ 5 \end{array}\right\} + \underbrace{\left\{\begin{array}{c} -1.5 \\ -0.5 \\ -0.75 \end{array}\right\}}_{\text{P}_{el}} = 10^6 \underbrace{\left[\begin{array}{ccc} 4.1139 & 0 & 0.0417 \\ 0 & 4.1139 & -0.0417 \\ 0.0417 & -0.0417 & 0.3333 \end{array}\right]}_{[\text{K}]} \left\{\begin{array}{c} \Delta_1 \\ \Delta_2 \\ \theta_3 \end{array}\right\}$$

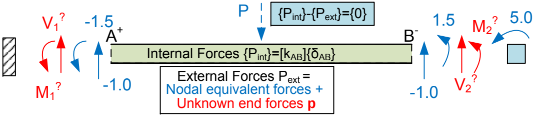
10 Solve for the displacements

$$\left\{\begin{array}{c}\Delta_1\\ \Delta_2\\ \theta_3\end{array}\right\} = 10^6\left[\begin{array}{ccc}4.1139 & 0 & 0.0417\\ 0 & 4.1139 & -0.0417\\ 0.0417 & -0.0417 & 0.3333\end{array}\right]^{-1}\left\{\begin{array}{c}-2\\ -0.5\\ 4.25\end{array}\right\}$$

or

$$\left\{\begin{array}{c}\Delta_1\\ \Delta_2\\ \theta_3\end{array}\right\} = 10^{-6}\left\{\begin{array}{c}-0.61\text{ m}\\ 0.0084\text{ m}\\ 12.82\text{ radian}\end{array}\right\}$$

11 To obtain the element internal forces, multiply each element stiffness matrix by the local displacements. For element AB, the local and global coordinates match, thus



$$\underbrace{\underbrace{\left\{\begin{array}{c} p_1^? \\ p_2^? \\ p_3^? \\ \hline p_4^? \\ p_5^? \\ p_6^? \end{array}\right\}}_{\mathbf{p}^{AB}_{int-for}} + \underbrace{\left\{\begin{array}{c} 0 \\ -\frac{P}{2} \\ -\frac{PL}{8} \\ 0 \\ -\frac{P}{2} \\ \frac{PL}{8} \end{array}\right\}}_{\mathbf{p}^{AB}_{el}}}_{\mathbf{p}^{AB}_{ext}} - \underbrace{\left[\begin{array}{ccc|ccc} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & 0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ 0 & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} & 0 & -\frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ \hline -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & 0 & \frac{12EI_y}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} & 0 & -\frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline \delta_1 \\ \delta_2 \\ \theta_3 \end{array}\right\}}_{\mathbf{p}^{AB}_{int}} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{c} p_1^? \\ p_2^? \\ p_3^? \\ \hline p_4^? \\ p_5^? \\ p_6^? \end{array}\right\} = \underbrace{10^6\left[\begin{array}{ccc|ccc} - & - & - & -4.1\times 10^6 & 0 & 0 \\ - & - & - & 0 & -13,889. & 41,667. \\ - & - & - & 0 & -41,667. & 83,333. \\ - & - & - & 4.1\times 10^6 & 0 & 0 \\ - & - & - & 0 & 13,889. & -41,667 \\ - & - & -. & 0 & -41,667 & 166,667. \end{array}\right]}_{\mathbf{k}^{AB}}$$

$$\underbrace{\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ -0.61 \\ 0.0084 \\ 12.82 \end{array}\right\}}_{\delta^{AB}} - \underbrace{\left\{\begin{array}{c} 0 \\ -0.5 \\ -0.75 \\ 0 \\ -0.5 \\ 0.75 \end{array}\right\}}_{\mathbf{p}^{AB}_{el}}$$

or

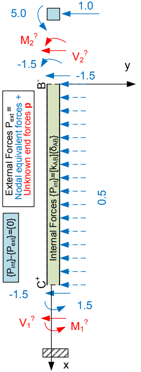
$$\left\{\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \hline p_4 \\ p_5 \\ p_6 \end{array}\right\} = \left\{\begin{array}{c} N_1 \\ V_1 \\ M_1 \\ \hline N_2 \\ V_2 \\ M_2 \end{array}\right\} = \left\{\begin{array}{c} 2.52 \text{ kN} \\ 1.03 \text{ kN} \\ 1.82 \text{ kN.m.} \\ -2.52 \text{ kN} \\ -0.034 \text{ kN} \\ 1.39 \text{ kN.m} \end{array}\right\}$$

12. For element BC, the local and global coordinates do not match, hence we will need to transform the displacements from their global to their local coordinate components. By inspection

Local	x	y	z
Global	$-Y$	$+X$	$+Z$

Note that there are no local or global displacements in dof 1-3, hence

$$\underbrace{\underbrace{\left\{\begin{array}{c} p_1^? \\ p_2^? \\ p_3^? \\ \hline p_4^? \\ p_5^? \\ p_6^? \end{array}\right\}}_{\mathbf{p}^{BC}_{int-for}} + \underbrace{\left\{\begin{array}{c} 0 \\ -1.5 \\ -1.5 \\ \hline 0 \\ -1.5 \\ 1.5 \end{array}\right\}}_{\mathbf{p}^{BC}_{el}}}_{\mathbf{p}^{BC}_{ext}} = \underbrace{\left[\begin{array}{ccc|ccc} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & 0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ 0 & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} & 0 & -\frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ \hline -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & 0 & \frac{12EI_y}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} & 0 & -\frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{array}\right]\left\{\begin{array}{c} \delta_4 \\ \delta_5 \\ \theta_6 \\ \hline 0 \\ 0 \\ 0 \end{array}\right\}}_{\mathbf{p}^{BC}_{int}} \tag{8}$$



$$\left\{\begin{array}{c} p_1^? \\ p_2^? \\ p_3^? \\ \hline p_4^? \\ p_5^? \\ p_6^? \end{array}\right\} = 10^6\left[\begin{array}{ccc|ccc} 4.1\times 10^6 & 0 & 0 & - & - & - \\ 0 & 13,888.9 & 41,666.7 & - & - & - \\ 0 & 41,666.7 & 16,6667. & - & - & - \\ -4.1\times 10^6 & 0 & 0 & - & - & - \\ 0 & -13,888.9 & -41,666.7 & - & - & - \\ 0 & 41,666.7 & 83,333.3 & - & - & - \end{array}\right] \tag{9}$$

$$\left\{\begin{array}{c} -0.0084 \\ -0.61 \\ 12.82 \\ 12.82 \\ 0 \\ 0 \\ 0 \end{array}\right\} - \left\{\begin{array}{c} 0 \\ -1.5 \\ -1.5 \\ 0 \\ -1.5 \\ 1.5 \end{array}\right\} = \left\{\begin{array}{c} N_1 \\ V_1 \\ M_1 \\ \hline N_2 \\ V_2 \\ M_2 \end{array}\right\} = \left\{\begin{array}{c} -0.034 \text{ kN} \\ 2.026 \text{ kN} \\ 3.612 \text{ kN.m} \\ 0.0344 \text{ kN} \\ 0.974 \text{ kN} \\ -0.456 \text{ kN.m} \end{array}\right\} \tag{10}$$

```
%% Stiffness Method Frame Example 09/18
% courtesy of Xiao Fu
clear all
clc

%% Elements properties
L_elem = [6; 6]; % m
A_elem = [0.123; 0.123]; % m^2
E_elem = [200E6; 200E6]; % kN/m^2
I_elem = [1250E-6; 1250E-6]; % m^4

%% Loads
P = 1;
M = 5;
w = 0.5;

%% Structure Displacements in GCS
% Assemble global stiffness matrix
K = [A_elem(1)*E_elem(1)/L_elem(1)+12*E_elem(2)*I_elem(2)/L_elem(2)^3, 0,...
6*E_elem(2)*I_elem(2)/L_elem(2)^2;
0, A_elem(2)*E_elem(2)/L_elem(2)+12*E_elem(1)*I_elem(1)/L_elem(1)^3,...
-6*E_elem(1)*I_elem(1)/L_elem(1)^2;
6*E_elem(2)*I_elem(2)/L_elem(2)^2, -6*E_elem(1)*I_elem(1)/L_elem(1)^2, ...
4*E_elem(1)*I_elem(1)/L_elem(1)+4*E_elem(2)*I_elem(2)/L_elem(2)]

% Determine vector of external forces
NEL = [-w*L_elem(2)/2; -P/2; P*L_elem(1)/8-w*L_elem(2)^2/12]; % Nodal Equivalent Load at DOFs
F = [-P/2; 0 ; M]; % Externally applied forces
F_ext = NEL + F;    % Total External Force

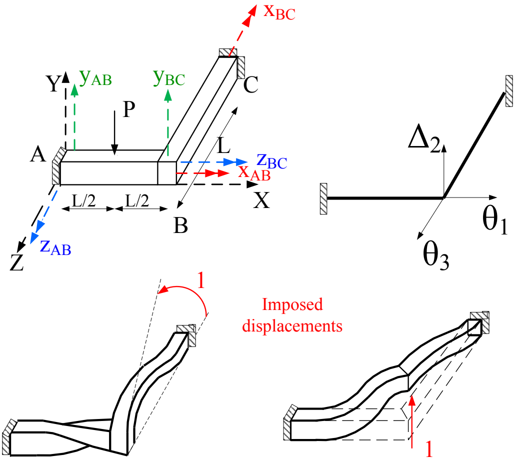
% Solve for Displacement
Disp = K\F_ext
```

```
%% Internal Forces

% Element-AB
 i = 1;
k_AB = stiff(E_elem(i), I_elem(i), L_elem(i), A_elem(i) ); % Element stiffness matrix in LCS
NEL_elem_AB = [0; -P/2; -P*L_elem(i)/8; 0; -P/2; P*L_elem(i)/8]; % nodal element forces in LCS
disp_elem_AB = [0; 0; 0; Disp(1); Disp(2); Disp(3)]; % global nodal displ. of AB in LCS
Force_elem_AB = k_AB*disp_elem_AB - NEL_elem_AB      % Internal forces of AB in LCS

% Element-BC
 i = 2;
k_BC = stiff(E_elem(2), I_elem(2), L_elem(2), A_elem(2) );
NEL_elem_BC = [0; -w*L_elem(i)/2; -w*L_elem(i)^2/12; 0; -w*L_elem(i)/2; w*L_elem(i)^2/12];
disp_elem_BC = [-Disp(2); Disp(1); Disp(3);0;0;0];
Force_elem_BC = k_BC*disp_elem_BC - NEL_elem_BC
```

```
function [k]=stiff(E,I,L,A)
EA=E*A; EI=E*I;
k=[
EA/L,          0,              0,    -EA/L,           0,           0;
0,     12*EI/L^3,   6*EI/L^2,        0,   -12*EI/L^3,   6*EI/L^2;
0,      6*EI/L^2,     4*EI/L,        0,    -6*EI/L^2,     2*EI/L;
-EA/L,         0,              0,     EA/L,           0,           0;
0, -12*EI/L^3,   -6*EI/L^2,        0,    12*EI/L^3,  -6*EI/L^2;
0,    6*EI/L^2,       2*EI/L,        0,    -6*EI/L^2,     4*EI/L];
```



The two elements have identical flexural and torsional rigidity, *EI* and *GJ*.

1. Identify the three degrees of freedom, θ_1, Δ_2, and θ_3.
2. Restrain all the degrees of freedom, and determine the nodal equivalent loads:

$$\left\{ \begin{array}{c} T_1 \\ V_2 \\ M_3 \end{array} \right\} = \underbrace{\left\{ \begin{array}{c} 0 \\ -\frac{P}{2} \\ -\frac{PL}{8} \end{array} \right\}}_{\text{@node A}} = \underbrace{\left\{ \begin{array}{c} 0 \\ -\frac{P}{2} \\ \frac{PL}{8} \end{array} \right\}}_{\text{@node B}}$$

3. Apply a unit displacement along each of the three degrees of freedom, and determine the internal forces:
   1. Apply unit rotation along global d.o.f. 1.
      1. AB (Torsion) $K_{11}^{AB} = \frac{GJ}{L}$, $K_{21}^{AB} = 0$, $K_{31}^{AB} = 0$
      2. BC (Flexure) $K_{11}^{BC} = \frac{4EI}{L}$, $K_{21}^{BC} = \frac{6EI}{L^2}$, $K_{31}^{BC} = 0$

2. Apply a unit translation along global d.o.f. 2.
   1. AB (Flexure): $K_{12}^{AB} = 0$, $K_{22}^{AB} = \frac{12EI}{L^3}$, $K_{32}^{AB} = -\frac{6EI}{L^2}$
   2. BC (Flexure): $K_{12}^{BC} = \frac{6EI}{L^2}$, $K_{22}^{BC} = \frac{12EI}{L^3}$, $K_{32}^{BC} = 0$
3. Apply unit rotation along global d.o.f. 3.
   1. AB (Flexure): $K_{13}^{AB} = 0$, $K_{23}^{AB} = -\frac{6EI}{L^2}$, $K_{33}^{AB} = \frac{4EI}{L}$
   2. BC (Torsion): $K_{13}^{BC} = 0$, $K_{23}^{BC} = 0$, $K_{33}^{BC} = \frac{GJ}{L}$

4. The structure stiffness matrix will now be assembled:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} = \begin{bmatrix} k_{44}^{AB}+k_{33}^{BC} & k_{45}^{AB}+k_{32}^{BC} & k_{46}^{AB}+k_{31}^{BC} \\ k_{54}^{AB}+k_{23}^{BC} & k_{55}^{AB}+k_{22}^{BC} & k_{56}^{AB}+k_{21}^{BC} \\ k_{64}^{AB}+k_{13}^{BC} & k_{55}^{AB}+k_{12}^{BC} & k_{66}^{AB}+k_{11}^{BC} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{GJ}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}}_{[\mathrm{K}_{AB}]} + \underbrace{\begin{bmatrix} \frac{4EI}{L} & \frac{6EI}{L^2} & 0 \\ \frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 \\ 0 & 0 & \frac{GJ}{L} \end{bmatrix}}_{[\mathrm{K}_{BC}]}$$

$$= \frac{EI}{L^3}\begin{bmatrix} \alpha L^2 & 0 & 0 \\ 0 & 12 & -6L \\ 0 & -6L & 4L^2 \end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix} 4L^2 & 6L & 0 \\ 6L & 12 & 0 \\ 0 & 0 & \alpha L^2 \end{bmatrix}$$

$$= \underbrace{\frac{EI}{L^3}\begin{bmatrix} (4+\alpha)L^2 & 6L & 0 \\ 6L & 24 & -6L \\ 0 & -6L & (4+\alpha)L^2 \end{bmatrix}}_{[\mathrm{K}_{Structure}]}$$

where $\alpha = \frac{GJ}{EI}$, and in the last equation it is assumed that for element BC, node 1 corresponds to C and 2 to B.

5. The structure equilibrium equation in matrix form:

$$\underbrace{\underbrace{\left\{\begin{array}{c}0\\0\\0\end{array}\right\}}_{\{\mathrm{P}_{nodes}\}}+\underbrace{\left\{\begin{array}{c}0\\-\frac{P}{2}\\\frac{PL}{8}\end{array}\right\}}_{P^B_{el}}}_{\mathrm{P}_{ext}}-\underbrace{\underbrace{\frac{EI}{L^3}\left[\begin{array}{ccc}(4+\alpha)L^2 & 6L & 0\\6L & 24 & -6L\\0 & -6L & (4+\alpha)L^2\end{array}\right]}_{[\mathrm{K}]}\underbrace{\left\{\begin{array}{c}\theta_1^?\\\Delta_2^?\\\theta_3^?\end{array}\right\}}_{\{\Delta\}}}_{\mathrm{P}_{int}}=\left\{\begin{array}{c}0\\0\\0\end{array}\right\}$$

or

$$\left\{\begin{array}{c}\theta_1\\\Delta_2\\\theta_3\end{array}\right\}=\left\{\begin{array}{c}\frac{PL^2}{16EI}\frac{5+2\alpha}{(1+\alpha)(4+\alpha)}\\-\frac{PL^3}{96EI}\frac{5+2\alpha}{1+\alpha}\\-\frac{3PL^2}{16EI}\frac{1}{(1+\alpha)(4+\alpha)}\end{array}\right\}$$

6. Internal forces: multiply each element stiffness matrix $[\mathrm{k}]$ with the vector of nodal displacement $\{\delta\}$. Note these operations should be accomplished in local coordinate system, and great care should be exercized in writing the nodal displacements in the same local coordinate system as the one used for the derivation of the element stiffness matrix.

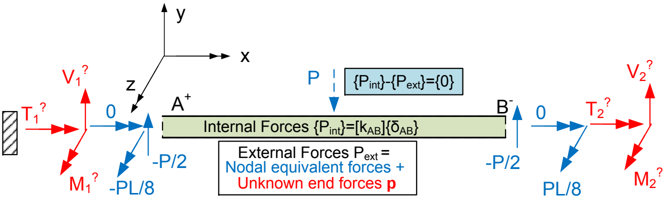
7. The mapping between local and global dof

$$\text{AB}\left\{\begin{array}{ccc} x & \leftarrow & X \\ y & \leftarrow & Y \\ z & \leftarrow & Z \end{array}\right. ; \qquad \text{BC}\left\{\begin{array}{ccc} x & \leftarrow & -Z \\ y & \leftarrow & Y \\ z & \leftarrow & X \end{array}\right.$$

8. For element AB and BC, the vector of nodal displacements are

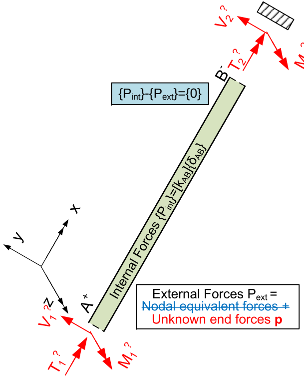
$$\left\{\begin{array}{c} \delta_1 \\ \delta_2 \\ \delta_3 \\ \hline \delta_4 \\ \delta_5 \\ \delta_6 \end{array}\right\} = \underbrace{\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline \theta_1 \\ \Delta_2 \\ \theta_3 \end{array}\right\}}_{AB} = \underbrace{\left\{\begin{array}{c} -\theta_3 \\ \Delta_2 \\ \theta_1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right\}}_{BC}$$

9. For element AB we have



$$
\underbrace{\underbrace{\left\{\begin{array}{c} p_1^? \\ p_2^? \\ p_3^? \\ \hline p_4^? \\ p_5^? \\ p_6^? \end{array}\right\}}_{\mathrm{p}^{AB}_{int-for}} + \underbrace{\left\{\begin{array}{c} 0 \\ -P/2 \\ -PL/8 \\ \hline 0 \\ -P/2 \\ PL/8 \end{array}\right\}}_{\mathrm{p}^{BC}_{el}}}_{\mathrm{p}^{AB}_{ext}} - \underbrace{\begin{array}{c} \begin{array}{cccccc} \alpha_{1x} & v_{1y} & \beta_{1z} & \alpha_{2x} & v_{2y} & \beta_{2z} \end{array} \\ \left[\begin{array}{cccccc} \frac{GI_{xx}}{L} & 0 & 0 & -\frac{GI_{xx}}{L} & 0 & 0 \\ 0 & \frac{12EI_{zz}}{L^3} & \frac{6EI_{zz}}{L^2} & 0 & -\frac{12EI_{zz}}{L^3} & \frac{6EI_{zz}}{L^2} \\ 0 & \frac{6EI_{zz}}{L^2} & \frac{4EI_{zz}}{L} & 0 & -\frac{6EI_{zz}}{L^2} & \frac{2EI_{zz}}{L} \\ -\frac{GI_{xx}}{L} & 0 & 0 & \frac{GI_{xx}}{L} & 0 & 0 \\ 0 & -\frac{12EI_{zz}}{L^3} & -\frac{6EI_{zz}}{L^2} & 0 & \frac{12EI_{zz}}{L^3} & -\frac{6EI_{zz}}{L^2} \\ 0 & \frac{6EI_{zz}}{L^2} & \frac{2EI_{zz}}{L} & 0 & -\frac{6EI_{zz}}{L^2} & \frac{4EI_{zz}}{L} \end{array}\right] \end{array} \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline \theta_1 \\ \Delta_2 \\ \theta_3 \end{array}\right\}}_{\mathrm{p}^{AB}_{int}} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right\}
$$

10. For element BC:



$$\underbrace{\underbrace{\left\{\begin{array}{c} p_1^? \\ p_2^? \\ p_3^? \\ \hline p_4^? \\ p_5^? \\ p_6^? \end{array}\right\}}_{\mathtt{p}^{BC}_{int-for}} + \underbrace{\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right\}}_{\mathtt{p}^{BC}_{el}}}_{\mathtt{p}^{BC}_{ext}} - \underbrace{\begin{array}{c} \begin{array}{cccccc} \alpha_{1x} & v_{1y} & \beta_{1z} & \alpha_{2x} & v_{2y} & \beta_{2z} \end{array} \\ \begin{bmatrix} \frac{GI_{xx}}{L} & 0 & 0 & -\frac{GI_{xx}}{L} & 0 & 0 \\ 0 & \frac{12EI_{zz}}{L^3} & \frac{6EI_{zz}}{L^2} & 0 & -\frac{12EI_{zz}}{L^3} & \frac{6EI_{zz}}{L^2} \\ 0 & \frac{6EI_{zz}}{L^2} & \frac{4EI_{zz}}{L} & 0 & -\frac{6EI_{zz}}{L^2} & \frac{2EI_{zz}}{L} \\ -\frac{GI_{xx}}{L} & 0 & 0 & \frac{GI_{xx}}{L} & 0 & 0 \\ 0 & -\frac{12EI_{zz}}{L^3} & -\frac{6EI_{zz}}{L^2} & 0 & \frac{12EI_{zz}}{L^3} & -\frac{6EI_{zz}}{L^2} \\ 0 & \frac{6EI_{zz}}{L^2} & \frac{2EI_{zz}}{L} & 0 & -\frac{6EI_{zz}}{L^2} & \frac{4EI_{zz}}{L} \end{bmatrix} \end{array} \left\{\begin{array}{c} -\theta_3 \\ \Delta_2 \\ \theta_1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right\}}_{\mathtt{p}^{BC}_{int}} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right\}$$

- Covered:
  - "rotation" of stiffness matrix from $\mathtt{k}^e$ to $\mathtt{K}^e$.
  - Displacements vectors from δ to Δ
  - Assembly of structural stiffness matrix $\mathtt{K}^S = \sum \mathtt{K}^e$.
- Next need to generalize the method to
  - Rotation matrices Γ for stiffness, displacements and forces.
  - Automate assembly process.
  - Write Matlab code.
- Address special topics
- Move to "classical" finite element method.

Intermediary Structural Analysis

Transformation Matrices

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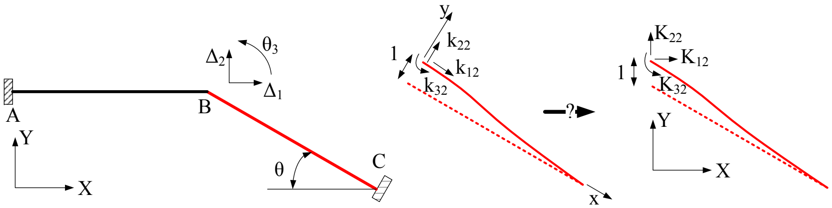
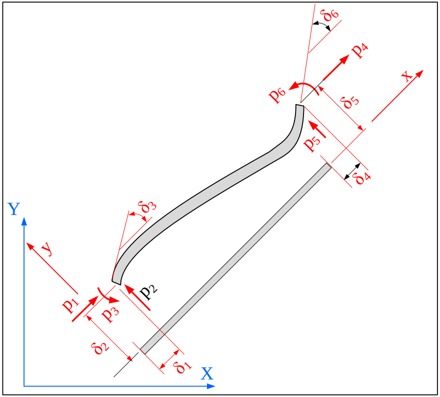
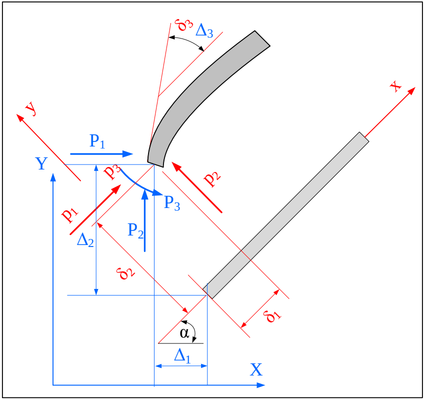
Fall 2021

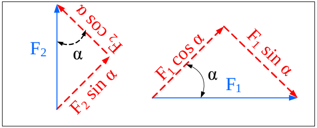
Table of Contents

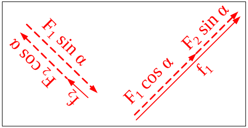
- Assembly of structure stiffness matrix is in global coordinate system, element stiffness matrix is first computed in local coordinate system.
- Need to transform $\mathtt{k}$ into $\mathtt{K}$ and δ into Δ for arbitrary structures.











$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\gamma} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

- Recall
$$\{\mathrm{p}^{(e)}\} = [\mathrm{k}^{(e)}]\{\delta^{(e)}\} \text{ and } \{\mathrm{P}^{(e)}\} = [\mathrm{K}^{(e)}]\{\Delta^{(e)}\} \quad (1)$$
- Let us define a vector transformation matrix $[\Gamma^{(e)}]$ such that:

$$\{\delta^{(e)}\} \stackrel{\text{def}}{=} [\Gamma^{(e)}]\{\Delta^{(e)}\} \text{ and } \{\mathrm{p}^{(e)}\} \stackrel{\text{def}}{=} [\Gamma^{(e)}]\{\mathrm{P}^{(e)}\} \quad (2)$$

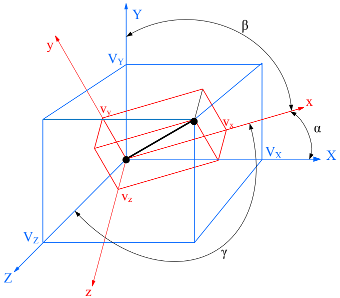
- Substituting we obtain $\{\mathrm{p}^{e}\} = [\Gamma^{(e)}]\{\mathrm{P}^{(e)}\} = [\mathrm{k}^{(e)}][\Gamma^{(e)}]\{\Delta^{(e)}\}$ premultiplying by $[\Gamma^{(e)}]^{-1}$: $\{\mathrm{P}^{(e)}\} = [\Gamma^{(e)}]^{-1}[\mathrm{k}^{(e)}][\Gamma^{(e)}]\{\Delta^{(e)}\}$
- But since the rotation matrix is orthogonal, we have $[\Gamma^{(e)}]^{-1} = [\Gamma^{(e)}]^{T}$ (and $\{\Delta^{(e)}\} = [\Gamma^{(e)}]^{T}\{\delta^{(e)}\}$)

$$\{\mathrm{P}^{(e)}\} = \underbrace{[\Gamma^{(e)}]^{T}[\mathrm{k}^{(e)}][\Gamma^{(e)}]}_{[\mathrm{K}^{(e)}]}\{\Delta^{(e)}\}$$

$$[\mathrm{K}^{(e)}] = [\Gamma^{(e)}]^{T}[\mathrm{k}^{(e)}][\Gamma^{(e)}] \quad (3)$$

which is the general relationship between element stiffness matrix in local and global coordinates.

$$\left\{ \begin{array}{c} v_x \\ v_y \\ v_z \end{array} \right\} = \underbrace{\left[\begin{array}{ccc} l_{xX} & l_{xY} & l_{xZ} \\ l_{yX} & l_{yY} & l_{yZ} \\ l_{zX} & l_{zY} & l_{zZ} \end{array} \right]}_{[\gamma]} \left\{ \begin{array}{c} V_X \\ V_Y \\ V_Z \end{array} \right\}$$



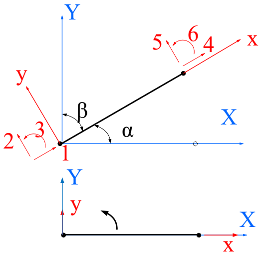
- l_{ij} is the direction cosine of axis i with respect to axis j.
- $l_{xX} = \cos(\alpha)$; $l_{xY} = \cos(\beta)$;
- Recall that $\cos(-\alpha) = \cos(\alpha)$, hence angle direction is irrelevant.
- The first row is given by (in terms of lower case $x - y - z$)

$$l_{xX} = C_X = \frac{x_j - x_i}{L}; \quad l_{xY} = C_Y = \frac{y_j - y_i}{L}; \quad l_{xZ} = C_Z = \frac{z_j - z_i}{L} \tag{4}$$

where $L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$.

- Determining other rows is best accomplished as follows in the next slides

At first, the local and global coordinate systems are superimposed, we then perform a rotation α with respect to the Z axis.

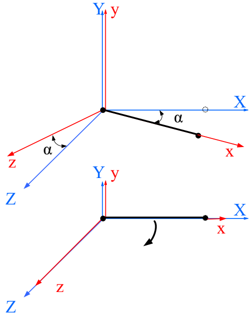


$$[\gamma] = \begin{bmatrix} l_{xX} & l_{xY} & l_{xZ} \\ l_{yX} & l_{yY} & l_{yZ} \\ l_{zX} & l_{zY} & 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \cos(\frac{\pi}{2}-\alpha) & 0 \\ \cos(\frac{\pi}{2}+\alpha) & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

We observe that the angles are defined from the second subscript to the first, and that counterclockwise angles are positive.

$$\left\{\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \hline p_4 \\ p_5 \\ p_6 \end{array}\right\} = \underbrace{\left[\begin{array}{ccc|ccc} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{[\Gamma]} \left\{\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ \hline P_4 \\ P_5 \\ P_6 \end{array}\right\} \tag{6}$$

At first, the local and global coordinate systems are superimposed, we then perform a rotation α with respect to the y axis.

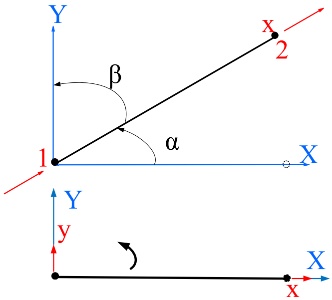


Rotation with respect to the Y axis.

$$[\gamma] = \begin{bmatrix} l_{xX} & 0 & l_{xZ} \\ l_{yX} & l_{yY} & l_{yZ} \\ l_{zX} & 0 & l_{zZ} \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \cos(\frac{\pi}{2}-\alpha) \\ 0 & 1 & 0 \\ \cos(\frac{\pi}{2}+\alpha) & 0 & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$\left\{ \begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \hline p_4 \\ p_5 \\ p_6 \end{array} \right\} = \underbrace{\left[\begin{array}{ccc|ccc} \cos\alpha & 0 & \sin\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos\alpha & 0 & \sin\alpha \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\sin\alpha & 0 & \cos\alpha \end{array} \right]}_{[\Gamma]} \left\{ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ \hline P_4 \\ P_5 \\ P_6 \end{array} \right\} \tag{7}$$



- At first, the local and global coordinate systems are superimposed, we then perform a rotation α with respect to the Z axis.
- Note that in local coordinate system, a truss has only 2 dof, while in the global one it has 2 or 3 (2D or 3D). Hence, γ will have only one row, and 2 or 3 columns.

Rotation with respect to the Z axis.

$$
\begin{array}{llllll}
[\gamma] & = \begin{bmatrix} l_{xX} & l_{xY} \end{bmatrix} & = \begin{bmatrix} Cx & Cy \end{bmatrix} & = \begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix} & \text{2D} \\
 & = \begin{bmatrix} l_{xX} & l_{xY} & l_{xZ} \end{bmatrix} & = \begin{bmatrix} Cx & Cy & Cz \end{bmatrix} & = \begin{bmatrix} \cos\alpha & \sin\alpha & 1 \end{bmatrix} & \text{3D}
\end{array}
$$

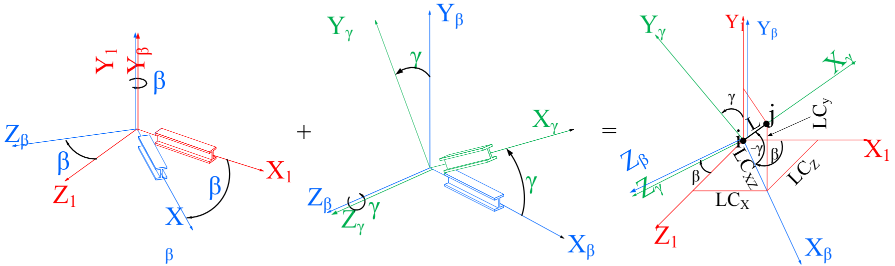
for 2D

$$
\left\{ \begin{array}{c} p_1 \\ \hline p_2 \end{array} \right\} = \begin{bmatrix} [\gamma] & 0 \\ 0 & [\gamma] \end{bmatrix} = \underbrace{\left[\begin{array}{cc|cc} \cos\alpha & \sin\alpha & 0 & 0 \\ \hline 0 & 0 & \cos\alpha & \sin\alpha \end{array} \right]}_{[\Gamma]} \left\{ \begin{array}{c} P_1 \\ P_2 \\ \hline P_3 \\ P_4 \end{array} \right\} \tag{8}
$$

- 2D elements are transformed through a single rotation (α).
- 3D elements are transformed through a minimum of 2, possibly 3 rotations through the eulerian angles β, γ and α.
- Start from X_1, Y_1, Z_1 and end with $X_\gamma, Y_\gamma, Z_\gamma$ or $X_\alpha, Y_\alpha, Z_\alpha$
- Start with the first row of the transformation matrix which corresponds to the direction cosines of the reference axis $(\mathrm{X}_1, \mathrm{Y}_1, \mathrm{Z}_1)$ with respect to X_2. This will define the first row of the vector rotation matrix $[\gamma]$:

$$[\gamma] = \begin{bmatrix} C_X & C_Y & C_Z \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \tag{9}$$

- Still have to define the second and third rows. This is achieved through through two successive rotations (assuming that $(X_1, Y_1, Z_1$ and $X_\beta, Y_\beta, Z_\beta$ are originally coincident) (assuming that the vertical axis of the member remains vertical)



From	To	With respect to	Angle
X_1, Y_1, Z_1	$X_\beta, Y_\beta, Z_\beta$	$Y_1 \equiv Y_\beta$	β
$X_\beta, Y_\beta, Z_\beta$	$X_\gamma, Y_\gamma, Z_\gamma$	$Z_\beta \equiv Z_\gamma$	γ
Optional			
$X_\gamma, Y_\gamma, Z_\gamma$	$X_\alpha, Y_\alpha, Z_\alpha$	$X_\gamma \equiv X_\alpha$	α

1. Rotation by β about the Y_1 axis, $X_1 \rightarrow X_\beta$. This rotation $[\mathrm{R}_\beta]$ is made of the direction cosines of the β axis $(X_\beta, Y_\beta, Z_\beta)$ with respect to (X_1, Y_1, Z_1):

$$[\mathrm{R}_\beta] = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} = \begin{bmatrix} \frac{C_X}{C_{XZ}} & 0 & \frac{C_Z}{C_{XZ}} \\ 0 & 1 & 0 \\ -\frac{C_Z}{C_{XZ}} & 0 & \frac{C_X}{C_{XZ}} \end{bmatrix}$$

$\cos\beta = \frac{C_X}{C_{XZ}}$, $\sin\beta = \frac{C_Z}{C_{XZ}}$, and from Eq. 4:

$$C_X = \frac{x_j - x_i}{L}; \quad C_Y = \frac{y_j - y_i}{L}; \quad C_Z = \frac{z_j - z_i}{L}; \quad C_{XZ} = \sqrt{C_X^2 + C_Z^2}$$

2. Rotation by γ about the Z axis

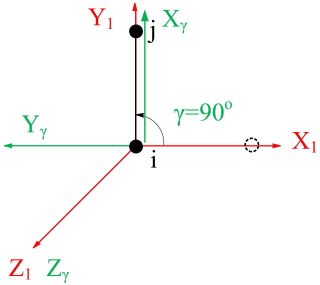
$$[\mathrm{R}_\gamma] = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{XZ} & C_Y & 0 \\ -C_Y & C_{XZ} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

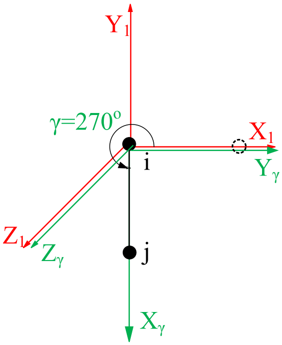
where $\cos\gamma = C_{XZ}$, and $\sin\gamma = C_Y$.

Combining yields:

$$[\gamma] = [R_\gamma][R_\beta] = \begin{bmatrix} C_X & C_Y & C_Z \\ \frac{-C_X C_Y}{C_{XZ}} & C_{XZ} & \frac{-C_Y C_Z}{C_{XZ}} \\ \frac{-C_Z}{C_{XZ}} & 0 & \frac{C_X}{C_{XZ}} \end{bmatrix} \tag{10}$$

For vertical member (along global Y) the preceding matrix is no longer valid as C_{XZ} is undefined ($X_i = X_j \Rightarrow C_X = 0$ and $Z_i = Z_j \Rightarrow C_Z = 0$). There is no rotation through β. Rotation is with respect to the Z_1 axis by an angle γ of 90^o or 270^o.





1. X_γ axis is aligned with Y_1

2. Y_γ axis is aligned with -X_1
3. Z_γ axis is aligned with Z_1

or

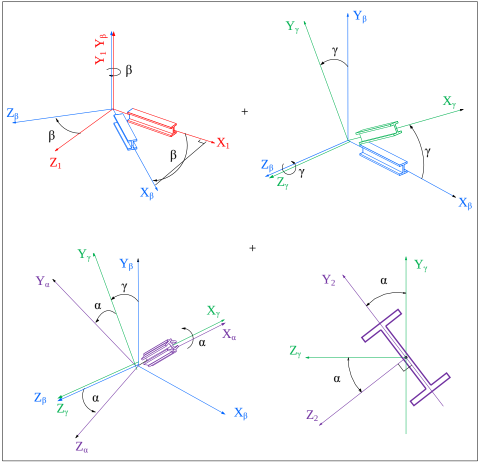
1. X_γ axis is aligned with -Y_1
2. Y_γ axis is aligned with -X_1
3. Z_γ axis is aligned with Z_1

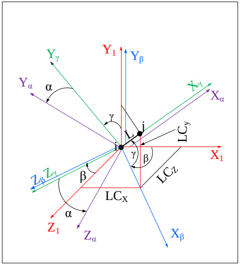
hence the rotation matrix with respect to the y axis, is similar to the one previously derived for rotation with respect to the z axis, except for the reordering of terms:

$$[R\gamma] = \begin{bmatrix} 0 & C_Y & 0 \\ -C_Y & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

which is valid for both cases ($C_Y = 1$ for $\gamma = 90^o$, and $C_Y = -1$ for $\gamma = 270^o$).

If the principal axes are to be rotated, then we need to define an additional rotation to the preceding transformation of an angle α about the X_γ axis.





This rotation is defined such that:

1. X_α is aligned with X_γ and normal to both Y_γ and Z_γ
2. Y_α makes an angle α with respect to Y_γ and $\beta = \frac{\pi}{2} - \alpha$
3. Z_α makes an angle 0, $\frac{\pi}{2} + \alpha$ and α, with respect to X_γ, Y_γ and Z_γ respectively

$\cos(\frac{\pi}{2} + \alpha) = -\sin\alpha$ and $\cos\beta = \sin\alpha$, the direction cosines of this transformation are given by:

$$[R_\alpha] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \tag{12}$$

causing the $Y_\gamma - Z_\gamma$ axis to coincide with the principal axes of the cross section. This will yield:

$$\begin{aligned}[\gamma] &= [R_\alpha][R_\gamma][R_\beta] \\ &= \begin{bmatrix} C_X & C_Y & C_Z \\ \frac{-C_X C_Y \cos\alpha - C_Z \sin\alpha}{C_{XZ}} & C_{XZ}\cos\alpha & \frac{-C_Y C_Z \cos\alpha + C_X \sin\alpha}{C_{XZ}} \\ \frac{C_X C_Y \sin\alpha - C_Z \cos\alpha}{C_{XZ}} & -C_{XZ}\sin\alpha & \frac{C_Y C_Z \sin\alpha + C_X \cos\alpha}{C_{XZ}} \end{bmatrix}\end{aligned} \tag{13}$$

As for the simpler case, the preceding equation is undefined for vertical members, and a counterpart to Eq. 11 must be derived. This will be achieved in two steps:

1. Rotate the member so that:
   1. X_γ axis aligned with Y_1
   2. Y_γ axis aligned with -X_1
   3. Z_γ axis aligned with Z_1

   this was previously done and resulted in Eq. 11

$$[R_\gamma] = \begin{bmatrix} 0 & C_Y & 0 \\ -C_Y & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. The second step consists in performing a rotation of angle α with respect to the new X_2 as defined in Eq. 12.

3. Finally, we multiply the two transformation matrices $[R_\gamma][R_\alpha]$ given by Eq. 14 to obtain:

$$[\gamma] = [R_\gamma][R_\alpha] = \begin{bmatrix} 0 & C_Y & 0 \\ -C_Y \cos\alpha & 0 & \sin\alpha \\ C_Y \sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{14}$$

Note with $\alpha = 0$, we recover Eq. 11.

$$\left\{ \begin{array}{c} F_{x1} \\ F_{y1} \\ F_{z1} \\ \hline M_{x1} \\ M_{y1} \\ M_{z1} \\ \hline F_{x2} \\ F_{y2} \\ F_{z2} \\ \hline M_{x2} \\ M_{y2} \\ M_{z2} \end{array} \right\} = \underbrace{\left[\begin{array}{c|c|c|c} [\gamma] & & & \\ \hline & [\gamma] & & \\ \hline & & [\gamma] & \\ \hline & & & [\gamma] \end{array} \right]}_{[\Gamma]} \left\{ \begin{array}{c} F_{X1} \\ F_{Y1} \\ F_{Z1} \\ \hline M_{X1} \\ M_{Y1} \\ M_{Z1} \\ \hline F_{X2} \\ F_{Y2} \\ F_{Z2} \\ \hline M_{X2} \\ M_{Y2} \\ M_{Z2} \end{array} \right\} \tag{15}$$

and should distinguish between the vector transformation $[\Gamma]$ and the element transformation matrix $[\gamma]$.

Intermediary Structural Analysis

Stiffness Method; II

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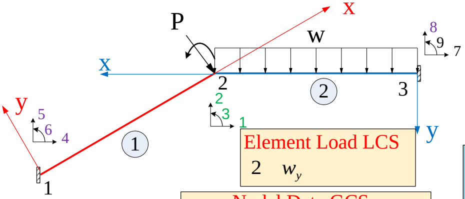
Fall 2021

Table of Contents I

- We know how to determine an individual element stiffness matrix in its local coordinate system.
- We have explored the stiffness method for orthogonal structures through a manual procedure, and assembled the global stiffness matrix in terms of free degrees of freedom (i.e. unconstrained).
- We have introduced the transformation matrices for various element types, and can determine the element stiffness matrix in system (i.e. global) coordinate system through $\mathtt{K}^{(e)} = \Gamma^{(e)T}\mathtt{k}^{(e)}\Gamma^{(e)}$.
- Next, we will generalize the stiffness method to
  1. address arbitrary structural geometries (i.e. non orthogonal).
  2. Determination of reactions through the use of augmented stiffness matrix.
  3. Describe algorithms to fully automate (i.e. write a computer program) the procedure.

- Numerical modeling of a structure requires that we can mathematically describe it (geometry, boundary conditions, geometry and properties of elements, and loads).



Element Load LCS

2	w_y

Node Computed ID

Node	$ID(i,1)$	$ID(i,2)$	$ID(i,3)$
1	4	5	6
2	1	2	3
3	7	8	9

Elem	LM					
1	4	5	6	1	2	3
2	7	8	9	1	2	3

Element Data

Elem	*i*	*j*	*Group*
1	1	2	1
2	3	2	1

Nodal Data GCS

Node	*X*	*Y*	*X*	*Y*	*Z*
1	0.	0.	1	1	1
2	7.42	3.	0	0	0
3	15.42	3.	1	1	1

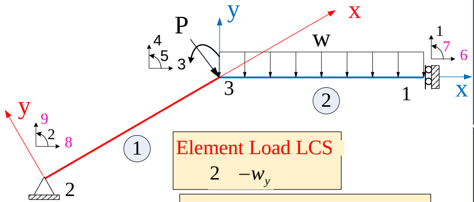
Group Data

Group	*E*	*I*	*A*
1	29000	500	12

Nodal Load GCS

2	P_X	P_Y	P_Z

Note: LM Locator Matrix



Element Load LCS

2	$-w_y$

Node Computed ID

Node	ID(i,1)	ID(i,2)	ID(i,3)
1	6	1	7
2	8	9	2
3	3	4	5

LM

Elem						
1	8	9	2	3	4	5
2	3	4	5	6	1	7

Element Data

Elem	i	j	Group
1	2	3	2
2	3	1	1

Nodal Data GCS

Node	X	Y	X	Y	Z
1	15.42	3.	1	0	1
2	0.	0.	1	1	0
3	7.42	3.	0	0	0

Group Data

Group	E	I	A
1	29000	500	12
2	29000	750	24

Nodal Load GCS

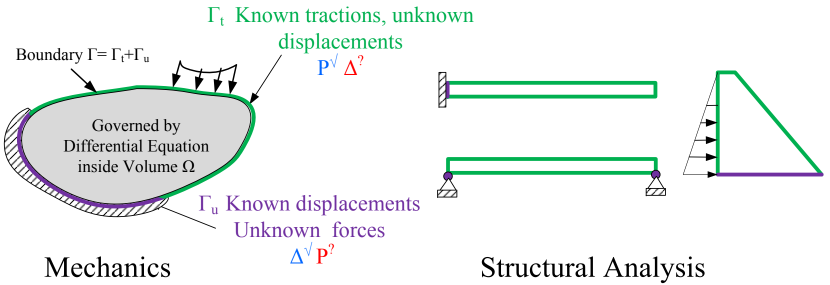
2	P_X	P_Y	P_Z

Structural idealization is as much an art as a science.

1. 2D vs 3D
2. Frame or truss
3. Rigid or semi-rigid connections
4. Rigid supports or elastic foundations
5. Include or not secondary members
6. Include or not axial deformation
7. Cross sectional properties
8. Neglect or not haunches
9. Linear or nonlinear analysis (linear analysis can not predict the peak or failure load, and will underestimate the deformations).
10. Small or large deformations
11. Time dependent effects
12. Partial collapse or local yielding
13. Static or dynamic
14. Wind load
15. Thermal load
16. Secondary stresses
17. ...

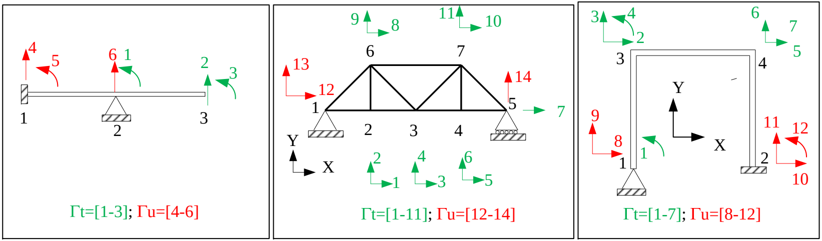
- Analysis of a structure is essentially solving a boundary value problem (governed by a differential equation over the volume Ω, and subjected to space/temporal boundary conditions along the boundary Γ).
- In our case we are discretizing our structure, and the governing differential equation (equilibrium) is embedded in $\mathrm{K}\Delta = \mathrm{P}$.
- $\Gamma = \Gamma_t \bigcup \Gamma_u$

Γ	Traction	Displ.	Math.	Struct.	DOF
Γ_t	$\mathrm{P}_t^{\surd}$	$\Delta_t^?$	Neuman	Essential	Free
Γ_u	$\mathrm{R}_u^?$	$\Delta_u^{\surd}$	Dirichlet	Natural	Fixed/Constrained



For the beam and the dam, we need to determine the displacements along Γ_t and the forces (reactions) along Γ_u.

- We have labeled the global dof associated with the unconstrained dof (Γ_t), where we solve for the displacements.
- We will need to label the global dof associated with the constrained dof (Γ_u) where we will solve for the reactions.
- We will label the dof along Γ_t first, and then those along Γ_u next.
- We have so far considered the reduced stiffness matrix (associated with Γ_t only).
- We will need to assemble the augmented stiffness matrix associated with $\Gamma = \Gamma_t \bigcup \Gamma_u$



- **Global** coordinate system

$$\mathrm{K}^e = \Gamma^T \mathrm{k}^e \Gamma \tag{1}$$

$$\mathrm{K}^S = \sum_{e=1}^{e=nelem} \mathrm{K}^e \tag{2}$$

$$\left\{\frac{\mathrm{P}_t^{\surd}}{\mathrm{R}_u^{?}}\right\} = \underbrace{\left[\begin{array}{c|c} \mathrm{K}_{tt} & \mathrm{K}_{tu} \\ \hline \mathrm{K}_{ut} & \mathrm{K}_{uu} \end{array}\right]}_{\text{Augmented Stiffness Matrix}} \left\{\frac{\Delta_t^{?}}{\Delta_u^{\surd}}\right\} \tag{3}$$

$$\mathrm{K}_{tt} = \mathrm{f}^{-1}; \text{ Reduced Stiffness Matrix} \tag{4}$$

$$\Delta_t = \mathrm{K}_{tt}^{-1} \underbrace{(\mathrm{P}_t - \mathrm{K}_{tu}\Delta_u)}_{\mathrm{P}_t'} \tag{5}$$

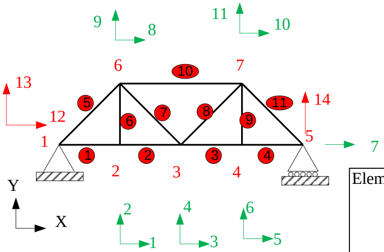
$$\mathrm{R}_u = \mathrm{K}_{ut}\Delta_t + \mathrm{K}_{uu}\Delta_u \tag{6}$$

- **Local** coordinate system

$$\delta^{(e)} = \Gamma^{(e)}\Delta^{(e)} \tag{7}$$

$$p_{int}^{(e)} = \mathrm{k}^{(e)}\delta^{(e)} \tag{8}$$

Note effect of P_{el} not included for clarity



$$
\begin{array}{c|cc}
\text{DOF} & 1 & 2
\end{array}
\qquad
[ID]=\begin{bmatrix}1&1\\0&0\\0&0\\0&0\\0&1\\0&0\\0&0\end{bmatrix}\Rightarrow
\begin{array}{l}
\begin{bmatrix}12&13\\1&2\\3&4\\5&6\\7&14\\8&9\\10&11\end{bmatrix}
\end{array}
\begin{array}{l}\text{Node 1}\\\text{Node 2}\\\text{Node 3}\\\text{Node 4}\\\text{Node 5}\\\text{Node 6}\\\text{Node 7}\end{array}
$$

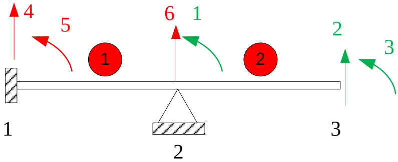
$$
\begin{array}{cc}
\multicolumn{2}{c}{\text{Nodes} \quad i \quad j}
\end{array}
\qquad
[LNODS]=\begin{bmatrix}1&2\\2&3\\3&4\\4&5\\1&6\\2&6\\3&6\\3&7\\7&4\\6&7\\7&5\end{bmatrix}
\begin{array}{l}\text{Element 1}\\\text{Element 2}\\\text{Element 3}\\\text{Element 4}\\\text{Element 5}\\\text{Element 6}\\\text{Element 7}\\\text{Element 8}\\\text{Element 9}\\\text{Element 10}\\\text{Element 11}\end{array}
$$

Element global dof

$$
\begin{array}{cccc}
1 & 2 & 3 & 4
\end{array}
\qquad
[LM]=\left[\begin{array}{cc|cc}
12&13&1&2\\
1&2&3&4\\
3&4&5&6\\
5&6&7&14\\
12&13&8&9\\
1&2&8&9\\
3&4&8&9\\
3&4&10&11\\
10&11&5&6\\
8&9&10&11\\
10&11&7&14
\end{array}\right]
\begin{array}{l}\text{Element 1}\\\text{Element 2}\\\text{Element 3}\\\text{Element 4}\\\text{Element 5}\\\text{Element 6}\\\text{Element 7}\\\text{Element 8}\\\text{Element 9}\\\text{Element 10}\\\text{Element 11}\end{array}
$$

LM is a mapping between element and structure global dof

[ ] User defined
[ ] Computed by program



$$
\begin{array}{cc}
 & \begin{array}{cc} i & j \end{array} \\
[LNODS] = & \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{array}{l} \text{Element 1} \\ \text{Element 2} \end{array}
\end{array}
$$

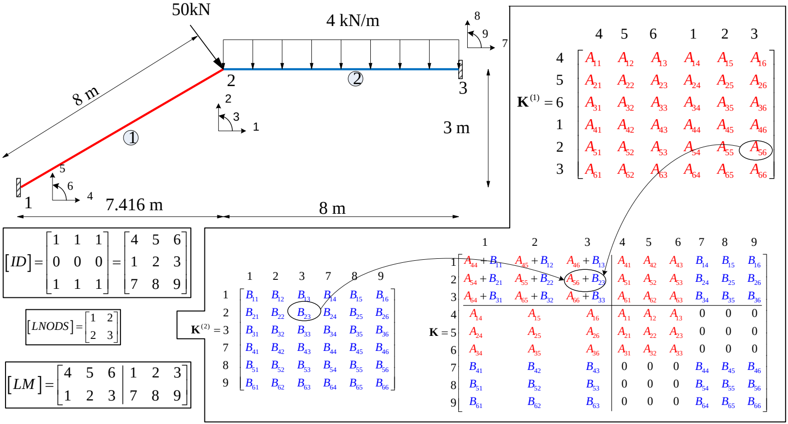
(Node: i, j)

Global DOF

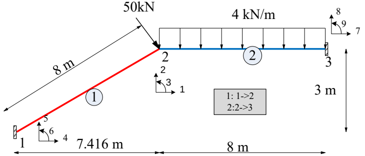
$$
\begin{array}{cc}
 & \begin{array}{cc} 1 & 2 \end{array} \\
[ID] = & \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 5 \\ 6 & 1 \\ 2 & 3 \end{bmatrix} \begin{array}{l} \text{Node 1} \\ \text{Node 2} \\ \text{Node 3} \end{array}
\end{array}
$$

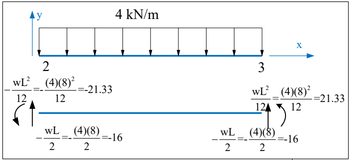
Element global dof

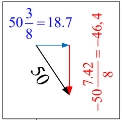
$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
[LM] = & \left[\begin{array}{cc|cc} 4 & 5 & 6 & 1 \\ 6 & 1 & 2 & 3 \end{array}\right] \begin{array}{l} \text{Element 1} \\ \text{Element 2} \end{array}
\end{array}
$$



$K_{ij}^{(e)} \rightarrow K_{st}^{(S)}$ and $\left\{ \begin{array}{ccc} s & = & LM(e,i) \\ t & = & LM(e,j) \end{array} \right.$ $[LM]$ is a mapping between the element global dof and the structure's (global) dof.

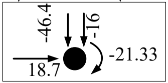






$$\{p_{El}^{(2)}\}=\begin{Bmatrix}0.\\-16.0\\-21.33\\0.0\\-16.0\\21.33\end{Bmatrix};\{P_{El}^{(2)}\}=\underbrace{\begin{bmatrix}\text{Cos }\alpha & \text{Sin }\alpha & 0 & 0 & 0 & 0\\ \text{-Sin }\alpha & \text{Cos }\alpha & 0 & & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \text{Cos }\alpha & \text{Sin }\alpha & 0\\ 0 & 0 & 0 & \text{-Sin }\alpha & \text{Cos }\alpha & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}}_{\Gamma^{(2)}}\underbrace{\begin{Bmatrix}0.\\-16.0\\-21.33\\0.0\\-16.0\\21.33\end{Bmatrix}}_{p_{El}^{(2)}}=\begin{Bmatrix}0.\\-16.0\\-21.33\\0.0\\-16.0\\21.33\end{Bmatrix}$$

$$[LM]=\left[\begin{array}{ccc|ccc}4 & 5 & 6 & 1 & 2 & 3\\ 1 & 2 & 3 & 7 & 8 & 9\end{array}\right]$$



$\Rightarrow P_t(LM^{(e)}(i))=P_{NEF}^{(e)}(i)+P_t(LM^{(e)}(i));\quad \forall\ LM^{(e)}(i)\le size(K_{tt})$
$size(K_{tt})=3$; Corresponds to number of unconstrained dof

$$LM(2,1)=1\le 3\Rightarrow\quad P_t(LM(2,1))=P_{El}^{(2)}(1)+P_{nod}(LM(2,1))\ \Rightarrow\ P_t(1)=0+P_{nod}(1)\quad =0+18.7\quad =18.7$$
$$LM(2,2)=2\le 3\Rightarrow\quad P_t(LM(2,2))=P_{El}^{(2)}(2)+P_{nod}(LM(2,2))\Rightarrow\ P_t(2)=-16+P_{nod}(2)\quad =-16-46.4\quad =-62.4$$
$$LM(2,3)=3\le 3\Rightarrow\quad P_t(LM(2,3))=P_{El}^{(2)}(3)+P_{nod}(LM(2,3))\ \Rightarrow\ P_t(3)=-21.33+P_{nod}(3)=0-21.33\quad =-21.33$$

1. Preliminaries
   1. Read the structure mathematical model (type, coordinates, connectivity, cross-sectional and material properties, loads)
   2. Determine the number of nodes (`nnode`), number of element (`nelem`), maximum number dof/node (`ndofpn`), size of K_{tt} (`sizet`), total number of dof (`ndoft`), update ID and determine LM matrices
2. Analysis, Global:
   1. For each element, determine
      1. Vector LM mapping local element to global structure degrees of freedoms.
      2. Element stiffness matrix $[\mathbf{k}^{(e)}]$
      3. Transformation matrix $[\Gamma^{(e)}]$
      4. Element stiffness matrix in global coordinates $[\mathrm{K}^{(e)}] = [\Gamma^{(e)}]^T[\mathbf{k}^{(e)}][\Gamma^{(e)}]$
   2. Assemble the augmented stiffness matrix $[\mathrm{K}^{(S)}] = \sum_{e=1}^{nelem} \mathbf{k}^{(e)}$ of unconstrained and constrained degree of freedom's.
   3. Extract $[\mathrm{K}_{tt}]$ from $[\mathrm{K}^{(S)}]$ and invert (actually decompose).
   4. Load Vector
      1. Compute nodal equivalent forces vectors for each element in local coordinate system $\mathbf{p}_{EI}^{(e)}$ and in global coordinate system $\mathrm{P}_{EI}^{(e)} = \Gamma^{(e)^T}\mathbf{p}_{EI}^{(e)}$

2. Assemble the nodal load vector to include nodal loads and nodal equivalent forces (note P is for the structure).
$\mathrm{P}_t = \sum_{e=1}^{nelem} \mathrm{P}_{El}^{(e)} + \mathrm{P}_{nodes}(LM^{(e)}(i)); \forall LM^{(e)} \leq \text{size}(K_{tt})$

5. Backsubstitute and obtain nodal displacements global coordinate system, $\Delta = \mathrm{K}_{tt}^{-1}\mathrm{P}_t$
6. Extract K_{ut}
7. Determine P_R that will store

$$\mathrm{P}_R(1\text{:ndof-sizet}) = \sum_{e=1}^{nelem} \Gamma^{(e)}\mathrm{p}_{el}^{(e)}; \forall LM^{(e)} > \text{size}(K_{tt}) \tag{9}$$

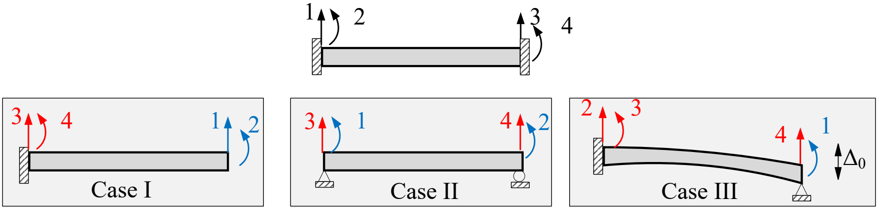
Those are the element load transformed to the global coordinate system for those degrees of freedom that are fixed. Hence they will affect the reaction.

8. Solve for the reactions, $\mathrm{R}_u = \mathrm{K}_{ut}\Delta_t + \mathrm{K}_{uu}\Delta_u - \mathrm{P}_R$

3. Analysis, Local; Internal forces: for each element

1. Determine the element nodal displacements in global coordinate system from the global nodal displacements
2. Transform its nodal displacement from global to local coordinates $\delta^{(e)} = [\Gamma^{(e)}]\Delta^{(e)}$.

3. Determine the internal forces $\mathbf{p}^{(e)} = \mathbf{k}^{(e)}\boldsymbol{\delta}^{(e)} - \mathbf{p}_{EI}^{(e)}$.



We consider the third case, a cantilevered Beam with initial Displacement and no other load.

1. The LM matrix is $LM = \lfloor \; 2 \quad 3 \quad 4 \quad 1 \; \rfloor$
2. The *element* stiffness matrix is

$$
\mathbf{k}^{(e)} = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 1 \quad \end{array} \\ \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & 6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}
$$

3. The augmented *structure* stiffness matrix is assembled

$$
\mathrm{K}^{(S)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} \\
\begin{bmatrix}
4EI/L & 6EI/L^2 & 2EI/L & -6EI/L^2 \\
6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\
2EI/L & 6EI/L^2 & 4EI/L & 6EI/L^2 \\
-6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3
\end{bmatrix}
\end{array}
$$

4. The global augmented matrix can be decomposed as

$$
\left\{ \begin{array}{c} M_1(=0)\surd \\ \hline R_2? \\ R_3? \\ R_4? \end{array} \right\} = \left[\begin{array}{c|ccc}
4EI/L & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline
6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\
2EI/L & 6EI/L^2 & 4EI/L & 6EI/L^2 \\
-6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3
\end{array} \right] \left\{ \begin{array}{c} \theta_1 ? \\ \hline \Delta_2 \surd \\ \theta_3 \surd \\ \Delta_4 \surd \end{array} \right\}
$$

5. K_{tt} is inverted (or actually decomposed) and stored in the same global matrix storage location

$$\left[\begin{array}{c|ccc} \boxed{L/4EI} & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array}\right]$$

6. Next we compute the equivalent load, $\mathrm{P}'_t = \mathrm{P}_t - \mathrm{K}_{tu}\Delta_u$, and overwrite P_t by P'_t (Note that we are boxing terms of interest only).

$$\mathrm{P}_t - \mathrm{K}_{tu}\Delta_u = \left\{\begin{array}{c} \boxed{M_1 = 0} \\ \hline R_2? \\ R_3? \\ R_4? \end{array}\right\} - \left[\begin{array}{c|ccc} L/4EI & \boxed{6EI/L^2} & \boxed{2EI/L} & \boxed{-6EI/L^2} \\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} \theta_1 \\ \hline \boxed{0} \\ \boxed{0} \\ \boxed{\Delta_0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} \boxed{6EI\Delta_0/L^2} \\ \hline R_2? \\ R_3? \\ R_4? \end{array}\right\}$$

7. Solve for the displacements from $\Delta_t = \mathrm{K}_{tt}^{-1}(\mathrm{P}_t - \mathrm{K}_{tu}\Delta_u)$ and overwrite P_t by Δ_t

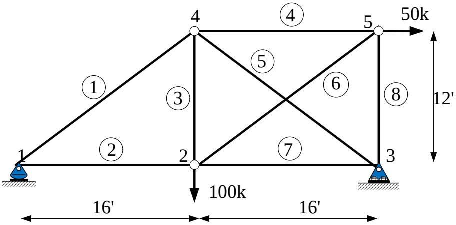
$$\left\{\begin{array}{c} \boxed{\theta_1^?} \\ \hline 0 \\ 0 \\ \Delta_0^{\surd} \end{array}\right\} = \left[\begin{array}{c|ccc} \boxed{L/4EI} & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline 6EI/L^2 & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 \\ 2EI/L & 6EI/L^2 & 4EI/L & -6EI/L^2 \\ -6EI/L^2 & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 \end{array}\right] \left\{\begin{array}{c} \boxed{6EI\Delta_0/L^2} \\ \hline R_2? \\ R_3? \\ R_4? \end{array}\right\}$$

$$= \left\{\begin{array}{c} 3\Delta_0/2L \\ \hline 0 \\ 0 \\ 0 \end{array}\right\}$$

8. Finally, we solve for the reactions, $\mathrm{R}_u = \mathrm{K}_{ut}\Delta_{tt} + \mathrm{K}_{uu}\Delta_u$, and overwrite Δ_u by R_u

$$\left\{\begin{array}{c} M_1 \\ \hline \boxed{R_2} \\ \boxed{R_3} \\ \boxed{R_4} \end{array}\right\} = \left[\begin{array}{c|ccc} L/4EI & 6EI/L^2 & 2EI/L & -6EI/L^2 \\ \hline \boxed{6EI/L^2} & \boxed{12EI/L^3} & \boxed{6EI/L^2} & \boxed{-12EI/L^3} \\ \boxed{2EI/L} & \boxed{6EI/L^2} & \boxed{4EI/L} & \boxed{-6EI/L^2} \\ \boxed{-6EI/L^2} & \boxed{-12EI/L^3} & \boxed{-6EI/L^2} & \boxed{12EI/L^3} \end{array}\right] \left\{\begin{array}{c} \boxed{3\Delta_0/2L} \\ \hline \boxed{0} \\ \boxed{0} \\ \boxed{\Delta_0} \end{array}\right\}$$

$$= \left\{\begin{array}{c} -6EI\Delta^0/L^2 \\ \hline \boxed{-3EI\Delta^0/L^3} \\ \boxed{-3EI\Delta^0/L^2} \\ \boxed{3EI\Delta^0/L^3} \end{array}\right\}$$



1. Degrees of freedom and LM (connectivity: from lower to higher node number)

$$ID = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 2 & 3 \\ 9 & 10 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}; \qquad [LM] = \begin{bmatrix} 1 & 8 & 4 & 5 \\ 1 & 8 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 4 & 5 \\ 2 & 3 & 6 & 7 \\ 2 & 3 & 9 & 10 \\ 9 & 10 & 6 & 7 \end{bmatrix}$$

2. Element stiffness matrix

$$[K^{(e)}] = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$$c = \cos\alpha = \frac{x_2 - x_1}{L}; \quad s = \sin\alpha = \frac{Y_2 - Y_1}{L}$$

3. Substitute

Element 1: $L = 20'$, $c = \frac{16-0}{20} = 0.8$, $s = \frac{12-0}{20} = 0.6$,
$\frac{EA}{L} = \frac{(30{,}000 ksi)(10 in^2)}{20'} = 15{,}000\ k/ft.$

$$[K_1] = \begin{array}{c} \\ 1 \\ 8 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} \quad 1 \quad & \quad 8 \quad & \quad 4 \quad & \quad 5 \quad \end{array} \\ \begin{bmatrix} 9{,}600 & 7200 & -9{,}600 & -7{,}200 \\ 7{,}200 & 5{,}400 & -7{,}200 & -5{,}400 \\ -9{,}600 & -7{,}200 & 9{,}600 & 7{,}200 \\ -7{,}200 & -5{,}400 & 7{,}200 & 5{,}400 \end{bmatrix} \end{array}$$

Element 2: $L = 16'$, $c = 1$, $s = 0$, $\frac{EA}{L} = 18{,}750$ k/ft.

$$
[K_2] = \begin{array}{c} \\ 1 \\ 8 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{cccc} 1 & 8 & 2 & 3 \end{array} \\
\begin{bmatrix} 18,750 & 0 & -18,750 & 0 \\ 0 & 0 & 0 & 0 \\ -18,750 & 0 & 18,750 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}
$$

Element 3 $L = 12'$, $c = 0$, $s = 1$, $\frac{EA}{L} = 25,000\ k/ft$

$$
[K_3] = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 25,000 & 0 & -25,000 \\ 0 & 0 & 0 & 0 \\ 0 & -25,000 & 0 & 25,000 \end{bmatrix} \end{array}
$$

Element 8 $L = 12'$, $c = 0$, $s = 1$, $\frac{EA}{L} = 25,000\ k/ft$

$$
[K_8] = \begin{array}{c} \\ 9 \\ 10 \\ 6 \\ 7 \end{array}
\begin{array}{c} \begin{array}{cccc} 9 & 10 & 6 & 7 \end{array} \\
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 25,000 & 0 & -25,000 \\ 0 & 0 & 0 & 0 \\ 0 & -25,000 & 0 & 25,000 \end{bmatrix} \end{array}
$$

4. Assemble the global stiffness matrix in k/ft Note that we are not assembling the augmented stiffness matrix, but rather its submatrix $[\mathrm{K}_{tt}]$.

5. Convert to k/in and simplify

$$\underbrace{\left\{\begin{array}{c} 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ 50 \\ 0 \end{array}\right\}}_{\mathrm{P}_t} = \underbrace{\left[\begin{array}{ccccccc} 2,362.5 & -1,562.5 & 0.00 & -800 & -600 & 0 & 0 \\ & 3,925.0 & 600 & 0 & 0 & -800 & -600 \\ & & 2,533.33 & 0.00 & -2,083.33 & -600 & -450 \\ & & & 3,162.5 & 0 & -1,562.5 & 0 \\ & & \mathrm{SYMMETRIC} & & 2,983.33 & 0 & 0 \\ & & & & & 2,362.5 & 600 \\ & & & & & & 2,533.33 \end{array}\right]}_{\mathrm{K}_{tt}} \underbrace{\left\{\begin{array}{c} U_1 \\ U_2 \\ V_3 \\ U_4 \\ V_5 \\ U_6 \\ V_7 \end{array}\right\}}_{\mathrm{u}_t}$$

6. Invert stiffness matrix and solve for displacements

$$\left\{\begin{array}{c} U_1 \\ U_2 \\ V_3 \\ U_4 \\ V_5 \\ U_6 \\ V_7 \end{array}\right\} = \left\{\begin{array}{c} -0.0223 \text{ in.} \\ 0.00433 \text{ in.} \\ -0.116 \text{ in.} \\ -0.0102 \text{ in.} \\ -0.0856 \text{ in.} \\ -0.00919 \text{ in.} \\ -0.0174 \text{ in.} \end{array}\right\}$$

7. Solve for member internal forces (in this case axial forces) in local coordinate systems

$$\left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\} = \underbrace{\frac{AE}{L}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right]}_{\mathbf{k}} \overbrace{\underbrace{\left[\begin{array}{cccc} C & S & 0 & 0 \\ 0 & 0 & C & S \end{array}\right]}_{\Gamma} \underbrace{\left\{\begin{array}{c} U_1 \\ V_1 \\ U_2 \\ V_2 \end{array}\right\}}_{\Delta}}_{\delta}$$

$$= \frac{AE}{L}\left[\begin{array}{cccc} c & s & -c & -s \\ -c & -s & c & s \end{array}\right]\left\{\begin{array}{c} U_1 \\ V_1 \\ U_2 \\ V_2 \end{array}\right\}$$

Element 1:

$$\left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\}^1 = (15,000 \text{ kpf})\left(\frac{1}{12}\frac{\text{ft.}}{\text{in.}}\right)\left[\begin{array}{cccc} 0.8 & 0.6 & -0.8 & -0.6 \\ -0.8 & -0.6 & 0.8 & 0.6 \end{array}\right]\left\{\begin{array}{c} -0.0223 \\ 0.00 \\ -0.0102 \\ -0.0856 \end{array}\right\}$$

$$= \left\{\begin{array}{c} 52.1 \text{ kip} \\ -52.1 \text{ kip} \end{array}\right\} \text{ Compression}$$

Element 2:

$$\left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\}^2 = 18,750 \text{ kpf} \left(\frac{1}{12}\frac{\text{ft}}{\text{in.}}\right) \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \left\{\begin{array}{c} -0.0233 \\ 0.00 \\ 0.00433 \\ -0.116 \end{array}\right\}$$

$$= \left\{\begin{array}{c} -43.2 \text{ kip} \\ 43.2 \text{ kip} \end{array}\right\} \text{ Tension}$$

Element 3:

$$\left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\}^3 = 25,000 \text{ kpf} \left(\frac{1}{12}\frac{\text{ft.}}{\text{in.}}\right) \left[\begin{array}{cccc} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{array}\right] \left\{\begin{array}{c} 0.00433 \\ -0.116 \\ -0.0102 \\ -0.0856 \end{array}\right\}$$

$$= \left\{\begin{array}{c} -63.3 \text{ kip} \\ 63.3 \text{ kip} \end{array}\right\} \text{ Tension}$$

Element 4:

$$\left\{\begin{array}{c} p_1 \\ p_2 \end{array}\right\}^4 = 18,750 \text{ kpf} \left(\frac{1}{12}\frac{\text{ft.}}{\text{in.}}\right) \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \left\{\begin{array}{c} -0.0102 \\ -0.0856 \\ -0.00919 \\ -0.0174 \end{array}\right\}$$

$$= \left\{\begin{array}{c} -1.58 \text{ kip} \\ 1.58 \text{ kip} \end{array}\right\} \text{ Tension}$$

Intermediary Structural Analysis

Mathematical Properties of Stiffness Matrix; Computational Issues;

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Fall 2019

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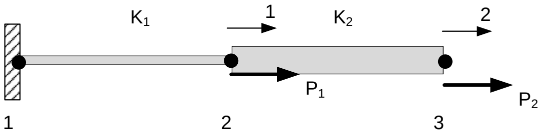


$$K^{(1)} = k1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad K^{(2)} = k2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix};$$

$$K^{(s)} = \begin{bmatrix} k1+k2 & -k2 \\ -k2 & k2 \end{bmatrix}$$

- The structure stiffness matrix and its inverse are given by

$$\mathbf{K} = \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}$$

$$\mathbf{K}^{-1} = \frac{1}{(K_1 + K_2)K_2 - K_2^2}\begin{bmatrix} K_2 & K_2 \\ K_2 & K_1 + K_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{K_1} & \frac{1}{K_1} \\ \frac{1}{K_1} & \frac{K_1+K_2}{K_1 K_2} \end{bmatrix}$$

where $\mathbf{K}.\mathbf{\Delta} = \mathbf{P}$

- Solution for the displacement vector is

$$\left\{ \begin{array}{c} \Delta_1 \\ \Delta_2 \end{array} \right\} = \left[\begin{array}{cc} \frac{1}{K_1} & \frac{1}{K_1} \\ \frac{1}{K_1} & \frac{K_1+K_2}{K_1 K_2} \end{array} \right] \left\{ \begin{array}{c} P_1 \\ P_2 \end{array} \right\}$$

- We rearrange to obtain two equations for $\Delta_2 = f(\Delta_1)$.

$$\left. \begin{array}{rcl} \Delta_1 & = & \frac{P_1}{K_1} + \frac{P_2}{K_1} \Rightarrow P_1 = K_1\Delta_1 - P_2 \\ \Delta_2 & = & \frac{P_1}{K_1} + \frac{P_2(K_1+K_2)}{K_1 K_2} \end{array} \right\} \Delta_2 = \Delta_1 + \frac{1}{K_2} P_2 \tag{1}$$

Likewise

$$\left. \begin{array}{rcl} \Delta_1 & = & \frac{P_1}{K_1} + \frac{P_2}{K_1} \Rightarrow P_2 = K_1\Delta_1 - P_1 \\ \Delta_2 & = & \frac{P_1}{K_1} + \frac{P_2(K_1+K_2)}{K_1 K_2} \end{array} \right\} \Delta_2 = \frac{K_1+K_2}{K_2}\Delta_1 - \frac{1}{K_2} P_1 \tag{2}$$

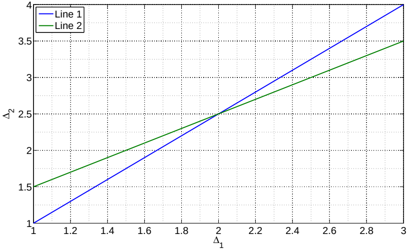
- Δ_2 can be expressed in terms of Δ_1, P_1 and P_2.

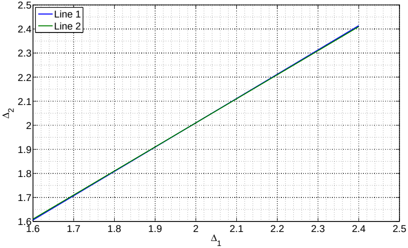
$$\left[\begin{array}{cc} -1 & 1 \\ \frac{K_1+K_2}{K_1 K_2} & -1 \end{array} \right] \left\{ \begin{array}{c} \Delta_1 \\ \Delta_2 \end{array} \right\} = \frac{1}{K_2} \left\{ \begin{array}{c} P_2 \\ P_1 \end{array} \right\} \tag{3}$$

- Let us consider two cases:

$$\begin{array}{lcll}\lfloor K_1 \quad K_2 \rfloor & = & \lfloor 1 \quad 2 \rfloor & \Rightarrow \begin{bmatrix} -1.0000 & 1.0000 \\ 1.5000 & -1.0000 \end{bmatrix} \\ \lfloor K_1 \quad K_2 \rfloor & = & \lfloor 1 \quad 10,000 \rfloor & \Rightarrow \begin{bmatrix} -1.0000 & 1.0000 \\ 1.0001 & 1.0000 \end{bmatrix}\end{array}$$

- plot the solutions for Δ_2 in terms of Δ_1 with $\lfloor P_1 \quad P_2 \rfloor = \lfloor 1 \quad 1 \rfloor$



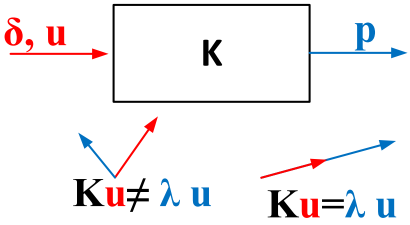


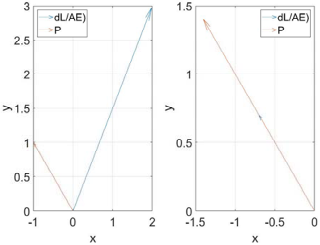
- Results, including eigenvalues λ_i give

	Well Conditioned		Ill Conditioned	
	K_1	K_2	K_1	K_2
	1.0	2.0	1.0	10,000
(Δ_1, Δ_2)	(4.0000, 5.0000)		10^4(1.999999999999668, 2.000099999999668)	
(λ_1, λ_2)	(0.2247,-2.2247)		(0.000049998750062, -2.000049998750063)	
C	9.8990		4.0002e+04	
Matlab 64 bit `eps`			2.2204e-16	

- The condition number (C) of a matrix is define as $\lambda_{max}/\lambda_{min}$.
- We lose accuracy with very large condition numbers. As rule of thumb a matrix is said to be ill-conditioned when the condition number ($\sim 1/(\texttt{eps})$ is larger than the reciprocal of the machine's precision, e.g., 10^7 for typical single precision (32 bit) arithmetic, and 10^{16} for 64 bit computer.
- Elements with drastically different stiffness values should not be connected together.
- For severely ill-conditioned matrices, use single value decomposition techniques.

- The stiffness matrix $[k]$ (or $[K]$) can be viewed as a mapping of the displacement vector $\{\delta\}$ into a force vector $\{p\}$.
- There is no reason for those vectors to be aligned.





for instance

$$\underbrace{\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\mathbf{k}^{truss}} \underbrace{\begin{Bmatrix} 2 \\ 3 \end{Bmatrix}}_{\mathbf{u}} = \underbrace{\frac{AE}{L}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}}_{\mathbf{p}} \tag{4}$$

- If on the other hand, those two vectors point in the same direction, then they are eigenvectors $\{u\}$ and we have
$$[k]\{u\} = \lambda\{u\}$$
In the preceding example, the eigenvector is $\lfloor -0.707 \quad 0.707 \rfloor$
- The internal strain energy stored in an element can be determined from $U = \frac{1}{2}\lfloor \mathbf{u} \rfloor \{\mathbf{p}\} = \frac{1}{2}\lfloor \mathbf{u} \rfloor [\mathbf{k}]\{\mathbf{u}\}$
- Consider a system where the load $\{\mathbf{p}\}$ applied to each node is proportional to the element nodal displacement $\{\mathbf{u}\}$ by a factor λ, we have: $[\mathbf{k}]\{\mathbf{u}\} = \lambda\{\mathbf{u}\}$ or $([\mathbf{k}] - \lambda[\mathbf{I}])\{\mathbf{u}\} = 0$
- This is by definition an eigenproblem. There will be as many eigenvalues λ_i as there are degrees of freedom (or rows in $[\mathbf{k}]$).
- To each eigenvalue λ_i corresponds an eigenvector $\{\mathbf{u}\}_i$.
- Eigenvectors are normalized such that: $\lfloor \mathbf{u} \rfloor_i \{\mathbf{u}\}_i = 1$, thus $\lfloor \mathbf{u} \rfloor_i [\mathbf{k}]\{\mathbf{u}\}_i = \lambda_i$
- Thus, the eigenvalue λ_i is equal to twice the internal strain energy stored in an element undergoing a (normalized) deformation defined by $\{\mathbf{u}\}_i$.
- In a rigid body motion, all nodes displace by the same amount, and there are no internal strains. Hence in a rigid body motion the strain energy U (and thus corresponding λ) must be equal to zero.

- There should be as many zero eigenvalues as there are possible independent rigid body motions (i.e. number of equations of equilibrium).
- For a two dimensional Lagrangian element, there should be three zero eigenvalues, corresponding to two translations and one rotation.
- Too few zero eigenvalues is an indication of an element lacking the capability of rigid body motion without strain.
- Too many zero eigenvalues is an indication of undesirable mechanism (or failure).
- Eigenvalues should not change when the element is rotated.
- Similar modes (such as flexure in two orthogonal directions) will have identical eigenvalues (for isotropic material).
- When comparing the stiffness matrices of two identical elements but based on different formulations, the one with the lowest strain energy (tr$[\mathbf{k}] = \Sigma\lambda_i$) is best.
- Hence, the element stiffness matrix will have:

  Order: corresponds to the number of degrees of freedom (i.e size of the matrix).

  Rank: corresponds to the total number of linearly independent equations which is equal to the order minus the number of rigid body motions.

  Rank Deficiency: would be equal to the total number of zero eigenvalues minus the rank.

- The augmented stiffness matrix may be expressed as (this equation will be derived late)

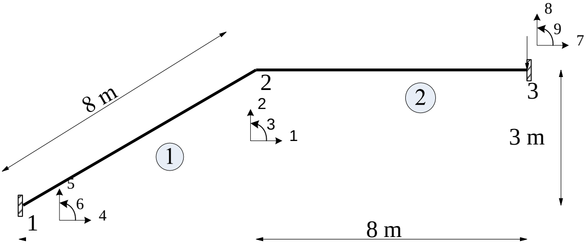
$$[\mathbf{K}] = \left[\begin{array}{c|c} [\mathbf{d}]^{-1} & [\mathbf{d}]^{-1}[\mathcal{B}]^T \\ \hline [\mathcal{B}][\mathbf{d}]^{-1} & [\mathcal{B}][\mathbf{d}]^{-1}[\mathcal{B}]^T \end{array}\right]$$

  where $\mathcal{B}$ is the statics (or equilibrium) matrix, relating external nodal forces to internal forces; $\mathbf{d}$ is a flexibility matrix, and $\mathbf{d}^{-1}$ is its inverse or reduced stiffness matrix.

- The stiffness matrix is obviously singular, since the second “row” is linearly dependent on the first one.

- The reduced stiffness matrix, which is the inverse of a flexibility matrix, is not.

- Hence there will be as many zero eigenvalues as the size of $\mathcal{B}$.



$$\lambda_{Kaug} = 10^5 \lfloor\; 0.0000 \quad 0.0000 \quad 0.0000 \quad 0.0059 \quad 0.1274 \quad 0.2201 \quad 0.4821 \quad 1.6109 \quad 4.3911 \;\rfloor$$

$$\lambda_{Ktt} = 10^5 \lfloor\; 0.1275 \quad 0.3999 \quad 2.8913 \;\rfloor$$

$$\lambda_{k} = 10^5 \lfloor\; 0.0000 \quad 0.0000 \quad 0.0000 \quad 0.1000 \quad 0.3188 \quad 3.0001 \;\rfloor$$

$$\lambda_{K^1} = 10^5 \lfloor\; 0.0000 \quad 0.0000 \quad 0.0000 \quad 0.1000 \quad 0.3188 \quad 3.0001 \;\rfloor$$

$$\lambda_{K^2} = 10^5 \lfloor\; 0.0000 \quad 0.0000 \quad 0.0000 \quad 0.1000 \quad 0.3187 \quad 3.0000 \;\rfloor$$

Add plots of eigenvectors showing rigid body motion

Intermediary Structural Analysis

A Brief Overview of Mechanics

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Fall 2017

Table of Contents I

- Generalize the concept of a vector by introducing the tensor ($\mathbf{T}$).
- A tensor is an operator which operates on tensors to produce other tensors.
- Designate this operation as $\mathbf{T}\cdot\mathbf{v}$ or simply $\mathbf{T}\mathbf{v}$.
- A tensor is also a physical quantity, independent of any particular coordinate system yet specified most conveniently by referring to an appropriate system of coordinates.
- A tensor is classified by the rank or order. A Tensor of order zero is specified in any coordinate system by one coordinate and is a scalar (such as temperature). A tensor of order one has three coordinate components in space, hence it is a vector (such as force). In general 3-D space the number of components of a tensor is 3^n where n is the order of the tensor.
- A force and a stress are tensors of order 1 and 2 respectively.

- Engineering notation may be the simplest and most intuitive one, it often leads to long and repetitive equations. Alternatively, tensor or the dyadic form will lead to shorter and more compact forms.
- The following rules define indicial notation:
  1. If there is one letter index (free index), that index goes from i to n (range of the tensor). For instance:

$$a_i = a^i = \lfloor \; a_1 \quad a_2 \quad a_3 \; \rfloor = \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right\} \qquad i = 1, 3$$

     assuming that $n = 3$.
  2. A repeated index or (dummy index) will take on all the values of its range, and the resulting tensors summed. In general no index occurs more than twice in a properly written expression.For instance:

$$a_{1i}x_i = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$

  3. Tensor's order:

- First order tensor (such as force) has only one free index:

$$a_i = a^i = \lfloor\ a_1 \quad a_2 \quad a_3\ \rfloor$$

  other first order tensors $a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$, F_{ikk}, $\varepsilon_{ijk}u_jv_k$ (note that there is only one free index).

- Second order tensor (such as stress or strain) will have two free indices.

$$T_{ij} = \begin{bmatrix} T_{11} & T_{22} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

  other examples A_{ijip}, $\delta_{ij}u_kv_k$.

- A fourth order tensor (such as Elastic constants) will have four free indices: $\sigma_{ij} = D_{ijkl}\varepsilon_{kl}$

4. Derivatives of tensor with respect to x_i is written as $, i$. For example:

$$\frac{\partial\phi}{\partial x_i} = \phi_{,i} \quad \frac{\partial v_i}{\partial x_i} = v_{i,i} \quad \frac{\partial v_i}{\partial x_j} = v_{i,j} \quad \frac{\partial T_{i,j}}{\partial x_k} = T_{i,j,k}$$

- Usefulness of the indicial notation is in presenting systems of equations in compact form. For instance:

$$x_i = c_{ij} z_j$$

this simple compacted equation, when expanded would yield:

$$\begin{aligned} x_1 &= c_{11} z_1 + c_{12} z_2 + c_{13} z_3 \\ x_2 &= c_{21} z_1 + c_{22} z_2 + c_{23} z_3 \\ x_3 &= c_{31} z_1 + c_{32} z_2 + c_{33} z_3 \end{aligned}$$

Similarly:

$$A_{ij} = B_{ip} C_{jq} D_{pq}$$

$$\begin{aligned} A_{11} &= B_{11} C_{11} D_{11} + B_{11} C_{12} D_{12} + B_{12} C_{11} D_{21} + B_{12} C_{12} D_{22} \\ A_{12} &= B_{11} C_{21} D_{11} + B_{11} C_{22} D_{12} + B_{12} C_{21} D_{21} + B_{12} C_{22} D_{22} \\ A_{21} &= B_{21} C_{11} D_{11} + B_{21} C_{12} D_{12} + B_{22} C_{11} D_{21} + B_{22} C_{12} D_{22} \\ A_{22} &= B_{21} C_{21} D_{11} + B_{21} C_{22} D_{12} + B_{22} C_{21} D_{21} + B_{22} C_{22} D_{22} \end{aligned}$$

- Using indicial notation, we may rewrite the definition of the dot product

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i = (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \cdot (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}) = a_x b_x + a_y b_y + a_z b_z$$

- Note that one can adopt the dyadic instead of the indicial notation for tensors as **linear vector operators** $\mathbf{u} = \mathbf{T}{\cdot}\mathbf{v}$ or $u_i = T_{ij}v_j$

- The sum of two tensors (must be of the same orde)is simply defined as:

$$\mathbf{S}_{ij} = \mathbf{T}_{ij} + \mathbf{U}_{ij}$$

- The scalar multiplication of a (second order) tensor is defined by:

$$\mathbf{S}_{ij} = \lambda \mathbf{T}_{ij}$$

- The outer product of two tensors is the tensor whose components are formed by multiplying each component of one of the tensors by every component of the other. This produces a tensor with an order equal to the sum of the orders of the factor tensors.

$$\begin{aligned} a_i b_j &= T_{ij} \quad \text{or} \left\{ \quad \right\}_{nx1} \lfloor \qquad \rfloor_{1xm} = \left[\qquad \right]_{nxm} \\ v_i F_{jk} &= b_{ijk} \\ D_{ij} T_{km} &= \phi_{ijkm} \end{aligned}$$

- The inner product of two tensors: contraction of one index from each tensor

$$
\begin{aligned}
&a_i b_i \\
a_i E_{ik} &= f_k \quad \text{or} \lfloor \quad \rfloor_{1xm} \begin{bmatrix} \quad \end{bmatrix}_{mxn} = \lfloor \quad \rfloor_{1xn} \\
E_{ij} F_{jm} &= G_{im} \quad \text{or} \begin{bmatrix} \quad \end{bmatrix}_{nxp} \begin{bmatrix} \quad \end{bmatrix}_{pxm} = \begin{bmatrix} \quad \end{bmatrix}_{nxm}
\end{aligned}
$$

- The cross product can be defined

$$\mathbf{a} \times \mathbf{b} = \varepsilon_{pqr} a_q b_r \mathbf{e}_p = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

In the second equation, there is one free index p thus there are three equations, there are two repeated (dummy) indices q and r, thus each equation has nine terms. ε_{pqr} is called the permutation symbol and is defined as

$$
\varepsilon_{pqr} = \begin{cases} 1 & \text{If the value of} i, j, k \text{are an even permutation of 1,2,3} \\ & \text{(i.e. if they appear as 1 2 3 1 2)} \\ -1 & \text{If the value of} i, j, k \text{are an odd permutation of 1,2,3} \\ & \text{(i.e. if they appear as 3 2 1 3 2)} \\ 0 & \text{If the value of} i, j, k \text{are not permutation of 1,2,3} \\ & \text{(i.e. if two or more indices have the same value)} \end{cases}
$$

- Two fundamental tensors in continuum mechanics are second order and symmetric (stress and strain), we examine some important properties of these tensors.
- For every symmetric tensor T_{ij} defined at some point in space, there is associated with each direction (specified by unit normal n_j) at that point, a vector given by the inner product

$$v_i = T_{ij} n_j$$

If the direction is one for which v_i is parallel to n_i, the inner product is

$$T_{ij} n_j = \lambda n_i$$

and the direction n_i is called principal direction of T_{ij}. Since $n_i = \delta_{ij} n_j$, this can be rewritten as

$$(T_{ij} - \lambda \delta_{ij}) n_j = 0$$

which represents a system of three equations for the four unknowns n_i and λ.

$$\begin{aligned} (T_{11} - \lambda) n_1 + T_{12} n_2 + T_{13} n_3 &= 0 \\ T_{21} n_1 + (T_{22} - \lambda) n_2 + T_{23} n_3 &= 0 \\ T_{31} n_1 + T_{32} n_2 + (T_{33} - \lambda) n_3 &= 0 \end{aligned}$$

To have a non-trivial solution ($n_i = 0$) the determinant of the coefficients must be zero,

$$|T_{ij} - \lambda\delta_{ij}| = 0$$

- Expansion of this determinant leads to the following characteristic equation

$$\lambda^3 - I_T\lambda^2 + II_T\lambda - III_T = 0$$

the roots are called the principal values of T_{ij} and

$$\begin{aligned} I_T &= T_{ij} = \text{tr } T_{ij} \\ II_T &= \frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ij}) \\ III_T &= |T_{ij}| = \det T_{ij} \end{aligned}$$

are called the first, second and third invariants respectively of T_{ij}.

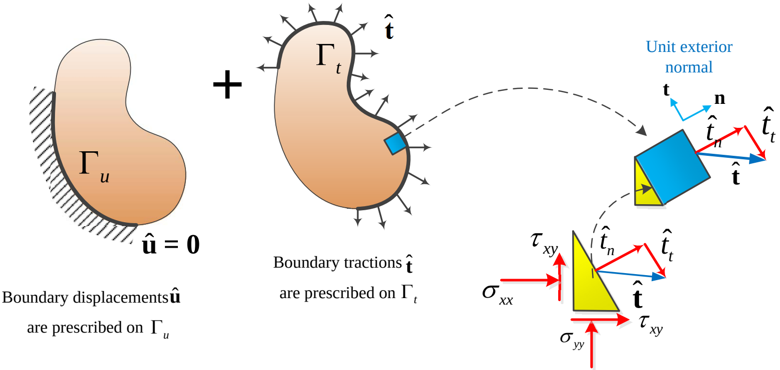
- It is customary to order those roots as $\lambda_{(1)} > \lambda_{(2)} > \lambda_{(3)}$
- For a symmetric tensor with real components, the principal values are also real. If those values are distinct, the three principal directions are mutually orthogonal.

- There are two kinds of forces in continuum mechanics

  body forces: act on the elements of volume or mass inside the body, e.g. gravity, electromagnetic fields. $\mathrm{d}\mathbf{F} = \rho \mathbf{b} \mathrm{d} Vol$.

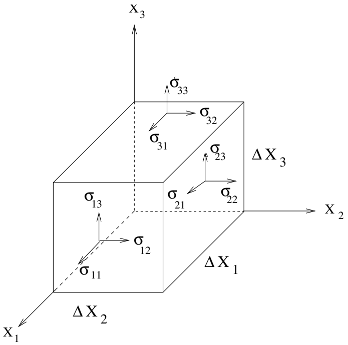
  Surface forces (or traction) are contact forces acting on the free body at its bounding surface. Those will be defined in terms of force per unit area.

$$\int_S \mathbf{t} \mathrm{d}S = \mathbf{i} \int_S t_x \mathrm{d}S + \mathbf{j} \int_S t_y \mathrm{d}S + \mathbf{k} \int_S t_z \mathrm{d}S$$



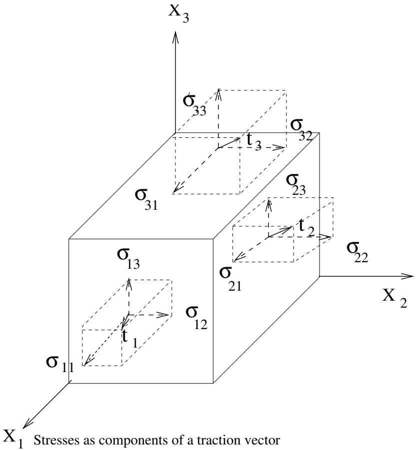
- Usually limit the term traction to an actual bounding surface of a body, and use the term stress vector for an imaginary interior surface.

- The traction vectors on planes perpendicular to the coordinate axes are particularly useful. When the vectors acting at a point on three such mutually perpendicular planes is given, the stress vector at that point on any other arbitrarily inclined plane can be expressed in terms of the first set of tractions.
- A stress is a second order cartesian tensor, σ_{ij} where the 1st subscript (i) refers to the direction of outward facing normal, and the second one (j) to the direction of component force.



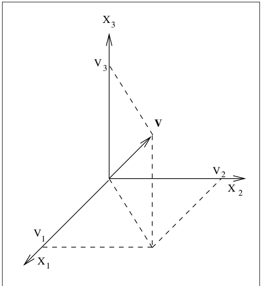
$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \left\{ \begin{array}{c} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{array} \right\}$$

- In fact the nine rectangular components σ_{ij} of $\boldsymbol{\sigma}$ turn out to be the three sets of three vector components $(\sigma_{11}, \sigma_{12}, \sigma_{13})$, $(\sigma_{21}, \sigma_{22}, \sigma_{23})$, $(\sigma_{31}, \sigma_{32}, \sigma_{33})$ which correspond to the three tractions $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$ which are acting on the x_1, x_2 and x_3 faces.
- Those tractions are not necessarily normal to the faces, and they can be decomposed into a normal and shear traction if need be. In other words, stresses are nothing else than the components of tractions (stress vector).



Stresses as components of a traction vector

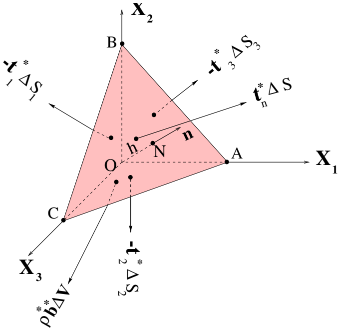
(Components of a tensor of order 2 are vectors)



(Components of a vector are scalars)

- The state of stress at a point cannot be specified entirely by a single vector with three components; it requires the second-order tensor with all nine components.

- We seek to determine the traction acting on the surface of an oblique plane (characterized by its normal $\mathbf{n}$) in terms of the known tractions normal to the three principal axis, $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$.
- Cauchy's tetrahedron



will be obtained without any assumption of equilibrium and it will apply in fluid dynamics as well as in solid mechanics.

- This equation is a vector equation, and the corresponding algebraic equations for the components of $\mathbf{t}_n$ are

$$
\begin{aligned}
t_{n_1} &= \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3 \\
t_{n_2} &= \sigma_{12}n_1 + \sigma_{22}n_2 + \sigma_{32}n_3 \\
t_{n_3} &= \sigma_{13}n_1 + \sigma_{23}n_2 + \sigma_{33}n_3
\end{aligned}
$$

or

Indicial notation $$t_{n_i} = \sigma_{ji}n_j$$

dyadic notation $$\mathbf{t}_n = \mathbf{n}\cdot\boldsymbol{\sigma} = \boldsymbol{\sigma}^T\cdot\mathbf{n}$$

- We have thus established that the nine components σ_{ij} are components of the second order tensor, Cauchy's stress tensor.

- For a stress tensor at point P given by

$$\boldsymbol{\sigma} = \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \left\{ \begin{array}{c} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{array} \right\}$$

  We seek to determine the traction (or stress vector) $\mathbf{t}$ passing through P and parallel to the plane ABC where $A(4,0,0)$, $B(0,2,0)$ and $C(0,0,6)$.

- The vector normal to the plane can be found by taking the cross products of vectors $\mathbf{AB}$ and $\mathbf{AC}$:

$$\begin{aligned} \mathbf{N} &= \mathbf{AB}\times\mathbf{AC} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -4 & 2 & 0 \\ -4 & 0 & 6 \end{vmatrix} \\ &= 12\mathbf{e}_1 + 24\mathbf{e}_2 + 8\mathbf{e}_3 \end{aligned}$$

- The unit normal of N is given by

$$\mathbf{n} = \frac{3}{7}\mathbf{e}_1 + \frac{6}{7}\mathbf{e}_2 + \frac{2}{7}\mathbf{e}_3$$

Hence the stress vector (traction) will be

$$\lfloor\ \frac{3}{7}\quad \frac{6}{7}\quad \frac{2}{7}\ \rfloor \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \lfloor\ -\frac{9}{7}\quad \frac{5}{7}\quad \frac{10}{7}\ \rfloor$$

and thus $\mathbf{t} = -\frac{9}{7}\mathbf{e}_1 + \frac{5}{7}\mathbf{e}_2 + \frac{10}{7}\mathbf{e}_3$

- The principal stresses are physical quantities, whose values do not depend on the coordinate system in which the components of the stress were initially given. They are therefore invariants of the stress state.
- When the determinant in the characteristic equation is expanded, the cubic equation takes the form

$$\lambda^3 - I_\sigma \lambda^2 - II_\sigma \lambda - III_\sigma = 0$$

where the symbols I_σ, II_σ and III_σ denote the following scalar expressions in the stress components:

$$\begin{aligned}
I_\sigma &= \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{ii} = \text{tr } \boldsymbol{\sigma} \\
II_\sigma &= -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2 \\
&= \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}) = \frac{1}{2}\sigma_{ij}\sigma_{ij} - \frac{1}{2}I_\sigma^2 \\
&= \frac{1}{2}(\boldsymbol{\sigma} : \boldsymbol{\sigma} - I_\sigma^2) \\
III_\sigma &= \det \boldsymbol{\sigma} = \frac{1}{6} e_{ijk} e_{pqr} \sigma_{ip}\sigma_{jq}\sigma_{kr}
\end{aligned}$$

- In terms of the principal stresses, those invariants can be simplified into

$$
\begin{aligned}
I_\sigma &= \sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)} \\
II_\sigma &= -(\sigma_{(1)}\sigma_{(2)} + \sigma_{(2)}\sigma_{(3)} + \sigma_{(3)}\sigma_{(1)}) \\
III_\sigma &= \sigma_{(1)}\sigma_{(2)}\sigma_{(3)}
\end{aligned}
$$

- let σ denote the mean normal stress p

$$\sigma = -p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{ii} = \frac{1}{3}\text{tr}\,\boldsymbol{\sigma}$$

then the stress tensor can be written as the sum of two tensors:

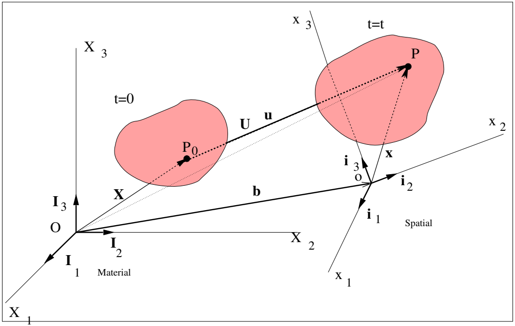
Hydrostatic stress in which each normal stress is equal to $-p$ and the shear stresses are zero. The hydrostatic stress produces volume change without change in shape in an isotropic medium.

$$\sigma_{hyd} = -p\mathbf{I} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

Deviatoric Stress: which causes the change in shape.

$$\sigma_{dev} = \begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{bmatrix}$$

- The undeformed configuration of a material continuum at time $t = 0$ together with the deformed configuration at $t = t$.



- In the initial configuration P_0 has the position vector

$$\mathbf{X} = X_1\mathbf{I}_1 + X_2\mathbf{I}_2 + X_3\mathbf{I}_3$$

which is here expressed in terms of the material coordinates (X_1, X_2, X_3).

- In the deformed configuration, the particle P_0 has now moved to the new position P and has the following position vector

$$\mathbf{x} = x_1 \mathbf{i}_1 + x_2 \mathbf{i}_2 + x_3 \mathbf{i}_3$$

which is expressed in terms of the spatial coordinates.

- The displacement vector $\mathbf{u}$ connecting P_0 and P is the displacement vector which can be expressed in both the material or spatial coordinates

$$\begin{aligned} \mathbf{U} &= U_K \mathbf{I}_K \\ \mathbf{u} &= u_k \mathbf{i}_k \end{aligned}$$

- From the preceding figure we can express motion as

$$\begin{array}{rl} x_i = x_i(X_1, X_2, X_3, t) & \text{Lagrangian formulation} \\ X_i = X_i(x_1, x_2, x_3, t) & \text{Eulerian formulation} \end{array}$$

- Ignoring a detailed analysis of large deformation, it is determined that

		Displacement gradient	
		Small	Large
Displacement	Small	Lagrangian small strain (Cauchy)	Lagrangian large strain (Green-Lagrange)
	Large	Eulerian small strain	Eulerian finite strain (Eulerian-Almansi)

- The Lagrangian finite strain tensor can be written as

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right)$$

- Alternatively these equations may be expanded as

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
\varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right)
\end{aligned}$$

- We define the engineering shear strain as

$$\gamma_{ij} = 2\varepsilon_{ij} \qquad (i \neq j)$$

- If $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ then we have six differential equations (in 3D the strain tensor has a total of 9 terms, but due to symmetry, there are 6 independent ones) for determining (upon integration) three unknowns displacements u_i. Hence the system is overdetermined, and there must be some linear relations between the strains.
- It can be shown (through appropriate successive differentiation) that the compatibility relation for strain reduces to:

$$\frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_j} + \frac{\partial^2 \varepsilon_{jj}}{\partial x_i \partial x_k} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ij}}{\partial x_j \partial x_k} = 0.$$

  In 3D, this would yield 9 equations in total, however only six are distinct.
- In 2D, this results in (by setting $i = 2$, $j = 1$ and $l = 2$):

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial^2 \gamma_{12}}{\partial x_1 \partial x_2}$$

  (recall that $2\varepsilon_{12} = \gamma_{12}$).

- We have thus far studied tensor fields (stress and strain).
- We have also obtained only one differential equation, that was the compatibility equation.
- Next we still derive additional differential equations governing the way stress and deformation vary at a point and with time. They will apply to any continuous medium, and yet we will not have enough equations to determine unknown tensor field. For that we need to wait for constitutive laws relating stress and strain will be introduced.
- The fundamental equations are:
  1. Conservation of mass (continuity equation)
  2. Conservation of momentum (Equation of motion; Equilibrium)
  3. Conservation of Energy.

- A conservation law establishes a balance of a scalar or tensorial quantity in volume V bounded by a surface S (inside a control surface). In its most general form, such a law may be expressed as

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathcal{A}\mathrm{d}V}_{\text{Rate of variation}} - \underbrace{\int_S \alpha \mathrm{d}S}_{\text{Exchange by Diffusion}} = \underbrace{\int_V \mathbf{A}\mathrm{d}V}_{\text{Source}}$$

- The preceding equation reads: rate of increase of $\mathcal{A}$ inside a control volume plus the rate of outward flux of $\mathcal{A}$ through the surface of the control volume is equal to the rate of increase of $\mathbf{A}$ inside the control volume
- The dimensions of various quantities are given by

$$\begin{aligned}\dim(\alpha) &= \dim(\mathcal{A}Lt^{-1})\\ \dim(\mathbf{A}) &= \dim(\mathcal{A}t^{-1})\end{aligned}$$

rightfully all expressed in terms of $\mathcal{A}$.

- the time rate of change of the total momentum of a given set of particles equals the vector sum of all external forces acting on the particles of the set, provided Newton's Third Law applies.
- The continuum form of this principle is a basic postulate of continuum mechanics (postulate: a statement, also known as an axiom, which is taken to be true without proof).
- Starting with

$$\int_S \mathbf{t} \mathrm{d}S + \int_V \rho \mathbf{b} \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \mathbf{v} \mathrm{d}V$$

- Divergence Theorem

$$\int_V v_{i,i} dV = \int_S \underbrace{v_i n_i}_{\text{flux}} dS$$

The flux of a vector function through some closed surface equals the integral of the divergence of that function over the volume enclosed by the surface.

- we substitute $t_i = T_{ij}n_j$ and apply the divergence theorem to obtain

$$\begin{aligned}\int_V \left(\frac{\partial T_{ij}}{\partial x_j} + \rho b_i\right) dV &= \int_V \rho \frac{dV_i}{dt} dV \\ \int_V \left[\frac{\partial T_{ij}}{\partial x_j} + \rho b_i - \rho \frac{dv_i}{dt}\right] dV &= 0\end{aligned}$$

or for an arbitrary volume

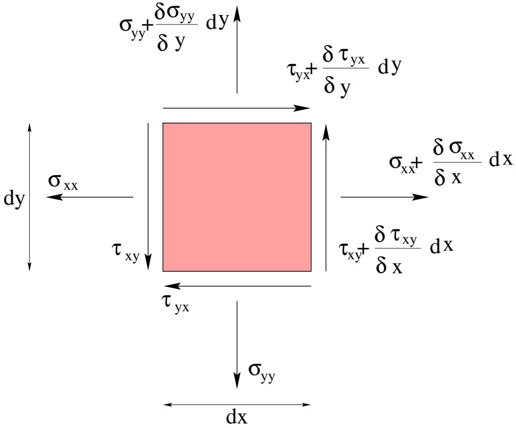
$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = \rho \frac{dv_i}{dt}$$

which is Cauchy's (first) equation of motion, or the linear momentum principle, or more simply equilibrium equation.

- When expanded in 3D, this equation yields:

$$\begin{aligned}\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} + \rho b_1 &= 0 \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} + \rho b_2 &= 0 \\ \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} + \rho b_3 &= 0\end{aligned}$$

- We note that these equations could also have been derived from the free body diagram with the assumption of equilibrium (via Newton's second law) considering an infinitesimal element of dimensions $dx_1 \times dx_2 \times dx_3$.



- If mechanical quantities only are considered, the principle of conservation of energy for the continuum may be derived directly from the equation of motion by taking the integral over the volume V of the scalar product and the velocity v_i.

$$\int_V v_i T_{ji,j} \mathrm{d}V + \int_V \rho b_i v_i \mathrm{d}V = \int_V \rho v_i \frac{\mathrm{d}v_i}{\mathrm{d}t} \mathrm{d}V$$

Applying the divergence theorem,

$$\frac{\mathrm{d}K}{\mathrm{d}t} + \frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}W}{\mathrm{d}t} + Q$$

this equation relates the time rate of change of total mechanical energy of the continuum on the left side to the rate of work done by the surface and body forces on the right hand side.

- If both mechanical and non mechanical energies are to be considered, the first principle states that the time rate of change of the kinetic plus the internal energy is equal to the sum of the rate of work plus all other energies supplied to, or removed from the continuum per unit time (heat, chemical, electromagnetic, etc.).

- For a thermomechanical continuum, it is customary to express the time rate of change of internal energy by the integral expression

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho u \mathrm{d}V$$

  where u is the internal energy per unit mass or specific internal energy.

- The dimension of U is one of energy dim $U = ML^2T^{-2}$, and the SI unit is the Joule, similarly dim $u = L^2T^{-2}$ with the SI unit of Joule/Kg.

Hooke

ceiiinosssttuu

Hooke, 1676

Ut tensio sic vis

Hooke, 1678

- The Generalized Hooke's Law can be written as:

$$\sigma_{ij} = D_{ijkl}\varepsilon_{kl} \qquad i, j, k, l = 1, 2, 3$$

- The (fourth order) tensor of elastic constants D_{ijkl} has 81 (3^4) components however, due to the symmetry of both σ and ε,there are at most 36 $\left(\frac{9(9-1)}{2}\right)$ distinct elastic terms.

- In terms of Lame's constants (which naturally are derived from coninuum mechanics consideration, but can not be both experimentally measured), Hooke's Law for an isotropic body is written as

$$T_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij}; \qquad E_{ij} = \frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\lambda+2\mu}\delta_{ij}T_{kk}\right)$$

- In terms of engineering constants (which can be measured in the laboratory)

$$\begin{array}{rcll} \frac{1}{E} & = & \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}; & \nu = \frac{\lambda}{2(\lambda+\mu)} \\ \lambda & = & \frac{\nu E}{(1+\nu)(1-2\nu)}; & \mu = G = \frac{E}{2(1+\nu)} \end{array}$$

- Hooke's law for isotropic material in terms of engineering constants becomes

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right); \qquad \varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}$$

- When the strain equation is expanded in 3D cartesian coordinates it would yield:

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}(2\varepsilon_{xy})\\ \gamma_{yz}(2\varepsilon_{yz})\\ \gamma_{zx}(2\varepsilon_{zx})\end{array}\right\} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1+\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1+\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix}\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{array}\right\}$$

- Plane Strain

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\end{array}\right\} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & 0\\ \nu & (1-\nu) & 0\\ \nu & \nu & 0\\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\}$$

- Axisymmetry

$$
\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u}{\partial r}; & \varepsilon_{\theta\theta} &= \frac{u}{r} \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z}; & \varepsilon_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}
\end{aligned}
$$

The constitutive relation is again analogous to 3D/plane strain

$$
\left\{ \begin{array}{c} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \tau_{rz} \end{array} \right\} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \left\{ \begin{array}{c} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\theta\theta} \\ \gamma_{rz} \end{array} \right\}
$$

- Plane Stress

$$
\begin{aligned}
\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{array} \right\} &= \frac{1}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \left\{ \begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{array} \right\} \\
\varepsilon_{zz} &= -\frac{1}{1-\nu} \nu (\varepsilon_{xx} + \varepsilon_{yy})
\end{aligned}
$$

Intermediary Structural Analysis

Special Topics

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Table of Contents I

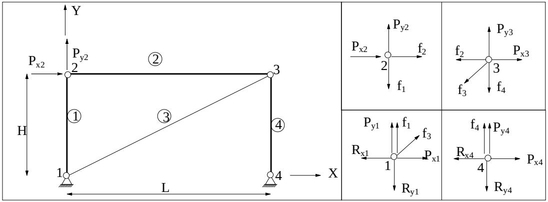
- The statics matrix $[\mathcal{B}]$ relates the vector of all the structure's $\{\mathtt{P}\}$ known nodal forces in global coordinates to all the unknown internal forces in their local coordinate system $\{\mathtt{F}\}$, through equilibrium relationship:

$$\{\mathtt{P}\} \equiv [\mathcal{B}]\,\{\mathtt{F}\} \tag{1}$$

- $[\mathcal{B}]$ would have as many rows as the total number of independent equations of equilibrium; and as many columns as independent internal forces.

Type	Internal Forces	Equations of Equilibrium
Truss	Axial force at one end	$\Sigma F_X = 0$, $\Sigma F_Y = 0$
Beam 1	Shear and moment at one end	$\Sigma F_y^A = 0$, $\Sigma M_z^A = 0$
Beam 2	Shear at each end	$\Sigma F_y^A = 0$, $\Sigma F_y^B = 0$
Beam 3	Moment at each end	$\Sigma M_z^A = 0$, $\Sigma M_z^B = 0$
2D Frame 1	Axial, Shear, Moment at one end	$\Sigma F_x^A = 0$, $\Sigma F_y^A = 0$, $\Sigma M_z^A = 0$

- $[\mathcal{B}]$ square matrix for a statically determinate structure, and rectangular (more columns than rows) otherwise.



8 unknown forces (4 internal member forces and 4 external reactions), and 8 equations of equilibrium (2 at each of the 4 nodes). Equilibrium equations ($\cos\alpha = \frac{L}{\sqrt{L^2+H^2}} = C$ and $\sin\alpha = \frac{H}{\sqrt{L^2+H^2}} = S$):

Node	$\Sigma F_X = 0$	$\Sigma F_Y = 0$
Node 1	$\underbrace{P_{x1}}_{0} + F_3 C - R_{x1} = 0$	$\underbrace{P_{y1}}_{0} + F_1 + F_3 S - R_{y1} = 0$
Node 2	$P_{x2} + F_2 = 0$	$P_{y2} - F_1 = 0$
Node 3	$\underbrace{P_{x3}}_{0} - F_2 - F_3 C = 0$	$\underbrace{P_{y3}}_{0} - F_4 - F_3 S = 0$
Node 4	$\underbrace{P_{x4}}_{0} + R_{x4} = 0$	$\underbrace{P_{y4}}_{0} + F_4 - R_{y4} = 0$

$$
\begin{array}{c} \Sigma F_x^1 \\ \Sigma F_y^1 \\ \Sigma F_x^2 \\ \Sigma F_y^2 \\ \Sigma F_x^3 \\ \Sigma F_y^3 \\ \Sigma F_x^4 \\ \Sigma F_y^4 \end{array}
\underbrace{\left\{ \begin{array}{c} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \\ P_{x3} \\ P_{y3} \\ P_{x4} \\ P_{y4} \end{array} \right\}}_{\{\mathrm{P}\}}
=
\underbrace{\left[\begin{array}{cccc|cccc}
0 & 0 & -C & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & -S & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & C & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & S & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1
\end{array} \right]}_{[\mathcal{B}]}
\underbrace{\left\{ \begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \hline R_{x1} \\ R_{y1} \\ R_{x4} \\ R_{y4} \end{array} \right\}}_{\{\mathrm{F}\}}
\tag{2}
$$

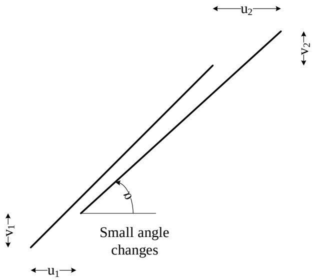
$$
\underbrace{\left\{ \begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \hline R_{x1} \\ R_{y1} \\ R_{x4} \\ R_{y4} \end{array} \right\}}_{\{\mathrm{F}\}}
=
\underbrace{\left[\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{C} & 0 & \frac{1}{C} & 0 & 0 & 0 \\
0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 0 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & \frac{S}{C} & 1 & \frac{S}{C} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -\frac{S}{C} & 0 & -\frac{S}{C} & 1 & 0 & 1
\end{array} \right]}_{[\mathcal{B}]^{-1}}
\underbrace{\left\{ \begin{array}{c} 0 \\ 0 \\ P_{x2} \\ P_{y2} \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\}}_{\{\mathrm{P}\}}
=
\left\{ \begin{array}{c} P_{y2} \\ -P_{x2} \\ \frac{P_{x2}}{C} \\ -\frac{S}{C} P_{x2} \\ \hline P_{x2} \\ \frac{S}{C} P_{x2} + P_{y2} \\ 0 \\ -\frac{S}{C} P_{x2} \end{array} \right\}
$$

$[\mathcal{B}]$ is independent of the external load

The kinematics matrix $[\mathcal{A}]$ relates all the structure's nodal *total displacements* in global coordinates $\{\Delta\}$ to the element *relative displacements* in their local coordinate system and the support displacement (which may not be zero if settlement occurs) $\{\Upsilon\}$ and is defined as:

$$\{\Upsilon\} \equiv [\mathcal{A}]\{\Delta\} \tag{3}$$

$[\mathcal{A}]$ is a rectangular matrix, number of rows is equal to the number of the element internal displacements, and the number of columns is equal to the number of nodal displacements.



Contrarily to the rotation matrix introduced earlier and which transforms the displacements from global to local coordinate for one *single* element, the kinematics matrix applies to the entire structure. It can be easily shown that for trusses (which corresponds to shortening or elongation of the member, and small change in angle α):

$$\Upsilon^e = (u_2 - u_1)\cos\alpha + (v_2 - v_1)\sin\alpha$$

Considering again the statically determinate truss of the previous example, the kinematic matrix will be given by:

$$\Delta_1^e = v_2 - v_1; \qquad \Delta_2^e = u_3 - u_2; \qquad \Delta_3^e = (u_3 - u_1)C + (v_3 - v_1)S; \qquad \Delta_i^e = \cdots$$

or in matrix form:

$$\left\{\begin{array}{c} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \\ \hline u_1 \\ v_1 \\ u_4 \\ v_4 \end{array}\right\} = \underbrace{\left[\begin{array}{cccccccc} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -C & -S & 0 & 0 & C & S & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{[\mathcal{A}]} \left\{\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}\right\}$$

Applying the constraints: $u_1 = 0$; $v_1 = 0$; $u_4 = 0$; and $v_4 = 0$ we obtain:

$$\left\{\begin{array}{c} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right\} = \underbrace{\left[\begin{array}{cccccccc} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -C & -S & 0 & 0 & C & S & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{[\mathcal{A}]} \left\{\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}\right\}$$

We should observe that $[\mathcal{A}]$ is the transpose of the $[\mathcal{B}]$ matrix in Eq. 2

Having defined both the statics $[\mathcal{B}]$ and kinematics $[\mathcal{A}]$ matrices, it is intuitive that those two matrices must be related.
The external work being defined as

$$\left.\begin{array}{rcl} W_{ext} & = & \frac{1}{2}\lfloor \mathrm{P} \rfloor \{\Delta\} \\ \{\mathrm{P}\} & = & [\mathcal{B}]\{\mathrm{F}\} \end{array}\right\} W_{ext} = \frac{1}{2}\lfloor \mathrm{F} \rfloor [\mathcal{B}]^T \{\Delta\}$$

Alternatively, the internal work is given by:

$$\left.\begin{array}{rcl} W_{int} & = & \frac{1}{2}\lfloor \mathrm{F} \rfloor \{\Upsilon\} \\ \{\Upsilon\} & = & [\mathcal{A}]\{\Delta\} \end{array}\right\} W_{int} = \frac{1}{2}\lfloor \mathrm{F} \rfloor [\mathcal{A}] \{\Delta\}$$

Equating the external to the internal work $W_{ext} = W_{int}$ we obtain:

$$\frac{1}{2}\lfloor \mathrm{F} \rfloor [\mathcal{B}]^T \{\Delta\} = \frac{1}{2}\lfloor \mathrm{F} \rfloor [\mathcal{A}] \{\Delta\}$$

$$[\mathcal{B}]^T = [\mathcal{A}] \tag{4}$$

The counterparts at the continuum level is

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{xz}\\ \varepsilon_{yz}\end{array}\right\} = \left[\begin{array}{ccc}\frac{\partial}{\partial x} & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}\\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y}\end{array}\right] \left\{\begin{array}{c}u_x\\ u_y\\ u_z\end{array}\right\}$$

or $\varepsilon = \mathtt{L}\mathtt{u}$ where $\mathtt{L}$ is called a Linear Operator, and

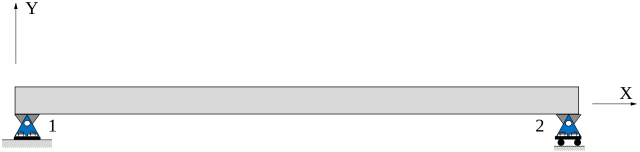
$$\left[\begin{array}{cccccc}\frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0\\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z}\\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{array}\right] \left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{array}\right\} + \left\{\begin{array}{c}b_x\\ b_y\\ b_z\end{array}\right\} = 0$$

or $\mathtt{L}^T\sigma + \mathtt{b} = 0$

Having introduced both the stiffness and flexibility methods, we shall rigorously consider the relationship among the two matrices $[\mathrm{K}]$ and $[\mathrm{D}]$ at the structure level.
Recall:

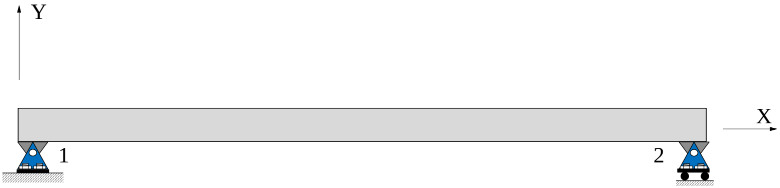
$$\left\{\begin{array}{c}\mathrm{P}_t\\ \hline \mathrm{P}_u\end{array}\right\}=\left[\begin{array}{c|c}\mathrm{K}_{tt} & \mathrm{K}_{tu}\\ \hline \mathrm{K}_{ut} & \mathrm{K}_{uu}\end{array}\right]\left\{\begin{array}{c}\Delta_t\\ \hline \Delta_u\end{array}\right\} \tag{5}$$

We seek $\mathtt{d}$, such that $\Delta = \mathtt{dp}$, for a structure supported in a stable and statically determinate way. For the following simple case:



$$\{\Delta_t\} = \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\}; \qquad \{\Delta_u\} = \left\{ \begin{array}{c} v_1 \\ v_2 \end{array} \right\}; \qquad \{\mathrm{P}_t\} = \left\{ \begin{array}{c} M_1 \\ M_2 \end{array} \right\}; \qquad \{\mathrm{P}_u\} = \left\{ \begin{array}{c} V_1 \\ V_2 \end{array} \right\}$$

Since $\{\Delta_u\} = \{0\} \Rightarrow \left\{ \begin{array}{c} \mathrm{P}_t \\ \mathrm{P}_u \end{array} \right\} = \left[\begin{array}{c} \mathrm{K}_{tt} \\ \mathrm{K}_{ut} \end{array} \right] \{\Delta_t\} \Rightarrow \{\mathrm{P}_t\} = [\mathrm{K}_{tt}]\{\Delta_t\} \Rightarrow [\mathtt{d}] = [\mathrm{K}_{tt}]^{-1}$



$$\left\{ \begin{array}{c} M_1 \\ M_2 \end{array} \right\} = \underbrace{\frac{EI}{l} \left[\begin{array}{cc} 4 & 2 \\ 2 & 4 \end{array} \right]}_{[\mathrm{K}_{tt}]} \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\}$$

$$[\mathrm{K}_{tt}]^{-1} = [\mathrm{d}] = \frac{l}{EI}\frac{1}{12} \left[\begin{array}{cc} 4 & -2 \\ -2 & 4 \end{array} \right] = \frac{l}{6EI} \left[\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right]$$

1. $[\mathrm{K}_{tt}]$: From Eq. 5, $[\mathrm{K}]$ was subdivided into free and supported d.o.f.'s, and we have shown that $[\mathrm{K}_{tt}] = [\mathrm{d}]^{-1}$, or $\{\mathrm{P}_t\} = [\mathrm{K}_{tt}]\{\Delta_t\}$ but we still have to determine $[\mathrm{K}_{tu}]$, $[\mathrm{K}_{ut}]$, and $[\mathrm{K}_{uu}]$.
2. $[\mathrm{K}_{ut}]$: Since $[\mathrm{d}]$ is obtained for a stable statically determinate structure, we have:

$$\{\mathrm{P}_u\} = [\mathcal{B}]\{\mathrm{P}_t\}; \qquad \{\mathrm{P}_u\} = \underbrace{[\mathcal{B}][\mathrm{K}_{tt}]}_{[\mathrm{K}_{ut}]}\{\Delta_t\}; \qquad [\mathrm{K}_{ut}] = [\mathcal{B}][\mathrm{d}]^{-1}$$

3. $[\mathrm{K}_{tu}]$: Equating the external to the internal work:
    1. External work: $W_{ext} = \frac{1}{2}\lfloor\Delta_t\rfloor\{\mathrm{P}_t\}$
    2. Internal work: $W_{int} = \frac{1}{2}\lfloor\mathrm{P}_u\rfloor\{\Delta_u\}$

Equating W_{ext} to W_{int} and combining with

$$\lfloor \mathrm{P}_u \rfloor = \lfloor \Delta_t \rfloor [\mathrm{K}_{ut}]^T$$

with $\{\Delta_u\} = \{0\}$ (zero support displacements) we obtain:

$$[\mathrm{K}_{tu}] = [\mathrm{K}_{ut}]^T = [\mathrm{d}]^{-1} [\mathcal{B}]^T \tag{6}$$

4. $[\mathrm{K}_{uu}]$:

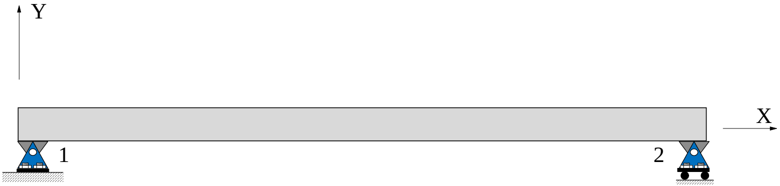
$$\{\mathrm{P}_u\} = [\mathcal{B}] \{\mathrm{P}_t\}; \qquad \{\mathrm{P}_t\} = [\mathrm{K}_{tu}] \{\Delta_u\}; \qquad [\mathrm{K}_{tu}] [\mathrm{d}]^{-1} [\mathcal{B}]^T$$

or:

$$\{\mathrm{P}_u\} = \underbrace{[\mathcal{B}][\mathrm{d}]^{-1}[\mathcal{B}]^T}_{[\mathrm{K}_{uu}]}\{\Delta_u\} \tag{7}$$

In summary we have:

$$[\mathrm{K}] = \left[\begin{array}{c|c} [\mathrm{d}]^{-1} & [\mathrm{d}]^{-1}[\mathcal{B}]^T \\ \hline [\mathcal{B}][\mathrm{d}]^{-1} & [\mathcal{B}][\mathrm{d}]^{-1}[\mathcal{B}]^T \end{array}\right] \tag{8}$$



Assuming that both M_1 and M_2 are positive (ccw):

1. The flexibility matrix is given by:

$$\left\{ \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right\} = \underbrace{\frac{l}{6EI} \left[\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right]}_{[\mathrm{d}]} \left\{ \begin{array}{c} M_1 \\ M_2 \end{array} \right\}$$

2. The statics matrix $[\mathcal{B}]$ relating external to internal forces is given by:

$$\left\{ \begin{array}{c} R_1 = V_1 \\ R_2 = V_2 \end{array} \right\} = \underbrace{\frac{1}{l} \left[\begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array} \right]}_{[\mathcal{B}]} \left\{ \begin{array}{c} M_1 \\ M_2 \end{array} \right\}$$

1. $[\mathrm{K}_{tt}]$: would simply be given by:

$$[\mathrm{K}_{tt}] = [\mathrm{d}]^{-1} = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

The statics matrix $[\mathcal{B}]$ relating external to internal forces is given by:

$$\begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = \underbrace{\frac{1}{l}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}}_{[\mathcal{B}]}\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$$

2. $[\mathrm{K}_{tu}]$: The upper off-diagonal

$$[\mathrm{K}_{tu}] = [\mathrm{d}]^{-1}[\mathcal{B}]^T = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\frac{1}{l}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \frac{EI}{l^2}\begin{bmatrix} 6 & -6 \\ 6 & -6 \end{bmatrix}$$

3. $[\mathrm{K}_{ut}]$: Lower off-diagonal term

$$[\mathrm{K}_{ut}] = [\mathcal{B}][\mathrm{d}]^{-1} = \frac{1}{l}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \frac{EI}{l^2}\begin{bmatrix} 6 & 6 \\ -6 & -6 \end{bmatrix}$$

4. $[\mathrm{K}_{uu}]$: Lower diagonal term

$$\begin{aligned} [\mathrm{K}_{uu}] &= [\mathcal{B}]\,[\mathrm{d}]^{-1}\,[\mathcal{B}]^T = [\mathrm{K}_{ut}]\,[\mathcal{B}]^T \\ &= \frac{EI}{l^2}\frac{1}{l}\begin{bmatrix} 6 & 6 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & -12 \\ -12 & 12 \end{bmatrix} \end{aligned}$$

Let us note that we can rewrite:

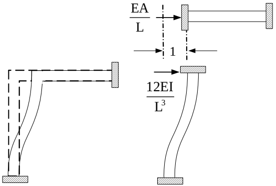
$$\left\{ \begin{array}{c} M_1 \\ M_2 \\ \hline V_1 \\ V_2 \end{array} \right\} = \frac{EI}{l^3}\left[\begin{array}{cc|cc} 4l^2 & 2l^2 & 6l & -6l \\ 2l^2 & 4l^2 & 6l & -6l \\ \hline 6l & 6l & 12 & -12 \\ -6l & -6l & -12 & 12 \end{array} \right]\left\{ \begin{array}{c} \theta_1 \\ \theta_2 \\ \hline v_1 \\ v_2 \end{array} \right\}$$

If we rearrange the stiffness matrix we would get:

$$\left\{ \begin{array}{c} V_1 \\ M_1 \\ \hline V_2 \\ M_2 \end{array} \right\} = \frac{EI}{l}\underbrace{\left[\begin{array}{cc|cc} \frac{12}{l^2} & \frac{6}{l} & \frac{-12}{l^2} & \frac{6}{l} \\ \frac{6}{l} & 4 & \frac{-6}{l} & 2 \\ \hline \frac{-12}{l^2} & \frac{-6}{l} & \frac{12}{l} & \frac{-6}{l} \\ \frac{6}{l} & 2 & \frac{-6}{l} & 4 \end{array} \right]}_{[\mathrm{K}]}\left\{ \begin{array}{c} v_1 \\ \theta_1 \\ \hline v_2 \\ \theta_2 \end{array} \right\}$$

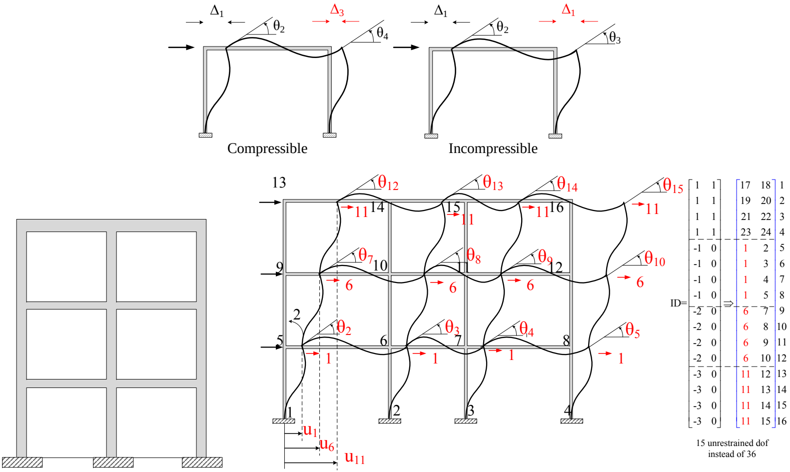
and is the same stiffness matrix earlier derived.

Insert from old lecture notes in Matrix, may be important.



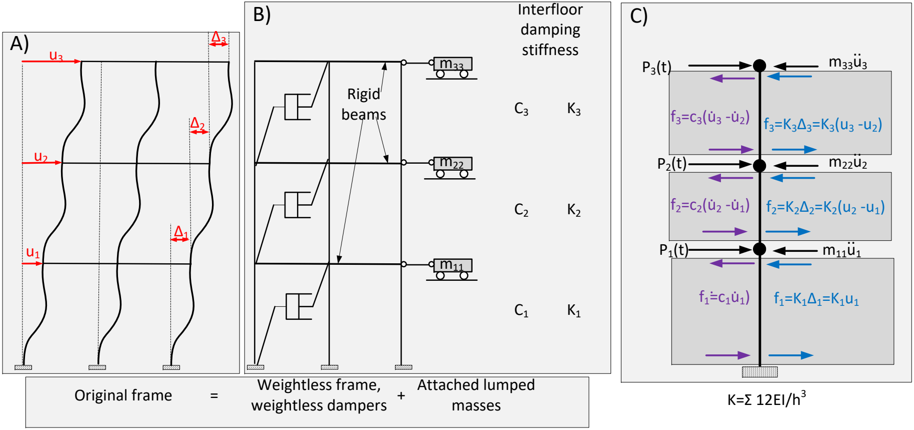
- Ratio of axial to flexural stiffness is:
  $\alpha = \frac{k_a}{k_f} = \frac{\frac{EA}{L}}{\frac{12EI}{L^3}} = \frac{AL^2}{12I}$.
- For a $b \times h$ rectangular section, with $b = h/2$, and $L = 10h$, $\Rightarrow \alpha = 100$
- For a W section
  $Z \approx \frac{wd}{9}$, $\frac{Z}{S} = \xi = 1.1$, $S = \frac{I}{\frac{d}{2}}$, $w = (490)\ \text{lbs/ft}^3 A$, or
  $I \approx 0.208Ad^2$, and $\alpha = \frac{\frac{EA}{L}}{\frac{12EI}{L^3}} = \frac{\frac{EA}{L}}{\frac{12E(0.208)Ad^2}{L^3}} = 0.4\left(\frac{L}{d}\right)^2$

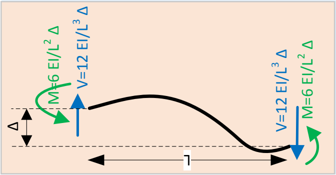
- For steel structure, we can assume
  $L = 20d$, $\Rightarrow \alpha = 160$ Axial stiffness is much higher than flexural stiffness. Note: we may have negligible axial deformations, however axial force is not negligible.
- This is often exploited in seismic analysis, enabling us to replace a (short) multi-bay building with a single column with lumped masses.



- Ignoring axial deformations in the columns.
- If $\alpha >\approx 10$ ignore axial deformation, and reduce number of degrees of freedom. When the frame is subjected to lateral (wind or earthquake) load, shear force in each column is proportional to its stiffness.

- Ignoring axial deformation, greatly facilitates the dynamic analysis of small rise building frames subjected to lateral load (wind, earthquakes).





- Using Newton's second law of motion for each of the three nodes:

$$
\begin{array}{llllr}
P_3(t) & -m_{33}\ddot{u}_3 & -c_3(\dot{u}_3-\dot{u}_2) & -K_3(u_3-u_2) & =0\\
P_2(t) & -m_{22}\ddot{u}_2 & +c_3(\dot{u}_3-\dot{u}_2)-c_2(\dot{u}_2-\dot{u}_1) & +K_3(u_3-u_2)-K_2(u_2-u_1) & =0\\
P_1(t) & -m_{11}\ddot{u}_1 & +c_2(\dot{u}_2-\dot{u}_1)-c_1\dot{u}_1 & +K_2(u_2-u_1)-K_1u_1 & =0
\end{array}
$$

- We can rewrite these equations as

$$
\begin{array}{llll}
m_{11}\ddot{u}_1 & +(c_1+c_2)\dot{u}_1 - c_2\dot{u}_2 & +(k_1+k_2)u_1 - k_2u_2 & = P_1(t) \\
m_{22}\ddot{u}_2 & -c_2\dot{u}_1 + (c_2+c_3)\dot{u}_2 - c_3\dot{u}_3 & -k_2u_1 + (k_2+k_3)u_2 - k_3u_3 & = P_2(t) \\
m_{33}\ddot{u}_3 & -c_3\dot{u}_2 + c_3\dot{u}_3 & -K_3u_2 + K_3u_3 & = P_3(t)
\end{array}
$$

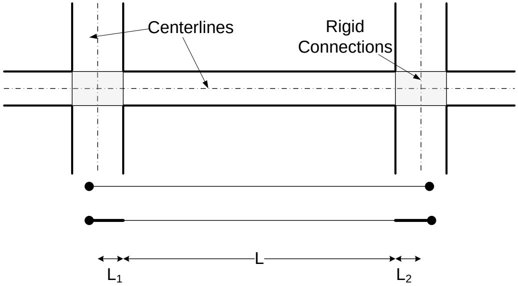
or

$$
[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{f(t)\} \tag{9}
$$

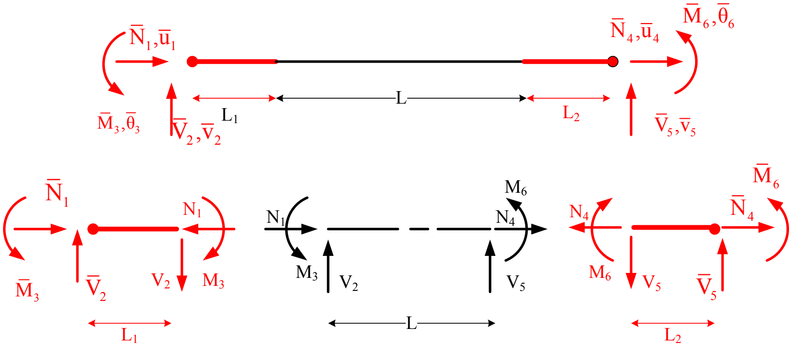
$$
\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix}
$$

$$
+ \begin{bmatrix} K_1+K_2 & -K_2 & 0 \\ -K_2 & K_2+K_3 & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{Bmatrix}
$$

So far all members were assumed to be rigidly connected and connecting center lines to center lines. In many instances, either we have a hinge, a semi-rigid connection and we may want to take into account the offset of the member.



$\lfloor \mathrm{p} \rfloor = \lfloor\ N_1 \quad V_2 \quad M_3 \quad N_4 \quad V_5 \quad M_6\ \rfloor$ and $\lfloor \overline{\mathrm{p}} \rfloor = \lfloor\ \overline{N}_1 \quad \overline{V}_2 \quad \overline{M}_3 \quad \overline{N}_4 \quad \overline{V}_5 \quad \overline{M}_6\ \rfloor$ are the forces acting on the interior and exterior sides of the rigid link respectively.
Similarly we denote by $\lfloor \overline{\mathrm{u}} \rfloor = \lfloor\ \overline{u}_1 \quad \overline{v}_2 \quad \overline{\theta}_3 \quad \overline{u}_4 \quad \overline{v}_5 \quad \overline{\theta}_6\ \rfloor$ the exterior displacements



- We need to express the exterior forces in terms of the interior ones. We consider equilibrium of the free body diagram:

$$\begin{array}{cccccccc} \overline{N}1 & = & N_1; & \overline{V}_2 & = & V_2; & \overline{M}_3 & = & L_1 V_2 + M_3 \\ \overline{N}4 & = & N_4; & \overline{V}_5 & = & V_5; & \overline{M}_6 & = & -L_2 V_5 + M_6 \end{array}$$

or

$$\overline{\mathrm{p}} = \mathcal{B}\mathrm{p} \tag{10}$$

- $\mathcal{B}$ is a Statics matrix:

$$\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & L_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -L_2 & 1 \end{bmatrix}$$

- Similarly, we can define a kinematics matrix $[\mathcal{A}]$ such that

$$\mathrm{u} = \mathcal{A}\overline{\mathrm{u}} \tag{11}$$

- It can be shown that $\mathcal{A} = \mathcal{B}^T$.
- We seek to determine the stiffness for the beam element with rigid offset in terms of the known

$$\mathrm{p} + \textbf{NEF} = \mathrm{k}\mathrm{u} \tag{12}$$

- If we multibly both sies by $\mathcal{B}$ and substitute Eq. 10 and 11 into 12:

$$\begin{aligned} \mathcal{B}p + \mathcal{B}\textbf{NEF} &= \mathcal{B}\mathrm{k}\mathcal{A}\overline{\mathrm{u}} && (13) \\ \mathcal{B}p + \mathcal{B}\textbf{NEF} &= \mathcal{B}\mathrm{k}\mathcal{B}^T\overline{\mathrm{u}} && (14) \\ \overline{\mathrm{p}} + \overline{\textbf{NEF}} &= \overline{\mathrm{k}}\overline{\mathrm{u}} && (15) \end{aligned}$$

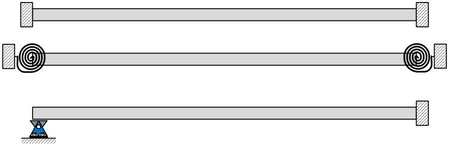
- where

$$\overline{\mathrm{k}} = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 12 & \alpha_1 & 0-12 & \alpha_2 & \\ 0 & \alpha_1 & \gamma & 0-\alpha_1 & \beta & \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -\alpha_1 & 012 & -\alpha_2 & \\ 0 & \alpha_2 & \beta & 0 & -\alpha_2 & \gamma \end{bmatrix}$$

where $\alpha_i = 6L + 12L_i$, $\beta = 2L^2 + 6LL_1 + 6LL_2 + 12L_1L_2$, and $\gamma = 4L^2 + 12LL_2 + 12L_2^2$

- thus

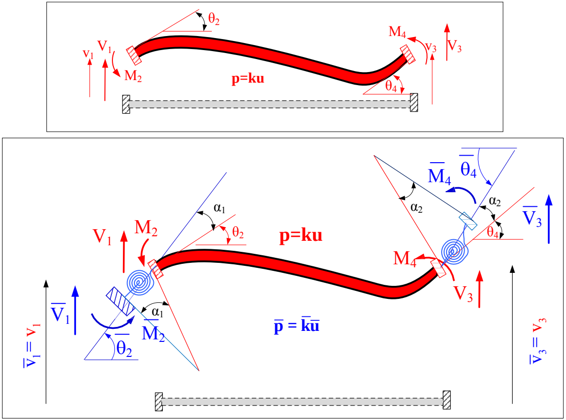
$$\underbrace{\begin{Bmatrix} \overline{FN_1} \\ \overline{FV_2} \\ \overline{FM_3} \\ \overline{FN_4} \\ \overline{FV_5} \\ \overline{FM_6} \end{Bmatrix}}_{\overline{NEF}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & L_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -L_2 & 1 \end{bmatrix} \underbrace{\begin{Bmatrix} FN_1 \\ FV_2 \\ FM_3 \\ FN_4 \\ FV_5 \\ FM_6 \end{Bmatrix}}_{NEF}$$



The force displacement relations is given by $\mathtt{p} + \mathbf{NEF} = \mathtt{ku}$.

We seek to determine the stiffness for the beam element with semi rigid connection $\overline{\mathtt{p}} + \overline{\mathbf{NEF}} = \overline{\mathtt{k}}\overline{\mathtt{u}}$ in terms of $\mathtt{k}$, **NEF** and the two spring stiffnesses k_1^s and k_2^s at the left and right end of the member (first and second node).

$\lfloor \mathrm{p} \rfloor = \lfloor\ V_1 \quad M_2 \quad V_3 \quad M_4\ \rfloor$ and $\lfloor \overline{\mathrm{p}} \rfloor = \lfloor\ \overline{V}_1 \quad \overline{M}_2 \quad \overline{V}_3 \quad \overline{M}_4\ \rfloor$ the forces acting on the interior and exterior sides of the springs respectively. Similarly we denote by $\lfloor \mathrm{u} \rfloor = \lfloor\ v_1 \quad \theta_2 \quad v_3 \quad \theta_4\ \rfloor$ and $\lfloor \overline{\mathrm{u}} \rfloor = \lfloor\ \overline{v}_1 \quad \overline{\theta}_2 \quad \overline{v}_3 \quad \overline{\theta}_4\ \rfloor$



- Considering the free body diagram of the spring, and assuming that the springs are infinitesimally small, equilibrium requires that $\mathrm{p} = \overline{\mathrm{p}}$, or

$$\begin{aligned}
v_1 &= \overline{v}_1 \\
\alpha_1 &= \overline{\theta}_2 - \theta_2 \Rightarrow \theta_2 = \overline{\theta}_2 - \alpha_1 \\
M_2 &= K_1^S \alpha_1 \Rightarrow \theta_2 = \overline{\theta}_2 - \underbrace{\frac{M_2}{k_1^s}}_{\alpha_1} \\
v_3 &= \overline{v}_3 \\
\alpha_2 &= \overline{\theta}_4 - \theta_4 \Rightarrow \theta_4 = \overline{\theta}_4 - \alpha_2 \\
M_4 &= K_2^S \alpha_2 \Rightarrow \theta_4 = \overline{\theta}_4 - \underbrace{\frac{M_4}{k_2^s}}_{\alpha_1}
\end{aligned}$$

where k_1^s and k_2^s are the left and right springs respectively.

- Substituting v_1, v_3, θ_2 and θ_4 into

$$\underbrace{\begin{Bmatrix} V_1 \\ M_2 \\ V_3 \\ M_4 \end{Bmatrix}}_{\{p\}} + \underbrace{\begin{Bmatrix} FV_1 \\ FM_2 \\ FV_3 \\ FM_4 \end{Bmatrix}}_{\{NEF\}} = \underbrace{\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}}_{[\mathrm{k}]} \underbrace{\begin{Bmatrix} v_1 \\ \theta_2 \\ v_3 \\ \theta_4 \end{Bmatrix}}_{\{u\}}$$

we obtain

$$\overline{V}_1 + FV_1 = \frac{EI}{L^3}\left[12\overline{v}_1 + 6L\left(\overline{\theta}_2 - \frac{\overline{M}_2}{k_1^s}\right) - 12\overline{v}_3 + 6L\left(\overline{\theta}_4 - \frac{\overline{M}_4}{k_2^s}\right)\right] \quad (16)$$

$$\overline{M}_2 + FM_2 = \frac{EI}{L^3}\left[6L\overline{v}_1 + 4L^2\left(\overline{\theta}_2 - \frac{\overline{M}_2}{k_1^s}\right) - 6L\overline{v}_3 + 2L^2\left(\overline{\theta}_4 - \frac{\overline{M}_4}{k_2^s}\right)\right] \quad (17)$$

$$\overline{V}_3 + FV_3 = \frac{EI}{L^3}\left[-12\overline{v}_1 - 6L\left(\overline{\theta}_2 - \frac{\overline{M}_2}{k_1^s}\right) + 12\overline{v}_3 - 6L\left(\overline{\theta}_4 - \frac{\overline{M}_4}{k_2^s}\right)\right] \quad (18)$$

$$\overline{M}_4 + FM_4 = \frac{EI}{L^3}\left[6L\overline{v}_1 + 2L^2\left(\overline{\theta}_2 - \frac{\overline{M}_2}{k_1^s}\right) - 6L\overline{v}_3 + 4L^2\left(\overline{\theta}_4 - \frac{\overline{M}_4}{k_2^s}\right)\right] \quad (19)$$

- These 4 equations are coupled ($\overline{M}_2$ and $\overline{M}_4$ are both inside and outside the stiffness matrix), we seek to uncouple them and express the forces exclusively in terms of the displacement.

- First we solve Eq. 17 and 19 simultaneously in terms of $\overline{u}$:

$$
\begin{aligned}
\overline{M}_2 &= \frac{EI}{L^3}\frac{\phi_b}{\Phi}\left[6L(2-\phi_2)\overline{v}_1 + 4L^2(3-2\phi_2)\overline{\theta}_2 - 6L(2-\phi_2)\overline{v}_3 + 2L^2\phi_2\overline{\theta}_4\right] \\
&\quad + \frac{\phi_1}{\Phi}\left[(4-3\phi_2)FM_2 - 2(1-\phi_2)FM_4\right] \\
\overline{M}_4 &= \frac{EI}{L^3}\frac{\phi_2}{\Phi}\left[6L(2-\phi_1)\overline{v}_1 + 2L^2\phi_1\overline{\theta}_2 - 6L(2-\phi_1)\overline{v}_3 + 4L^2(3-2\phi_1)\overline{\theta}_4\right] \\
&\quad + \frac{\phi_2}{\Phi}\left[-2(1-\phi_1)FM_2 + (4-3\phi_1)FM_4\right]
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_1 &= \frac{k_1^s L}{EI + k_1^s L} \\
\phi_2 &= \frac{k_2^s L}{EI + k_2^s L} \\
\Phi &= 12 - 8\phi_1 - 8\phi_2 + 5\phi_1\phi_2
\end{aligned}
$$

- ϕ can be interpreted as a "rigidity factor". For rigid connection $\phi = 1$, whereas for hinged ones $\phi = 0$.
- Next we substitute these last equation into Eq. 16-18:

$$\begin{aligned}
\overline{V}_1 &= \frac{EI}{\Phi L^3}\left[12(\phi_1+\phi_2-\phi_b\phi_2)\overline{v}_1 + 6L\phi_1(2-\phi_2)\overline{\theta}_2 - 12(\phi_1+\phi_2-\phi_1\phi_2)\overline{v}_3 + 6L\phi_2(2-\phi_1)\overline{\theta}_4\right] \\
&\quad + FV_1 - \frac{6}{\Phi L}\left[(1-\phi_1)(2-\phi_2)FM_2 + (1-\phi_2)(2-\phi_1)FM_4\right] \qquad (20 \\
\overline{V}_3 &= \frac{EI}{\Phi L^3}\left[-12(\phi_1+\phi_2-\phi_b\phi_2)\overline{v}_1 - 6L\phi_1(2-\phi_2)\overline{\theta}_2 + 12(\phi_1+\phi_2-\phi_1\phi_2)\overline{v}_3 - 6L\phi_2(2-\phi_1)\overline{\theta}_4\right] \\
&\quad + FV_3 + \frac{6}{\Phi L}\left[(1-\phi_1)(2-\phi_2)FM_2 + (1-\phi_2)(2-\phi_1)FM_4\right]
\end{aligned}$$

- We can express these expressions as

$$\left\{\overline{P}\right\} + \left\{\overline{NEF}\right\} = [\overline{\mathrm{k}}]\,\{\overline{u}\}$$

where

$$[\overline{\mathrm{k}}] = \frac{EI}{\Phi L^3}\begin{bmatrix} 12(\phi_1+\phi_2-\phi_1\phi_2) & 6L\phi_1(2-\phi_2) & -12(\phi_1+\phi_2-\phi_1\phi_2) & 6L\phi_2(2-\phi_1) \\ 6L\phi_1(2-\phi_2) & 4L^2\phi_1(3-2\phi_2) & -6L\phi_1(2-\phi_2) & 2L^2\phi_1\phi_2 \\ -12(\phi_1+\phi_2-\phi_1\phi_2) & -6L\phi_1(2-\phi_2) & 12(\phi_1+\phi_2-\phi_1\phi_2) & -6L\phi_2(2-\phi_b) \\ 6L\phi_2(2-\phi_1) & 2L^2\phi_1\phi_2 & -6L\phi_2(2-\phi_1) & 4L^2\phi_2(3-2\phi_1) \end{bmatrix}$$

$$\underbrace{\left\{\begin{array}{c} \overline{FV_1} \\ \overline{FM_2} \\ \overline{FV_3} \\ \overline{FM_4} \end{array}\right\}}_{\overline{NEF}} = \begin{bmatrix} 1 & -\frac{6}{\Phi L}[(1-\phi_1)(2-\phi_2)] & 0 & -\frac{6}{\Phi L}[(1-\phi_2)(2-\phi_1)] \\ 0 & \frac{\phi_1}{\Phi}[(4-3\phi_2)] & 0 & \frac{\phi_1}{\Phi}[-2(1-\phi_2)] \\ 0 & \frac{6}{\Phi L}[(1-\phi_1)(2-\phi_2)] & 1 & \frac{6}{\Phi L}[(1-\phi_2)(2-\phi_1)] \\ 0 & \frac{\phi_2}{\Phi}[-2(1-\phi_1)] & 0 & \frac{\phi_2}{\Phi}[(4-3\phi_1)] \end{bmatrix} \underbrace{\left\{\begin{array}{c} FV_1 \\ FM_2 \\ FV_3 \\ FM_4 \end{array}\right\}}_{NEF}$$

- For fully rigid connections, $\phi = 1$, we recover the original stiffness matrix of the beam.
- If we set $\phi_1 = 0$ and $\phi_2 = 1$ then we have a hinge on the left, and a rigid connection on the right and the corresponding stiffness matrix is:

$$[\overline{\mathrm{k}}] = \frac{EI}{L^3}\begin{bmatrix} 3 & 0 & -3 & 3L \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 3 & -3L \\ 3L & 0 & -3L & 3L^2 \end{bmatrix}$$

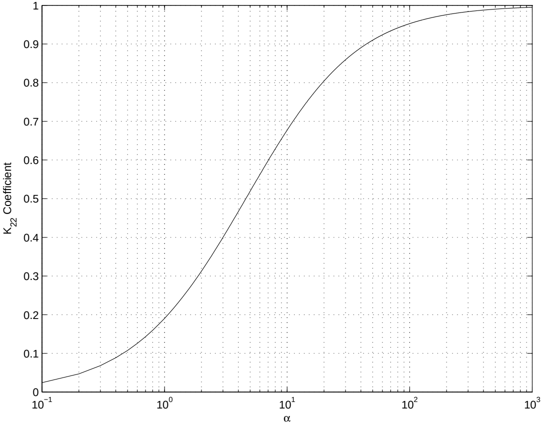
- The stiffness matrix of a beam column with a hinge at its right will then be:

$$[\overline{\mathrm{k}}] = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 3EI/L^3 & 3EI/L^2 & 0 & -3EI/L^3 & 0 \\ 0 & 3EI/L^2 & 3EI/L & 0 & -3EI/L^2 & 0 \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -3EI/L^3 & -3EI/L^2 & 0 & 3EI/L^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

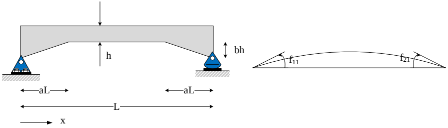
- If the hinge is on the left end

$$[\bar{\mathrm{k}}] = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 3EI/L^3 & 0 & 0 & -3EI/L^2 & 3EI/L^2 \\ 0 & 0 & {\color{red}0} & 0 & 0 & 0 \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -3EI/L^3 & 0 & 0 & 3EI/L^3 & -3EI/L^2 \\ 0 & 3EI/L^2 & 0 & 0 & -3EI/L^2 & 3EI/L \end{bmatrix}$$

- Careful: the global dof corresponding to a zero local dof should not be zero, i.e.e another element should "contribute" to the global term.
- if we express the spring stiffness k^s as $k^s = \alpha EI/L$, then $\phi = \alpha/(1+\alpha)$. The dependance of the $\overline{K}_{22}$ coefficient on α (assuming both springs having the same stiffness).



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
K_{22} Coefficient
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
α


h
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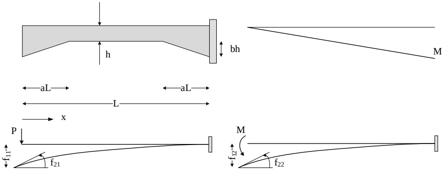
x
f_{11}
f_{21}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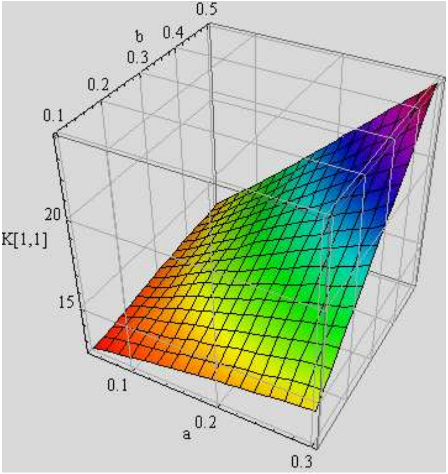
f_{21}
M
f_{12}
f_{22}


0.5
b
0.4
0.3
0.2
0.1
20
K[1,1]
15
0.1
0.2
a
0.3

Intermediary Structural Analysis

Finite Element Formulation

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Fall 2014

Table of Contents I

Introduction

- So far we have considered continuous systems, in this chapter we seek to apply the previously derived relations to discretized systems.
- Primary solutions only at the nodes only (as opposed to a continuous solution inside Ω).
- Application of the Principle of Virtual Displacement requires an assumed displacement field. This displacement field can be approximated by interpolation functions written in terms of:
  1. Unknown polynomial coefficients, most appropriate for continuous systems, and the Rayleigh-Ritz method $v(x) = \underbrace{a_1}_{c^{(1)}} \underbrace{x(L-x)}_{\phi^{(1)}} + \underbrace{a_2}_{c^{(2)}} \underbrace{x^2(L-x)^2}_{\phi^{(2)}} + \dots$ A major drawback of this approach, is that the coefficients have no physical meaning.
  2. Unknown nodal deformations, most appropriate for discrete systems and Potential Energy based formulations $v(\overline{\Delta}_i) = \Delta = N_1\overline{\Delta}_1 + N_2\overline{\Delta}_2 + \dots + N_n\overline{\Delta}_n$ where $\overline{\Delta}_i$ is the known displacement at dof i.

Shape Functions; Definitions I

Expression for the generalized known displacement (translation or rotation), Δ at any point in terms of all its known nodal ones, $\overline{\boldsymbol{\Delta}}$.

$$\Delta = \sum_{i=1}^{n} N_i(x)\overline{\Delta}_i = \lfloor \mathbf{N}(x) \rfloor \{\overline{\boldsymbol{\Delta}}\}$$

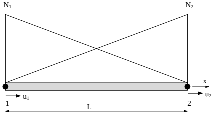
$\overline{\Delta}_i$ is the (generalized) nodal displacement corresponding to d.o.f i

1. N_i is an interpolation function, or shape function which has the following characteristics: $N_i = 1$ at dof i and $N_i = 0$ at dof j where $i \neq j$.
2. Summation of N at any point is equal to unity $\Sigma N = 1$.
3. $\mathbf{N}$ can be derived on the bases of:
   1. Assumed deformation state defined in terms of polynomial series.
   2. Interpolation function (Lagrangian or Hermitian).
4. As with the Rayleigh-Ritz method, polynomial functions should
   1. Be continuous, of the type required by the variational principle.
   2. Exhibit rigid body motion (i.e. $v = a_1 + \ldots$)
   3. Exhibit constant strain.

Shape Functions; Definitions II

- Shape functions should be complete, and meet the same requirements as the coefficients of the Rayleigh Ritz method.
- Shape functions can often be written in non-dimensional coordinates (i.e. $\xi = \frac{x}{l}$). This will be exploited later by the so-called isoparametric elements.

C^0, Axial/Torsional Shape Functions



- Let $u(x) = N_1(x)\overline{u}_1 + N_2(x)\overline{u}_2$ or $\theta_x = N_1\overline{\theta}_{x1} + N_2\overline{\theta}_{x2}$
- We have 2 d.o.f's, we will assume a linear deformation state $u(x) = a_1 x + a_2$ where u can be either Δ or θ, and the essential B.C.'s are given by: $u = \overline{u}_1$ at $x = 0$, and $u = \overline{u}_2$ at $x = L$. Thus we have:

$$\overline{u}_1 = a_2; \qquad \overline{u}_2 = a_1 L + a_2$$

- Solving for a_1 and a_2 in terms of $\overline{u}_1$ and $\overline{u}_2$ we obtain:

$$a_1 = \frac{\overline{u}_2}{L} - \frac{\overline{u}_1}{L}; \qquad a_2 = \overline{u}_1$$

- Substituting and rearranging those expressions we obtain

$$\begin{aligned} u(x) &= (\frac{\overline{u}_2}{L} - \frac{\overline{u}_1}{L})x + \overline{u}_1 \\ &= \underbrace{(1 - \frac{x}{L})}_{N_1(x)}\overline{u}_1 + \underbrace{\frac{x}{L}}_{N_2(x)}\overline{u}_2 \end{aligned}$$

Note that $N1(x) + N2(x) = 1 \quad \forall x \in [0 \;\; L]$

Generalization

- The previous derivation can be generalized by writing:

$$u(x) = a_1 x + a_2 = \underbrace{\lfloor\; x \quad 1 \;\rfloor}_{\lfloor \mathbf{p}(x) \rfloor} \underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\}}_{\{\mathbf{a}\}}$$

  where $\lfloor \mathbf{p}(x) \rfloor$ corresponds to the polynomial approximation, and $\{\mathbf{a}\}$ is the coefficient vector.

- Apply the boundary conditions:

$$\underbrace{\left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \end{array} \right\}}_{\{\overline{\boldsymbol{\Delta}}\}} = \underbrace{\left[\begin{array}{cc} 0 & 1 \\ L & 1 \end{array} \right]}_{[\mathcal{L}]} \underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\}}_{\{\mathbf{a}\}}$$

- Following inversion of $[\mathcal{L}]$, this leads to

$$\underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\}}_{\{\mathbf{a}\}} = \underbrace{\frac{1}{L} \left[\begin{array}{cc} -1 & 1 \\ L & 0 \end{array} \right]}_{[\mathcal{L}]^{-1}} \underbrace{\left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \end{array} \right\}}_{\{\overline{\boldsymbol{\Delta}}\}}$$

- Substituting this last equation, we obtain:

$$u(x) = \underbrace{\underbrace{\lfloor\; (1 - \tfrac{x}{L}) \quad \tfrac{x}{L} \;\rfloor}_{\lfloor \mathbf{p}(x) \rfloor [\mathcal{L}]^{-1}}}_{[\mathbf{N}(x)]} \underbrace{\left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \end{array} \right\}}_{\{\overline{\boldsymbol{\Delta}}\}}$$

- Hence, the shape functions $[\mathbf{N}]$ can be directly obtained from

$$[\mathbf{N}(x)] = \lfloor \mathbf{p}(x) \rfloor [\mathcal{L}]^{-1}$$

C^1, Flexural Shape Functions I

- We have 4 d.o.f.'s, $\{\boldsymbol{\Delta}\}_{4\times 1}$:and hence will need 4 shape functions, N_1 to N_4, and those will be obtained through 4 boundary conditions.
- With four essential boundary conditions (two on each node), we must assume a polynomial with four coefficients

$$\begin{aligned} v(x) &= a_1x^3 + a_2x^2 + a_3x + a_4 \\ \theta(x) &= \frac{dv}{dx} = 3a_1x^2 + 2a_2x + a_3 \end{aligned}$$

C^1, Flexural Shape Functions II

- Note that v can be rewritten as:

$$\{\ v(x)\ \} = \underbrace{\lfloor\ x^3 \quad x^2 \quad x \quad 1\ \rfloor}_{\lfloor \mathbf{p}(x) \rfloor} \underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \right\}}_{\{\mathbf{a}\}}$$

- We now apply the boundary conditions:

$$\begin{array}{lll} v = \overline{v}_1 & \text{at} & x = 0 \\ v = \overline{v}_2 & \text{at} & x = L \\ \theta = \overline{\theta}_1 = \frac{dv}{dx} & \text{at} & x = 0 \\ \theta = \overline{\theta}_2 = \frac{dv}{dx} & \text{at} & x = L \end{array}$$

C^1, Flexural Shape Functions III

or:

$$\underbrace{\begin{Bmatrix} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{Bmatrix}}_{\{\overline{\mathbf{\Delta}}\}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3L^2 & 2L & 1 & 0 \end{bmatrix}}_{[\mathcal{L}]} \underbrace{\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}}_{\{\mathbf{a}\}}$$

- Inverting

$$\underbrace{\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}}_{\{\mathbf{a}\}} = \underbrace{\frac{1}{L^3}\begin{bmatrix} 2 & L & -2 & L \\ -3L & -2L^2 & 3L & -L^2 \\ 0 & L^3 & 0 & 0 \\ L^3 & 0 & 0 & 0 \end{bmatrix}}_{[\mathcal{L}]^{-1}} \underbrace{\begin{Bmatrix} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{Bmatrix}}_{\{\overline{\mathbf{\Delta}}\}}$$

C^1, Flexural Shape Functions IV

- Combining, we obtain:

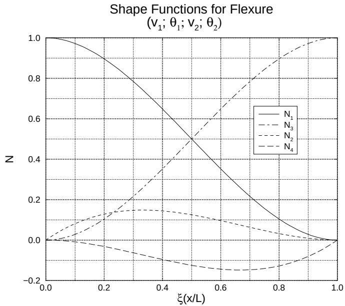
$$
\begin{aligned}
\Delta(x) &= \underbrace{\lfloor\, x^3 \quad x^2 \quad x \quad 1 \,\rfloor}_{\lfloor \mathbf{p}(x) \rfloor} \underbrace{\frac{1}{L^3}\begin{bmatrix} 2 & L & -2 & L \\ -3L & -2L^2 & 3L & -L^2 \\ 0 & L^3 & 0 & 0 \\ L^3 & 0 & 0 & 0 \end{bmatrix}}_{[\mathcal{L}]^{-1}} \underbrace{\begin{Bmatrix} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{Bmatrix}}_{\{\overline{\boldsymbol{\Delta}}\}} \\
&= \underbrace{\lfloor \underbrace{(1+2\xi^3-3\xi^2)}_{N_1} \quad \underbrace{x(1-\xi)^2}_{N_2} \quad \underbrace{(3\xi^2-2\xi^3)}_{N_3} \quad \underbrace{x(\xi^2-\xi)}_{N_4} \rfloor}_{\underbrace{[\mathbf{p}][\mathcal{L}]^{-1}}_{[\mathbf{N}]}} \underbrace{\begin{Bmatrix} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{Bmatrix}}_{\{\overline{\boldsymbol{\Delta}}\}}
\end{aligned}
$$

where $\xi = \frac{x}{L}$.

C^1, Flexural Shape Functions V

- Hence, the shape functions for the flexural element are given by:

$$
\begin{aligned}
N_1 &= (1 + 2\xi^3 - 3\xi^2) \\
N_2 &= x(1 - \xi)^2 \\
N_3 &= (3\xi^2 - 2\xi^3) \\
N_4 &= x(\xi^2 - \xi)
\end{aligned}
$$



C^1, Flexural Shape Functions VI

- Note that Shape function associated with dof 1 is equal to one a $\xi = 0$, equal to zero at $\xi = 1$, and its slopes at those two points is equal to zero. Similarly, shape function 2 is zero at the two end points, slope equal to 1 at $\xi = 0$, and zero at $\xi = 1$.
- Summary

Function	$\xi = 0$		$\xi = 1$	
	N_i	$N_{i,x}$	N_i	$N_{i,x}$
$N_1 = (1 + 2\xi^3 - 3\xi^2)$	1	0	0	0
$N_2 = \xi(1 - \xi)^2$	0	1	0	0
$N_3 = (3\xi^2 - 2\xi^3)$	0	0	1	0
$N_4 = \xi(\xi^2 - \xi)$	0	0	0	1

- Since the transverse displacements and the rotations are uncoupled, we can write

$$\left\{ \begin{array}{c} v \\ \theta \end{array} \right\} = \left[\begin{array}{cccc} N_1 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_4 \end{array} \right] \left\{ \begin{array}{c} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{array} \right\}$$

Finite Element; Introduction

- Earlier in the semester, we derived the stiffness matrices of one dimensional rod elements, the approach used could not be generalized to general finite element. Alternatively, the derivation of this chapter will be applicable to both one dimensional rod (or nearly continuum) elements or contnuum (2D or 3D) elements.
- It is important to note that whereas the previously presented method to derive the stiffness matrix yielded an exact solution, it can not be generalized to continuum (2D/3D elements). On the other hands, the method presented here is an approximate method, which happens to result in an exact stiffness matrix for flexural one dimensional elements. Despite its approximation, this so-called finite element method will yield excellent results if enough elements are used.

Strain Displacement Relations

- The displacement Δ at any point inside an element can be written in terms of the shape functions $\lfloor \mathbf{N} \rfloor$ and the nodal displacements $\{\overline{\boldsymbol{\Delta}}\}$ as $\Delta(x) \stackrel{\text{def}}{=} \lfloor \mathbf{N}(x) \rfloor \{\overline{\boldsymbol{\Delta}}\}$
- The strain is then defined as: $\varepsilon(x) \stackrel{\text{def}}{=} [\mathbf{B}(x)]\{\overline{\boldsymbol{\Delta}}\}$ where $[\mathbf{B}]$ is the matrix which relates nodal displacements to strain field and is clearly expressed in terms of derivatives of $\mathbf{N}$.

Strain Displacement Relations; Axial

$$u(x) = \underbrace{\lfloor \underbrace{(1 - \frac{x}{L})}_{N_1} \quad \underbrace{\frac{x}{L}}_{N_2} \rfloor}_{\lfloor \mathbf{N} \rfloor} \underbrace{\left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \end{array} \right\}}_{\{\overline{\mathbf{\Delta}}\}}$$

$$\varepsilon(x) = \varepsilon_{xx} = \frac{\mathrm{d}u}{\mathrm{d}x} = \underbrace{\left\lfloor \underbrace{-\frac{1}{L}}_{\frac{\partial N_1}{\partial x}} \quad \underbrace{\frac{1}{L}}_{\frac{\partial N_2}{\partial x}} \right\rfloor}_{[\mathbf{B}]} \underbrace{\left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \end{array} \right\}}_{\{\overline{\mathbf{\Delta}}\}}$$

Strain Displacement Relations; Flexural Members

Using the shape functions for flexural elements previously derived in

$$
\begin{aligned}
\varepsilon &= \frac{y}{\rho} = y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \\
&= y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \\
&= \underbrace{y\left\lfloor \underbrace{\frac{6}{L^2}(2\xi - 1)}_{\frac{\partial^2 N_1}{\partial x^2}} \quad \underbrace{-\frac{2}{L}(3\xi - 2)}_{\frac{\partial^2 N_2}{\partial x^2}} \quad \underbrace{\frac{6}{L^2}(-2\xi + 1)}_{\frac{\partial^2 N_3}{\partial x^2}} \quad \underbrace{-\frac{2}{L}(3\xi - 1)}_{\frac{\partial^2 N_4}{\partial x^2}} \right\rfloor}_{[\mathbf{B}]} \underbrace{\left\{ \begin{array}{c} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{array} \right\}}_{\{\overline{\boldsymbol{\Delta}}\}}
\end{aligned}
$$

Virtual Displacement and Strain

In anticipation of the application of the principle of virtual displacement, we define the vectors of virtual displacements and strain in terms of nodal displacements and shape functions:

$$\delta\Delta(x) = [\mathbf{N}(x)]\{\delta\overline{\mathbf{\Delta}}\} \quad (1)$$

$$\delta\varepsilon(x) = [\mathbf{B}(x)]\{\delta\overline{\mathbf{\Delta}}\} \quad (2)$$

Element Stiffness Matrix I

- Recall

$$\{\boldsymbol{\sigma}\} = [\mathbf{D}]\{\boldsymbol{\varepsilon}\} - [\mathbf{D}]\{\boldsymbol{\varepsilon}^0\} \tag{3}$$

where $[\mathbf{D}]$ is the constitutive matrix which relates stress and strain vectors. and $q(x)$ is the load acting on its surface.

- Let us now apply the principle of virtual displacement and restate some known relations (careful with matrices):

$$\begin{aligned}
\delta U &= \delta W && (4)\\
\delta U &= \int_\Omega \lfloor \delta\boldsymbol{\varepsilon} \rfloor \{\boldsymbol{\sigma}\} d\Omega && (5)\\
\{\boldsymbol{\sigma}\} &= [\mathbf{D}]\{\boldsymbol{\varepsilon}\} - [\mathbf{D}]\{\boldsymbol{\varepsilon}^0\} && (6)\\
\{\boldsymbol{\varepsilon}\} &= [\mathbf{B}]\{\overline{\boldsymbol{\Delta}}\} && (7)\\
\{\delta\boldsymbol{\varepsilon}\} &= [\mathbf{B}]\{\delta\overline{\boldsymbol{\Delta}}\} && (8)\\
\lfloor \delta\boldsymbol{\varepsilon} \rfloor &= \lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor [\mathbf{B}]^T && (9)
\end{aligned}$$

- Combining Eqns. 4, 5, 6, 9, and 7, the internal virtual strain energy is given by:

$$\begin{aligned}
\delta U &= \int_\Omega \underbrace{\lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor [\mathbf{B}]^T}_{\lfloor \delta\boldsymbol{\varepsilon} \rfloor} \underbrace{[\mathbf{D}][\mathbf{B}]\{\overline{\boldsymbol{\Delta}}\}}_{\{\boldsymbol{\sigma}\}} \, d\Omega - \int_\Omega \underbrace{\lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor [\mathbf{B}]^T}_{\lfloor \delta\boldsymbol{\varepsilon} \rfloor} \underbrace{[\mathbf{D}]\{\boldsymbol{\varepsilon}^0\}}_{\left\{\boldsymbol{\sigma}^0\right\}} d\Omega \\
&= \lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor \int_\Omega [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}] \, d\Omega \{\overline{\boldsymbol{\Delta}}\} - \lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor \int_\Omega [\mathbf{B}]^T [\mathbf{D}]\{\boldsymbol{\varepsilon}^0\} d\Omega
\end{aligned} \tag{10}$$

Element Stiffness Matrix II

- The virtual external work in turn is given by:

$$\delta W = \underbrace{\lfloor \delta \overline{\boldsymbol{\Delta}} \rfloor}_{\text{Virt. Nodal Displ.}} \underbrace{\{\overline{\mathbf{F}}\}}_{\text{Nodal Force}} + \int_l \lfloor \delta \Delta \rfloor q(x) dx \tag{11}$$

- Combining this equation with $\{\delta\Delta\} = [\mathbf{N}]\{\delta\overline{\boldsymbol{\Delta}}\}$ yields:

$$\delta W = \lfloor \delta \overline{\boldsymbol{\Delta}} \rfloor \{\overline{\mathbf{F}}\} + \lfloor \delta \overline{\boldsymbol{\Delta}} \rfloor \int_0^l [\mathbf{N}]^T q(x)\, dx \tag{12}$$

- Equating the internal strain energy Eqn. 10 with the external work Eqn. 12, we obtain:

$$\underbrace{\lfloor \delta \overline{\boldsymbol{\Delta}} \rfloor \underbrace{\int_\Omega [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}]\, d\Omega}_{[\mathbf{k}]} \{\overline{\boldsymbol{\Delta}}\} - \lfloor \delta \overline{\boldsymbol{\Delta}} \rfloor \underbrace{\int_\Omega [\mathbf{B}]^T [\mathbf{D}]\{\varepsilon^0\} d\Omega}_{\{\overline{\mathbf{F}}^0\}}}_{\delta U} = \underbrace{\lfloor \delta \overline{\boldsymbol{\Delta}} \rfloor \{\overline{\mathbf{F}}\} + \lfloor \delta \overline{\boldsymbol{\Delta}} \rfloor \underbrace{\int_0^l [\mathbf{N}]^T q(x)\, dx}_{\{\overline{\mathbf{F}}^e\}}}_{\delta W} \tag{13}$$

Element Stiffness Matrix III

or

$$\boxed{[\mathbf{k}]\{\overline{\boldsymbol{\Delta}}\} - \{\overline{\mathbf{F}}^{o}\} = \{\overline{\mathbf{F}}\} + \{\overline{\mathbf{F}}^{e}\}} \tag{14}$$

which is the counterpart of Eq. 3.

- Canceling out the $\lfloor \delta\overline{\boldsymbol{\Delta}} \rfloor$ term, this is the same equation of equilibrium as the one written earlier on. It relates the (unknown) nodal displacement $\left\{\overline{\boldsymbol{\Delta}}\right\}$, the structure stiffness matrix $[\mathbf{k}]$, the external nodal force vector $\left\{\overline{\mathbf{F}}\right\}$, the distributed element force $\left\{\overline{\mathbf{F}}^{e}\right\}$, and the vector of initial displacement.
- From this relation we define:

The element stiffness matrix:

$$\boxed{[\mathbf{k}] = \int_{\Omega} [\mathbf{B}]^{T}[\mathbf{D}][\mathbf{B}]d\Omega} \tag{15}$$

Element initial force vector:

$$\boxed{\{\overline{\mathbf{F}}^{0}\} = \int_{\Omega} [\mathbf{B}]^{T}[\mathbf{D}]\{\varepsilon^{0}\}d\Omega} \tag{16}$$

Element Stiffness Matrix IV

Element equivalent load vector:

$$\boxed{\{\overline{\mathbf{F}}^e\} = \int_0^L [\mathbf{N}]\, q(x)\, dx} \tag{17}$$

- The general equation of equilibrium can be written as:

$$\boxed{[\mathbf{k}]\{\overline{\boldsymbol{\Delta}}\} - \{\overline{\mathbf{F}}^0\} = \{\overline{\mathbf{F}}\} + \{\overline{\mathbf{F}}^e\}} \tag{18}$$

Stress Recovery I

- Whereas from the preceding section, we derived a general relationship in which the nodal displacements are the primary unknowns, we next seek to determine the internal (generalized) stresses which are most often needed for design.
- Recalling that we have:

$$\{\boldsymbol{\sigma}\} = [\mathbf{D}]\{\boldsymbol{\varepsilon}\} \tag{19}$$

$$\{\boldsymbol{\varepsilon}\} = [\mathbf{B}]\{\overline{\boldsymbol{\Delta}}\} \tag{20}$$

- With the vector of nodal displacement $\{\boldsymbol{\Delta}\}$ known, those two equations would yield:

$$\boxed{\{\boldsymbol{\sigma}\} = [\mathbf{D}]\cdot[\mathbf{B}]\{\overline{\boldsymbol{\Delta}}\}} \tag{21}$$

- We note that the secondary variables (strain and stresses) are derivatives of the primary variables (displacement), and as such may not always be determined with the same accuracy.

Stiffness Matrix of the Truss Element

- The shape functions of the truss element were derived earlier:

$$\begin{aligned} N_1 &= 1 - \frac{x}{L} \\ N_2 &= \frac{x}{L} \end{aligned}$$

- The corresponding strain displacement relation $[\mathbf{B}]$ is given by:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\mathbf{du}}{\mathrm{d}x} \\ &= \left[\begin{array}{cc} \frac{dN_1}{\mathrm{d}x} & \frac{dN_2}{\mathrm{d}x} \end{array} \right] \\ &= \underbrace{\left[\begin{array}{cc} -\frac{1}{L} & \frac{1}{L} \end{array} \right]}_{[\mathbf{B}]} \end{aligned}$$

- For the truss element, the constitutive matrix $[\mathbf{D}]$ reduces to the scalar E; Hence, substituting into Eq. 15, with $\mathrm{d}\Omega = \mathrm{d}A\mathrm{d}x$: $[\mathbf{k}] = \int_\Omega [\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]\mathrm{d}\Omega$
- But $\mathrm{d}\Omega = A\mathrm{d}x$ and for element with constant cross sectional area we obtain:

$$[\mathbf{k}] = A\int_0^L \left\{ \begin{array}{c} -\frac{1}{L} \\ \frac{1}{L} \end{array} \right\} \cdot E \cdot \lfloor \begin{array}{cc} -\frac{1}{L} & \frac{1}{L} \end{array} \rfloor \mathrm{d}x$$

$$\boxed{\begin{aligned} [\mathbf{k}] &= \frac{AE}{L^2}\int_0^L \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \mathrm{d}x \\ &= \frac{AE}{L} \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \end{aligned}}$$

Stiffness Matrix of Beam Element I

- For a beam element, for which we have previously derived the shape functions and the $[\mathbf{B}]$ matrix. Substituting in Eq. 15:

$$[\mathbf{k}] = \int_0^L \int_A [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}]\, y^2 \,\mathrm{d}A\, \mathrm{d}x$$

- Noting that $\int_A y^2 \,\mathrm{d}A = I_z$ Eq. 15 reduces to

$$[\mathbf{k}] = \int_0^L [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}] I_z \,\mathrm{d}x$$

- For this simple case, we have: $[\mathbf{D}] = E$, thus:

$$[\mathbf{k}] = EI_z \int_0^l [\mathbf{B}]^T [\mathbf{B}] \,\mathrm{d}x$$

Stiffness Matrix of Beam Element II

- Using the shape function for the beam element, and noting the change of integration variable from dx to dξ, we obtain

$$[\mathbf{k}] = EI_z \int_0^1 \left\{ \begin{array}{c} \frac{6}{L^2}(2\xi - 1) \\ -\frac{2}{L}(3\xi - 2) \\ \frac{6}{L^2}(-2\xi + 1) \\ -\frac{2}{L}(3\xi - 1) \end{array} \right\} \left\lfloor \begin{array}{cccc} \frac{6}{L^2}(2\xi - 1) & -\frac{2}{L}(3\xi - 2) & \frac{6}{L^2}(-2\xi + 1) & -\frac{2}{L}(3\xi - 1) \end{array} \right\rfloor \underbrace{L d\xi}_{dx}$$

or

$$\boxed{[\mathbf{k}] = \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{array}{cccc} \overline{V}_1 & \overline{\theta}_1 & \overline{V}_2 & \overline{\theta}_2 \end{array} \atop \left[\begin{array}{cccc} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{array} \right]}$$

Identical to the matrix previously derived earlier in the semester ☺

Intermediary Structural Analysis

A Brief Overview of Mechanics

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Table of Contents I

- Generalize the concept of a vector by introducing the tensor ($\mathtt{T}$).
- A tensor is an operator which operates on tensors to produce other tensors.
- Designate this operation as $\mathtt{T}\cdot\mathtt{v}$ or simply $\mathtt{T}\mathtt{v}$.
- A tensor is also a physical quantity, independent of any particular coordinate system yet specified most conveniently by referring to an appropriate system of coordinates.
- A tensor is classified by the rank or order. A Tensor of order zero is specified in any coordinate system by one coordinate and is a scalar (such as temperature). A tensor of order one has three coordinate components in space, hence it is a vector (such as force). In general 3-D space the number of components of a tensor is 3^n where n is the order of the tensor.
- A force and a stress are tensors of order 1 and 2 respectively.

- Engineering notation may be the simplest and most intuitive one, it often leads to long and repetitive equations. Alternatively, tensor or the dyadic form will lead to shorter and more compact forms.
- The following rules define indicial notation:
  1. If there is one letter index (free index), that index goes from i to n (range of the tensor). For instance:

$$a_i = a^i = \lfloor \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \rfloor = \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right\} \qquad i = 1, 3$$

     assuming that $n = 3$.
  2. A repeated index or (dummy index) will take on all the values of its range, and the resulting tensors summed. In general no index occurs more than twice in a properly written expression.For instance:

$$a_{1i}x_i = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$

  3. Tensor's order:

- First order tensor (such as force) has only one free index:

$$a_i = a^i = \lfloor \; a_1 \quad a_2 \quad a_3 \; \rfloor$$

  other first order tensors $a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$, F_{ikk}, $\varepsilon_{ijk}u_jv_k$ (note that there is only one free index).
- Second order tensor (such as stress or strain) will have two free indices.

$$T_{ij} = \begin{bmatrix} T_{11} & T_{22} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

  other examples A_{ijip}, $\delta_{ij}u_kv_k$.
- A fourth order tensor (such as Elastic constants) will have four free indices: $\sigma_{ij} = D_{ijkl}\varepsilon_{kl}$

4. Derivatives of tensor with respect to x_i is written as $,i$. For example:

$$\frac{\partial\Phi}{\partial x_i} = \Phi_{,i} \quad \frac{\partial v_i}{\partial x_i} = v_{i,i} \quad \frac{\partial v_i}{\partial x_j} = v_{i,j} \quad \frac{\partial T_{i,j}}{\partial x_k} = T_{i,j,k}$$

- Usefulness of the indicial notation is in presenting systems of equations in compact form. For instance:

$$x_i = c_{ij} z_j$$

this simple compacted equation, when expanded would yield:

$$\begin{aligned} x_1 &= c_{11} z_1 + c_{12} z_2 + c_{13} z_3 \\ x_2 &= c_{21} z_1 + c_{22} z_2 + c_{23} z_3 \\ x_3 &= c_{31} z_1 + c_{32} z_2 + c_{33} z_3 \end{aligned}$$

Similarly:

$$A_{ij} = B_{ip} C_{jq} D_{pq}$$

$$\begin{aligned} A_{11} &= B_{11} C_{11} D_{11} + B_{11} C_{12} D_{12} + B_{12} C_{11} D_{21} + B_{12} C_{12} D_{22} \\ A_{12} &= B_{11} C_{21} D_{11} + B_{11} C_{22} D_{12} + B_{12} C_{21} D_{21} + B_{12} C_{22} D_{22} \\ A_{21} &= B_{21} C_{11} D_{11} + B_{21} C_{12} D_{12} + B_{22} C_{11} D_{21} + B_{22} C_{12} D_{22} \\ A_{22} &= B_{21} C_{21} D_{11} + B_{21} C_{22} D_{12} + B_{22} C_{21} D_{21} + B_{22} C_{22} D_{22} \end{aligned}$$

- Using indicial notation, we may rewrite the definition of the dot product

$$\mathbf{a}\cdot\mathbf{b} = a_i b_i = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) = a_x b_x + a_y b_y + a_z b_z$$

- Note that one can adopt the dyadic instead of the indicial notation for tensors as **linear vector operators** $\mathbf{u} = \mathbf{T}\cdot\mathbf{v}$ or $u_i = T_{ij} v_j$

- The sum of two tensors (must be of the same orde)is simply defined as:

$$\mathrm{S}_{ij} = \mathrm{T}_{ij} + \mathrm{U}_{ij}$$

- The scalar multiplication of a (second order) tensor is defined by:

$$\mathrm{S}_{ij} = \lambda \mathrm{T}_{ij}$$

- The outer product of two tensors is the tensor whose components are formed by multiplying each component of one of the tensors by every component of the other. This produces a tensor with an order equal to the sum of the orders of the factor tensors.

$$\begin{aligned} a_i b_j &= T_{ij} \\ v_i F_{jk} &= b_{ijk} \\ D_{ij} T_{km} &= \phi_{ijkm} \end{aligned} \quad \text{or} \left\{ \quad \right\}_{nx1} \lfloor \quad \rfloor_{1xm} = \left[\quad \right]_{nxm}$$

- The inner product of two tensors: contraction of one index from each tensor

$$
\begin{aligned}
& a_i b_i \\
a_i E_{ik} &= f_k \quad \text{or} \lfloor \quad \rfloor_{1xm} \begin{bmatrix} & \\ & \end{bmatrix}_{mxn} = \lfloor \quad \rfloor_{1xn} \\
E_{ij} F_{jm} &= G_{im} \quad \text{or} \begin{bmatrix} & \\ & \end{bmatrix}_{nxp} \begin{bmatrix} & \\ & \end{bmatrix}_{pxm} = \begin{bmatrix} & \\ & \end{bmatrix}_{nxm}
\end{aligned}
$$

- The cross product can be defined

$$\mathrm{a} \times \mathrm{b} = \varepsilon_{pqr} a_q b_r \mathrm{e}_p = (a_y b_z - a_z b_y)\mathrm{i} + (a_z b_x - a_x b_z)\mathrm{j} + (a_x b_y - a_y b_x)\mathrm{k}$$

In the second equation, there is one free index p thus there are three equations, there are two repeated (dummy) indices q and r, thus each equation has nine terms. ε_{pqr} is called the permutation symbol and is defined as

$$
\varepsilon_{pqr} = \begin{cases}
1 & \text{If the value of} i, j, k \text{are an even permutation of 1,2,3} \\
 & \text{(i.e. if they appear as 1 2 3 1 2)} \\
-1 & \text{If the value of} i, j, k \text{are an odd permutation of 1,2,3} \\
 & \text{(i.e. if they appear as 3 2 1 3 2)} \\
0 & \text{If the value of} i, j, k \text{are not permutation of 1,2,3} \\
 & \text{(i.e. if two or more indices have the same value)}
\end{cases}
$$

- Two fundamental tensors in continuum mechanics are second order and symmetric (stress and strain), we examine some important properties of these tensors.
- For every symmetric tensor T_{ij} defined at some point in space, there is associated with each direction (specified by unit normal n_j) at that point, a vector given by the inner product

$$v_i = T_{ij} n_j$$

If the direction is one for which v_i is parallel to n_i, the inner product is

$$T_{ij} n_j = \lambda n_i$$

and the direction n_i is called principal direction of T_{ij}. Since $n_i = \delta_{ij} n_j$, this can be rewritten as

$$(T_{ij} - \lambda \delta_{ij}) n_j = 0$$

which represents a system of three equations for the four unknowns n_i and λ.

$$\begin{aligned}
(T_{11} - \lambda) n_1 + T_{12} n_2 + T_{13} n_3 &= 0 \\
T_{21} n_1 + (T_{22} - \lambda) n_2 + T_{23} n_3 &= 0 \\
T_{31} n_1 + T_{32} n_2 + (T_{33} - \lambda) n_3 &= 0
\end{aligned}$$

To have a non-trivial solution ($n_i = 0$) the determinant of the coefficients must be zero,

$$|T_{ij} - \lambda\delta_{ij}| = 0$$

- Expansion of this determinant leads to the following characteristic equation

$$\lambda^3 - I_T\lambda^2 + II_T\lambda - III_T = 0$$

the roots are called the principal values of T_{ij} and

$$\begin{aligned} I_T &= T_{ii} = \text{tr } T_{ii} \\ II_T &= \frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ij}) \\ III_T &= |T_{ij}| = \det T_{ij} \end{aligned}$$

are called the first, second and third invariants respectively of T_{ij}.

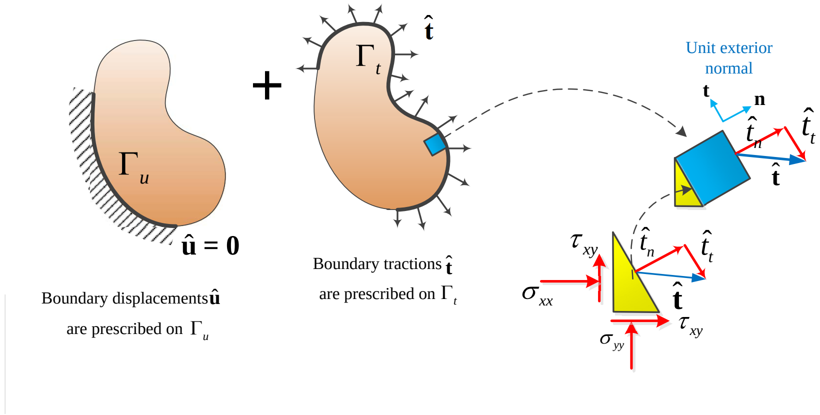
- It is customary to order those roots as $\lambda_{(1)} > \lambda_{(2)} > \lambda_{(3)}$
- For a symmetric tensor with real components, the principal values are also real. If those values are distinct, the three principal directions are mutually orthogonal.

- There are two kinds of forces in continuum mechanics

body forces: act on the elements of volume or mass inside the body, e.g. gravity, electromagnetic fields. $\mathrm{d}\mathbf{F} = \rho \mathbf{b} \mathrm{d}Vol$.

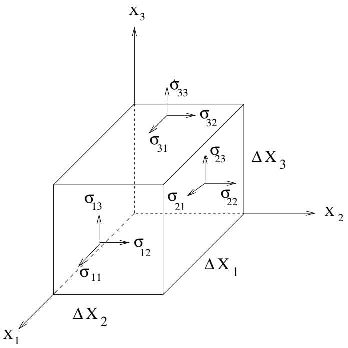
Surface forces (or traction) are contact forces acting on the free body at its bounding surface. Those will be defined in terms of force per unit area.

$$\int_S \mathbf{t} \mathrm{d}S = \mathbf{i} \int_S t_x \mathrm{d}S + \mathbf{j} \int_S t_y \mathrm{d}S + \mathbf{k} \int_S t_z \mathrm{d}S$$



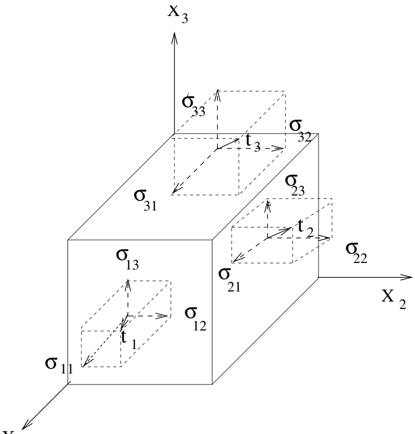
- Usually limit the term traction to an actual bounding surface of a body, and use the term stress vector for an imaginary interior surface.

- The traction vectors on planes perpendicular to the coordinate axes are particularly useful. When the vectors acting at a point on three such mutually perpendicular planes is given, the stress vector at that point on any other arbitrarily inclined plane can be expressed in terms of the first set of tractions.
- A stress is a second order cartesian tensor, σ_{ij} where the 1st subscript (i) refers to the direction of outward facing normal, and the second one (j) to the direction of component force.



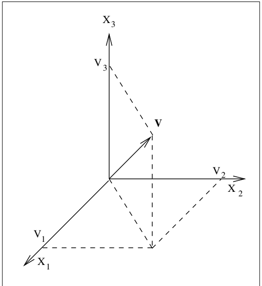
$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \left\{ \begin{array}{c} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{array} \right\}$$

- In fact the nine rectangular components σ_{ij} of $\boldsymbol{\sigma}$ turn out to be the three sets of three vector components $(\sigma_{11}, \sigma_{12}, \sigma_{13})$, $(\sigma_{21}, \sigma_{22}, \sigma_{23})$, $(\sigma_{31}, \sigma_{32}, \sigma_{33})$ which correspond to the three tractions $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$ which are acting on the x_1, x_2 and x_3 faces.
- Those tractions are not necessarily normal to the faces, and they can be decomposed into a normal and shear traction if need be. In other words, stresses are nothing else than the components of tractions (stress vector).



Stresses as components of a traction vector

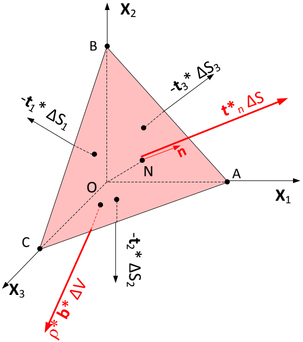
(Components of a tensor of order 2 are vectors)



(Components of a vector are scalars)

- The state of stress at a point cannot be specified entirely by a single vector with three components; it requires the second-order tensor with all nine components.

- We seek to determine the traction acting on the surface of an oblique plane (characterized by its normal n) in terms of the known tractions normal to the three principal axis, $\mathrm{t}_1, \mathrm{t}_2$ and t_3.
- Cauchy's tetrahedron



will be obtained without any assumption of equilibrium and it will apply in fluid dynamics as well as in solid mechanics.

- This equation is a vector equation, and the corresponding algebraic equations for the components of $\mathbf{t}_n$ are

$$
\begin{array}{lrcl}
 & t_{n_1} & = & \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3 \\
 & t_{n_2} & = & \sigma_{12}n_1 + \sigma_{22}n_2 + \sigma_{32}n_3 \\
 & t_{n_3} & = & \sigma_{13}n_1 + \sigma_{23}n_2 + \sigma_{33}n_3 \\
 & \text{or} & & \\
\text{Indicial notation} & t_{n_i} & = & \sigma_{ji}n_j \\
\text{dyadic notation} & \mathbf{t}_n & = & \mathbf{n}\cdot\boldsymbol{\sigma} = \boldsymbol{\sigma}^T\cdot\mathbf{n}
\end{array}
$$

- We have thus established that the nine components σ_{ij} are components of the second order tensor, Cauchy's stress tensor.

- For a stress tensor at point P given by

$$\sigma = \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{Bmatrix} \mathrm{t}_1 \\ \mathrm{t}_2 \\ \mathrm{t}_3 \end{Bmatrix}$$

  We seek to determine the traction (or stress vector) t passing through P and parallel to the plane ABC where $A(4,0,0)$, $B(0,2,0)$ and $C(0,0,6)$.

- The vector normal to the plane can be found by taking the cross products of vectors AB and AC:

$$\begin{aligned} \mathrm{N} &= \mathrm{AB}\times\mathrm{AC} = \begin{vmatrix} \mathrm{e}_1 & \mathrm{e}_2 & \mathrm{e}_3 \\ -4 & 2 & 0 \\ -4 & 0 & 6 \end{vmatrix} \\ &= 12\mathrm{e}_1 + 24\mathrm{e}_2 + 8\mathrm{e}_3 \end{aligned}$$

- The unit normal of N is given by

$$\mathbf{n} = \frac{3}{7}\mathbf{e}_1 + \frac{6}{7}\mathbf{e}_2 + \frac{2}{7}\mathbf{e}_3$$

Hence the stress vector (traction) will be

$$\lfloor \begin{array}{ccc} \frac{3}{7} & \frac{6}{7} & \frac{2}{7} \end{array} \rfloor \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \lfloor \begin{array}{ccc} -\frac{9}{7} & \frac{5}{7} & \frac{10}{7} \end{array} \rfloor$$

and thus $\mathbf{t} = -\frac{9}{7}\mathbf{e}_1 + \frac{5}{7}\mathbf{e}_2 + \frac{10}{7}\mathbf{e}_3$

- The principal stresses are physical quantities, whose values do not depend on the coordinate system in which the components of the stress were initially given. They are therefore invariants of the stress state.
- When the determinant in the characteristic equation is expanded, the cubic equation takes the form

$$\lambda^3 - I_\sigma \lambda^2 - II_\sigma \lambda - III_\sigma = 0$$

where the symbols I_σ, II_σ and III_σ denote the following scalar expressions in the stress components:

$$\begin{aligned}
I_\sigma &= \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{ii} = \mathrm{tr}\,\boldsymbol{\sigma} \\
II_\sigma &= -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2 \\
&= \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}) = \frac{1}{2}\sigma_{ij}\sigma_{ij} - \frac{1}{2}I_\sigma^2 \\
&= \frac{1}{2}(\boldsymbol{\sigma} : \boldsymbol{\sigma} - I_\sigma^2) \\
III_\sigma &= \det\boldsymbol{\sigma} = \frac{1}{6}e_{ijk}e_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr}
\end{aligned}$$

- In terms of the principal stresses, those invariants can be simplified into

$$\begin{aligned} I_\sigma &= \sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)} \\ II_\sigma &= -(\sigma_{(1)}\sigma_{(2)} + \sigma_{(2)}\sigma_{(3)} + \sigma_{(3)}\sigma_{(1)}) \\ III_\sigma &= \sigma_{(1)}\sigma_{(2)}\sigma_{(3)} \end{aligned}$$

- let σ denote the mean normal stress p

$$\sigma = -p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{ii} = \frac{1}{3}\text{tr } \boldsymbol{\sigma}$$

then the stress tensor can be written as the sum of two tensors:

Hydrostatic stress in which each normal stress is equal to $-p$ and the shear stresses are zero. The hydrostatic stress produces volume change without change in shape in an isotropic medium.

$$\sigma_{hyd} = -p\mathrm{I} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

Deviatoric Stress: which causes the change in shape.

$$\sigma_{dev} = \begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{bmatrix}$$

- From Eq. **??** and **??**, the stress transformation for the second order stress tensor is given by

$$\boxed{\begin{array}{rcl} \overline{\sigma}_{ip} & = & a_i^j a_p^q \sigma_{jq} \text{ in Matrix Form } [\overline{\sigma}] = [A]^T[\sigma][A] \\ \sigma_{jq} & = & a_i^j a_p^q \overline{\sigma}_{ip} \text{ in Matrix Form } [\sigma] = [A][\overline{\sigma}][A]^T \end{array}} \tag{1}$$

- For the 2D plane stress case we rewrite Eq. **??**

$$\boxed{\left\{ \begin{array}{c} \overline{\sigma}_{xx} \\ \overline{\sigma}_{yy} \\ \overline{\sigma}_{xy} \end{array} \right\} = \left[\begin{array}{ccc} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \cos\alpha\sin\alpha & \cos^2\alpha - \sin^2\alpha \end{array} \right] \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\}} \tag{2}$$

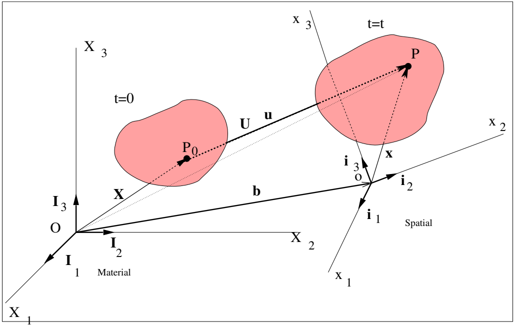
It is often necessary to express cartesian stresses in terms of polar stresses and vice versa. This can be done through the following relationships

$$\begin{array}{rcl} \sigma_{xx} & = & \sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - \sigma_{r\theta}\sin 2\theta \\ \sigma_{yy} & = & \sigma_{rr}\sin^2\theta + \sigma_{\theta\theta}\cos^2\theta + \sigma_{r\theta}\sin 2\theta \\ \sigma_{xy} & = & (\sigma_{rr} - \sigma_{\theta\theta})\sin\theta\cos\theta + \sigma_{r\theta}(\cos^2\theta - \sin^2\theta) \end{array}$$

and

$$
\begin{aligned}
\sigma_{rr} &= \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right)\left(1-\frac{a^2}{r^2}\right)+\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta \\
&\quad +\sigma_{xy}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\sin 2\theta \\
\sigma_{\theta\theta} &= \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right)\left(1+\frac{a^2}{r^2}\right)-\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta \\
&\quad -\sigma_{xy}\left(1+\frac{3a^4}{r^4}\right)\sin 2\theta \\
\sigma_{r\theta} &= -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta+\sigma_{xy}\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)
\end{aligned}
$$

- The undeformed configuration of a material continuum at time $t = 0$ together with the deformed configuration at $t = t$.



- In the initial configuration P_0 has the position vector

$$\mathrm{X} = X_1 \mathrm{I}_1 + X_2 \mathrm{I}_2 + X_3 \mathrm{I}_3$$

which is here expressed in terms of the material coordinates (X_1, X_2, X_3).

- In the deformed configuration, the particle P_0 has now moved to the new position P and has the following position vector

$$\mathbf{x} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3$$

which is expressed in terms of the spatial coordinates.

- The displacement vector $\mathbf{u}$ connecting P_0 and P is the displacement vector which can be expressed in both the material or spatial coordinates

$$\begin{aligned} \mathbf{U} &= U_K\mathbf{I}_K \\ \mathbf{u} &= u_k\mathbf{i}_k \end{aligned}$$

- From the preceding figure we can express motion as

$$\begin{array}{ll} x_i = x_i(X_1, X_2, X_3, t) & \text{Lagrangian formulation} \\ X_i = X_i(x_1, x_2, x_3, t) & \text{Eulerian formulation} \end{array}$$

- Ignoring a detailed analysis of large deformation, it is determined that

		Displacement gradient	
		Small	Large
Displacement	Small	Lagrangian small strain (Cauchy)	Lagrangian large strain (Green-Lagrange)
	Large	Eulerian small strain	Eulerian finite strain (Eulerian-Almansi)

- The Lagrangian finite strain tensor can be written as

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right)$$

- Alternatively these equations may be expanded as

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
\varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right)
\end{aligned}$$

- We define the engineering shear strain as

$$\gamma_{ij} = 2\varepsilon_{ij} \qquad (i \neq j)$$

- If $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ then we have six differential equations (in 3D the strain tensor has a total of 9 terms, but due to symmetry, there are 6 independent ones) for determining (upon integration) three unknowns displacements u_i. Hence the system is overdetermined, and there must be some linear relations between the strains.

- It can be shown (through appropriate successive differentiation) that the compatibility relation for strain reduces to:

$$\frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_j} + \frac{\partial^2 \varepsilon_{jj}}{\partial x_i \partial x_k} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ij}}{\partial x_j \partial x_k} = 0.$$

  In 3D, this would yield 9 equations in total, however only six are distinct.

- In 2D, this results in (by setting $i = 2$, $j = 1$ and $l = 2$):

$$\begin{aligned} \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} &= 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \\ &= \frac{\partial^2 \gamma_{12}}{\partial x_1 \partial x_2} \end{aligned}$$

  (recall that $2\varepsilon_{12} = \gamma_{12}$).

- We have thus far studied tensor fields (stress and strain).
- We have also obtained only one differential equation, that was the compatibility equation.
- Next we still derive additional differential equations governing the way stress and deformation vary at a point and with time. They will apply to any continuous medium, and yet we will not have enough equations to determine unknown tensor field. For that we need to wait for constitutive laws relating stress and strain will be introduced.
- The fundamental equations are:
  1. Conservation of mass (continuity equation)
  2. Conservation of momentum (Equation of motion; Equilibrium)
  3. Conservation of Energy.

- A conservation law establishes a balance of a scalar or tensorial quantity in volume V bounded by a surface S (inside a control surface). In its most general form, such a law may be expressed as

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathcal{A}\mathrm{d}V}_{\text{Rate of variation}} - \underbrace{\int_S \alpha \mathrm{d}S}_{\text{Exchange by Diffusion}} = \underbrace{\int_V \mathtt{A}\mathrm{d}V}_{\text{Source}}$$

- The preceding equation reads: rate of increase of $\mathcal{A}$ inside a control volume plus the rate of outward flux of $\mathcal{A}$ through the surface of the control volume is equal to the rate of increase of $\mathtt{A}$ inside the control volume
- The dimensions of various quantities are given by

$$\begin{aligned} \dim(\alpha) &= \dim(\mathcal{A}Lt^{-1}) \\ \dim(\mathtt{A}) &= \dim(\mathcal{A}t^{-1}) \end{aligned}$$

rightfully all expressed in terms of $\mathcal{A}$.

- The time rate of change of the total momentum of a given set of particles equals the vector sum of all external forces acting on the particles of the set, provided Newton's Third Law applies.
- The continuum form of this principle is a basic postulate of continuum mechanics (postulate: a statement, also known as an axiom, which is taken to be true without proof).
- Starting with (Newton's second law)

$$\underbrace{\int_S \mathbf{t} dS + \int_V \rho \mathbf{b} dV}_{\mathbf{F}} = \underbrace{\frac{d}{dt}\int_V \rho \mathbf{v} dV}_{m\mathbf{a}} \tag{3}$$

- Divergence Theorem

$$\int_V v_{i,i} dV = \int_S \underbrace{v_i n_i}_{\text{flux}} dS$$

The flux of a vector function through some closed surface equals the integral of the divergence of that function over the volume enclosed by the surface.

- we substitute $t_i = T_{ij} n_j$ and apply the divergence theorem to obtain

$$\begin{aligned}\int_V \left(\frac{\partial T_{ij}}{\partial x_j} + \rho b_i \right) \mathrm{d}V &= \int_V \rho \frac{\mathrm{d}V_i}{\mathrm{d}t} \mathrm{d}V \\ \int_V \left[\frac{\partial T_{ij}}{\partial x_j} + \rho b_i - \rho \frac{\mathrm{d}v_i}{\mathrm{d}t} \right] \mathrm{d}V &= 0\end{aligned}$$

or for an arbitrary volume

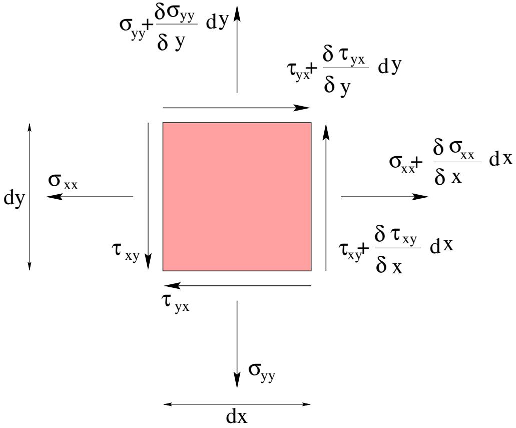
$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = \rho \frac{\mathrm{d}v_i}{\mathrm{d}t} \tag{4}$$

which is Cauchy's (first) equation of motion, or the linear momentum principle, or more simply equilibrium equation.

- When expanded in 3D, this equation yields:

$$\begin{aligned}\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} + \rho b_1 &= 0 \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} + \rho b_2 &= 0 \\ \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} + \rho b_3 &= 0\end{aligned}$$

- We note that these equations could also have been derived from the free body diagram with the assumption of equilibrium (via Newton's second law) considering an infinitesimal element of dimensions $dx_1 \times dx_2 \times dx_3$.



- If mechanical quantities only are considered, the principle of conservation of energy for the continuum may be derived directly from the equation of motion given by Eq. 4. This is accomplished by taking the integral over the volume V of the scalar product between Eq. 4 and the velocity v_i.

$$\int_V \rho v_i \frac{\mathrm{d}v_i}{\mathrm{d}t}\mathrm{d}V = \int_V v_i T_{ji,j}\mathrm{d}V + \int_V \rho b_i v_i \mathrm{d}V \tag{5}$$

- If we consider the left hand side

$$\int_V \rho v_i \frac{\mathrm{d}v_i}{\mathrm{d}t}\mathrm{d}V = \frac{d}{\mathrm{d}t}\int_V \frac{1}{2}\rho v_i v_i \mathrm{d}V = \frac{d}{\mathrm{d}t}\int_V \frac{1}{2}\rho v^2 \mathrm{d}V = \frac{\mathrm{d}K}{\mathrm{d}t} \tag{6}$$

which represents the time rate of change of the kinetic energy K in the continuum.

- If we consider thermal processes, the rate of increase of total heat into the continuum is given by

$$Q = -\int_S q_i n_i \mathrm{d}S + \int_V \rho r \mathrm{d}V \tag{7}$$

Q has the dimension[1] of power, that is ML^2T^{-3}, and the SI unit is the Watt (W). q is the heat flux per unit area by conduction, its dimension is MT^{-3} and the corresponding SI unit is Wm^{-2}. Finally, r is the radiant heat constant per unit mass, its dimension is $MT^{-3}L^{-4}$ and the corresponding SI unit is Wm^{-6}.

- We thus have

$$\frac{\mathrm{d}K}{\mathrm{d}t} + \int_V D_{ij} T_{ij} \mathrm{d}V = \int_V (v_i T_{ji})_{,j} \mathrm{d}V + \int_V \rho v_i b_i \mathrm{d}V + Q \tag{8}$$

- We next convert the first integral on the right hand side to a surface integral by the divergence theorem ($\int_V \boldsymbol{\nabla}\cdot\mathrm{T}dV = \int_S \mathrm{T.n}dS$) and since $t_i = T_{ij}n_j$ we obtain

$$\frac{\mathrm{d}K}{\mathrm{d}t} + \int_V D_{ij}T_{ij}\mathrm{d}V = \int_S v_i t_i \mathrm{d}S + \int_V \rho v_i b_i \mathrm{d}V + Q \tag{9}$$

$$\frac{\mathrm{d}K}{\mathrm{d}t} + \frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{W}}{\mathrm{d}t} + Q \tag{10}$$

  this equation relates the time rate of change of total mechanical energy of the continuum on the left side to the rate of work done by the surface and body forces on the right hand side.

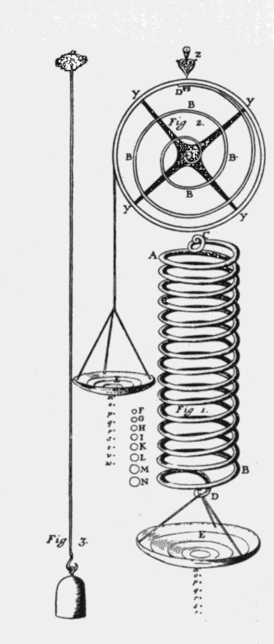
- If both mechanical and non mechanical energies are to be considered, the first principle states that the time rate of change of the kinetic plus the internal energy is equal to the sum of the rate of work plus all other energies supplied to, or removed from the continuum per unit time (heat, chemical, electromagnetic, etc.).

- For a thermomechanical continuum, it is customary to express the time rate of change of internal energy by the integral expression

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{d}{\mathrm{d}t}\int_V \rho u \mathrm{d}V \tag{11}$$

where u is the internal energy per unit mass or specific internal energy. We note that U appears only as a differential in the first principle, hence if we really need to evaluate this quantity, we need to have a reference value for which U will be null. The dimension of U is one of energy dim $U = ML^2T^{-2}$, and the SI unit is the Joule, similarly dim $u = L^2T^{-2}$ with the SI unit of Joule/Kg.

[1] Work=$FL = ML^2T^{-2}$; Power=Work/time



Hooke

ceiinosssttuu

Hooke, 1676

Ut tensio sic vis

Hooke, 1678

- The Generalized Hooke's Law can be written as:

$$\sigma_{ij} = D_{ijkl}\varepsilon_{kl} \qquad i,j,k,l = 1,2,3$$

- The (fourth order) tensor of elastic constants D_{ijkl} has 81 (3^4) components however, due to the symmetry of both σ and ε,there are at most 36 $\left(\frac{9(9-1)}{2}\right)$ distinct elastic terms.

- In terms of Lame's constants (which naturally are derived from coninuum mechanics consideration, but can not be both experimentally measured), Hooke's Law for an isotropic body is written as

$$T_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij}; \qquad E_{ij} = \frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\lambda+2\mu}\delta_{ij}T_{kk}\right)$$

- In terms of engineering constants (which can be measured in the laboratory)

$$\begin{array}{rcll} \frac{1}{E} & = & \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}; & \nu = \frac{\lambda}{2(\lambda+\mu)} \\ \lambda & = & \frac{\nu E}{(1+\nu)(1-2\nu)}; & \mu = G = \frac{E}{2(1+\nu)} \end{array}$$

- Hooke's law for isotropic material in terms of engineering constants becomes

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right); \qquad \varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}$$

- When the strain equation is expanded in 3D cartesian coordinates it would yield:

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}(2\varepsilon_{xy})\\ \gamma_{yz}(2\varepsilon_{yz})\\ \gamma_{zx}(2\varepsilon_{zx})\end{array}\right\} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1+\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1+\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1+\nu\end{bmatrix}\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{array}\right\}$$

- Plane Strain

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\end{array}\right\} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu) & \nu & 0\\ \nu & (1-\nu) & 0\\ \nu & \nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\}$$

- **Axisymmetry**

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u}{\partial r}; & \varepsilon_{\theta\theta} &= \frac{u}{r} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z}; & \varepsilon_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{aligned}$$

The constitutive relation is again analogous to 3D/plane strain

$$\left\{ \begin{array}{c} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \tau_{rz} \end{array} \right\} = \frac{E}{(1+\nu)(1-2\nu)} \left[\begin{array}{cccc} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{array} \right] \left\{ \begin{array}{c} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\theta\theta} \\ \gamma_{rz} \end{array} \right\}$$

- **Plane Stress**

$$\begin{aligned} \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{array} \right\} &= \frac{1}{1-\nu^2} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{array} \right] \left\{ \begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{array} \right\} \\ \varepsilon_{zz} &= -\frac{1}{1-\nu}\nu(\varepsilon_{xx} + \varepsilon_{yy}) \end{aligned}$$

Intermediary Structural Analysis

Variational and Energy Methods

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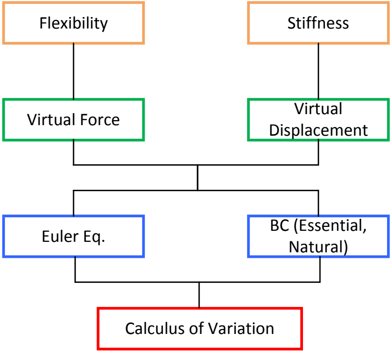
- Structural engineering (and mechanics) can be approached from two different angles:
  1. Newtonian approach, equations of equilibrium.
  2. Lagrangian approach: thermodynamics (balance of energy).
- So far we have pursued the former, from this point onward, we shall focus on the second which will provide the formalism needed to develop the finite element method.
- Some of the concepts will look familiar (first law of thermodynamic, principle of virtual force, minimum potential energy) at first.
- This chapter will
  1. Provide a rigorous framework for variational methods which are the basis of so-called “energy” methods. In so doing, formalize the definition of Natural and Essential boundary conditions.
  2. Bring together the various "energy methods" and show that they are all (essentially) the same.
  3. Develop the principle of virtual displacement as a prelude to the finite element method.

4. Show the duality between the so-called strong form (differential equation) and the weak form (satisfy a principle in an average sense).

- So far, analyses based on the solution of a specific partial differential equations.
- Alternatively, we can use of direct methods in the calculus of variations, that exploits minimum principles.
- Broadly speaking, previous methods can be labeled as Newtonian, whereas methods based on energy considerations (as will be the case in this chapter) are labelled as Lagrangian.



- Vector or scalar

Newton	Force $\vec{F}$	Momentum $m\vec{v}$	Vectors	Newtonian	Equation
Leibniz	Work (potential energy)	Kinetic Energy *vis viva**	2 scalars	Lagrangian	Principle

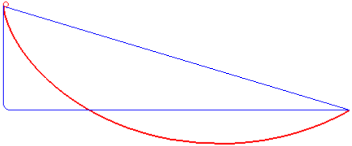
* *Living force*

- Consider a particle at point P_1 at time t_1, and assume that we know the velocity at that time.
- Euler-Lagrange: P_1, t_1, P_2 known, t_2 unknown.
    - Assume particle will be at P_2 after a given time.
    - Connect P_1 and P_2 by any arbitrary tentative path. In all likelihood, this will be the wrong one.
    - Gradually correct the tentative path according to the energy principle: sum of kinetic and potential energies must be kept constant.
    - This will impose a definite velocity to any point of the path and thus will determine the motion (which will end at P_2.
    - For each path we can define action time integral of the *vis viva* (double the kinetic energy) over the entire motion from P_1 to P_2.

- Once all possible paths have been determined, the one with smallest action is the actual path of motion.

- Hamiltonian: P_1, t_1, t_2 known, P_2 unknown.
  - when the work function is a function not only of the particle position but also of time.
  - Laws of conservation of energy does not hold, Euler-Lagrange not applicable, but Hamilton principle is.
  - Require that tentative motion starts at P_1 and t_1 and motion ends at unknown point at time t_2.
- Calculus of Variation
  - Final results can be established without considering an infinity of solution, but we will achieve a solution infinitesimally near the actual solution (a variation of the actual path).
  - Many elementary problems can be solved by vectorial mechanics specially in cartesian coordinates.
  - Scalar mechanics far superior for curvilinear coordinates.
- Applications of calculus of variation

- Greatest projectile range that can be achieved (Newton, Euler).
- Optimal shape to minimize water resistance (Newton).
- Shortest time of descent by varying shape of a wire on which beads are sliding (Galileo, Bernouilli, von Leibniz) brachistochrone.



- Differential calculus (DC) involves a function of one or more variable, whereas variational calculus (VC) involves a function of a function, or a functional.
- Fundamental theorem of calculus

$$F(x) = \int_a^x f(t)dt \quad \text{and} \quad F'(x) = f(x) \tag{1}$$

- Fundamental problem of the calculus of variation is to find a function $u(x)$ such that

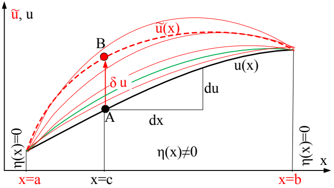
$$\Pi(u) = \int_a^b F(x, u(x), u'(x))dx \tag{2}$$

$$\delta\Pi = 0 \tag{3}$$

where δ indicates the variation operator.

- Define $u(x)$ to be a function of x in the interval (a, b), and F to be a known real function (such as the energy density).
- Define the domain of a functional as the collection of admissible functions belonging to a class of functions in function space rather than a region in coordinate space (as is the case for a function).

- $\longrightarrow$ Seek the function $u(x)$ which extremizes Π.
- Let $\tilde{u}(x)$ be a family of neighboring paths of the extremizing function $u(x)$ and assume that at the end points $x = a, b$ they coincide.
- Define $u(\tilde{x})$ as the sum of the extremizing path and some arbitrary variation.



$$\tilde{u}(x, \varepsilon) = u(x) + \varepsilon\eta(x) = u(x) + \delta u(x) \quad (4)$$

where ε is a small parameter, and $\delta u(x)$ is the variation of $u(x)$

$$\delta u = \tilde{u}(x, \varepsilon) - u(x) \quad (5)$$
$$= \varepsilon\eta(x) \quad (6)$$

and $\eta(x)$ is twice differentiable, has undefined amplitude but is such that $\eta(a) = \eta(b) = 0$. Note that $\tilde{u}$ coincides with u if $\varepsilon = 0$.

- Note that:

- The necessary condition to extremize a value in DC is that the first derivative be equal to zero, and that the first variation be zero in VC.
- The result of the extremization is a single variable x in DC, and $u(x)$ in VC.
- The variational operator δ is analogous to the δ associated with virtual displacement later.
- It can be shown that the variation and derivation operators are commutative

$$\left.\begin{array}{rcl} \frac{d}{dx}(\delta u) & = & \tilde{u}'(x,\varepsilon) - u'(x) \\ \delta u' & = & \tilde{u}'(x,\varepsilon) - u'(x) \end{array}\right\} \frac{d}{dx}(\delta u) = \delta\left(\frac{du}{dx}\right)$$

- Variational operator δ and the differential calculus operator d can be similarly used, i.e.

$$\begin{aligned} \delta(u')^2 &= 2u'\delta u' \\ \delta(u+v) &= \delta u + \delta v \\ \delta\left(\int u dx\right) &= \int(\delta u) dx \\ \delta u &= \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y \end{aligned}$$

however, they have clearly different meanings. du is associated with a neighboring point at a distance dx, however δu is a small arbitrary change in the function u for a given x (there is no associated δx).

- For boundaries where u is specified, its variation must be zero, and it is arbitrary elsewhere. The variation δu of u is said to undergo a virtual change.

- Cast the variational formulation ($\delta\Pi = 0$) into a differential one $\frac{d\Phi(\varepsilon)}{d\varepsilon} = 0$ and use basic calculus.
- Define $\Phi(\varepsilon)$ as

$$\Phi(\varepsilon) \stackrel{\text{def}}{=} \Pi(u + \varepsilon\eta(x)) = \int_a^b F(x, u(x) + \varepsilon\eta(x), u'(x) + \varepsilon\eta'(x))dx \tag{7}$$

Note that this will be referred as the weak form ("weak" because it needs derivative of one lesser order)

- Since $\tilde{u}(x) \to u(x)$ as $\varepsilon \to 0$, the necessary condition for Π to be an extremum is

$$\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

- From Eq. 4 $\tilde{u} = u + \varepsilon\eta$, and $\tilde{u(x)}' = u'(x) + \varepsilon\eta'(x)$, and applying the chain rule

$$\frac{d\Phi(\varepsilon)}{d\varepsilon} = \int_a^b \left(\frac{\partial F}{\partial \tilde{u}} \frac{d\tilde{u}}{d\varepsilon} + \frac{\partial F}{\partial \tilde{u}'} \frac{d\tilde{u}'}{d\varepsilon} \right) dx = \int_a^b \left(\eta \frac{\partial F}{\partial \tilde{u}} + \eta' \frac{\partial F}{\partial \tilde{u}'} \right) dx$$

for $\varepsilon = 0$, $\tilde{u} = u$, thus

$$\left. \frac{d\Phi(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = \int_a^b \left(\eta \frac{\partial F}{\partial u} + \eta' \frac{\partial F}{\partial u'} \right) dx = 0 \tag{8}$$

- Integration by part ($\int fg'dx = fg - \int f'gdx$) of the second term leads to

$$\int_a^b \left(\eta' \frac{\partial F}{\partial u'} \right) dx = \eta \frac{\partial F}{\partial u'} \bigg|_a^b - \int_a^b \eta(x) \left(\frac{d}{dx} \frac{\partial F}{\partial u'} \right) dx \tag{9}$$

- Substituting,

$$\left. \frac{d\Phi(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = \underbrace{\int_a^b \eta(x) \left[\frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} \right]}_{\mathrm{I}\ (x \in [a,b])} + \underbrace{\eta(x) \frac{\partial F}{\partial u'} \bigg|_a^b}_{\mathrm{II}\ (x = a, b)} = 0 \tag{10}$$

- Each term must be zero.
  1. First part will give us the Euler equation.
  2. Second part will enable us to define the boundary conditions.

- Fundamental lemma of the calculus of variation states that for continuous $\Psi(x)$ in $a \leq x \leq b$, and with arbitrary continuous function $\eta(x)$ which vanishes at a and b, then

$$\int_a^b \eta(x)\Psi(x)dx = 0 \Leftrightarrow \Psi(x) = 0 \tag{11}$$

Thus, part I in Eq. 10 yields Strong Form

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} = 0 \quad \text{in} \quad a < x < b \tag{12}$$

- This differential equation is called the Euler-Lagrange equation associated with Π and is a necessary condition for $u(x)$ to extremize Π.
- Note that the weak form is in terms of u' (Eq. 7) and the strong form in terms of u'' Eq. 12.

- Generalizing for a functional Π which depends on two field variables, $u = u(x, y)$ and $v = v(x, y)$

$$\Pi = \int\int F(x, y, u, v, u_{,x}, u_{,y}, v_{,x}, v_{,y}, \cdots, v_{,yy})dxdy \tag{13}$$

There would be as many Euler equations as dependent field variables

$$\left\{ \begin{array}{lcl} \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_{,x}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial u_{,y}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial u_{,xx}} + \frac{\partial^2}{\partial x \partial y}\frac{\partial F}{\partial u_{,xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial u_{,yy}} & = & 0 \\ \frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\frac{\partial F}{\partial v_{,x}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial v_{,y}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial v_{,xx}} + \frac{\partial^2}{\partial x \partial y}\frac{\partial F}{\partial v_{,xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial v_{,yy}} & = & 0 \end{array} \right. \tag{14}$$

- Note that the Functional and the corresponding Euler Equations, Eq. 2 and 12, or Eq. 13 and 14 describe the same problem.
- The Euler equations usually correspond to the governing differential equation and are referred to as the strong form (or classical form).
- The functional is referred to as the weak form (or generalized solution).
- In Mechanics, equilibrium is enforced in an average sense over the body (and the field variable is differentiated m times in the weak form, and $2m$ times in the strong form) Eq. 7 *v.s.* Eq. 12.

- It can be shown that in the principle of virtual displacements, the Euler equations are the equilibrium equations, whereas in the principle of virtual forces, they are the compatibility equations.
- Euler equations are differential equations which can not always be solved by exact methods. An alternative method consists in bypassing the Euler equations and go directly to the variational statement of the problem to the solution of the Euler equations.
- Finite Element formulation are based on the weak form, whereas the formulation of Finite Differences are based on the strong form.

- In preceding section we have just shown that $d\Phi(\varepsilon)/d\varepsilon$ leads to the Euler-Lagrange equation (Eq. 10)

$$\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \underbrace{\int_a^b \eta(x)\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'}\right]}_{\text{I } (x\in[a,b])} + \underbrace{\left.\eta(x)\frac{\partial F}{\partial u'}\right|_a^b}_{\text{II } (x=a,b)} = 0$$

We still have to define $\delta\Pi$.

- The first variation of a functional expression is

$$\left.\begin{array}{rcl} \delta F &=& \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \\ \delta\Pi &=& \int_a^b \delta F dx \end{array}\right\} \delta\Pi = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right) dx \qquad (15)$$

Integration by parts of the second term (as in Eq. 8) yields

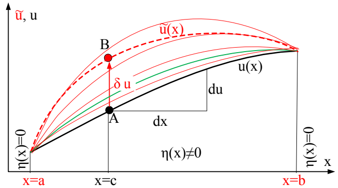
$$\delta\Pi = \int_a^b \delta u \left(\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'}\right) dx \qquad (16)$$

- Just shown that finding the stationary value of Π by setting $\delta\Pi = 0$ is equivalent to finding the extremal value of Π by setting $\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}$ equal to zero.
- Similarly, it can be shown that as with second derivatives in calculus, the second variation $\delta^2\Pi$ can be used to characterize the extremum as either a minimum or maximum.
- An important observation is that the variational formulation is a scalar one, whereas the Eulerian one is vectorial.

- Revisiting the second part of Eq. 10,

$$\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \underbrace{\int_a^b \eta(x)\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'}\right]}_{\mathrm{I}\ (x\in[a,b])} + \underbrace{\left.\eta(x)\frac{\partial F}{\partial u'}\right|_a^b}_{\mathrm{II}\ (x=a,b)} = 0 \qquad (17)$$

enables us to define the boundary conditions



$$\underbrace{\underbrace{\eta(x)}_{\text{Ess.}}\ \underbrace{\left.\frac{\partial F}{\partial u'}\right|_a^b}_{\text{Nat.}}}_{\text{Boundary Cond.}} = 0 \qquad (18)$$

This can be achieved through the following combinations

$$\begin{array}{llllll} \eta(a)=0 & \text{and} & \eta(b) & =0 & \text{Essential} & \Gamma_u \\ \eta(a)=0 & \text{and} & \frac{\partial F}{\partial u'}(b) & =0 & \text{Mixed} & \Gamma_u\bigcup\Gamma_t \\ \frac{\partial F}{\partial u'}(a)=0 & \text{and} & \eta(b) & =0 & \text{Mixed} & \Gamma_u\bigcup\Gamma_t \\ \frac{\partial F}{\partial u'}(a)=0 & \text{and} & \frac{\partial F}{\partial u'}(b) & =0 & \text{Natural} & \Gamma_t \end{array} \qquad (19)$$

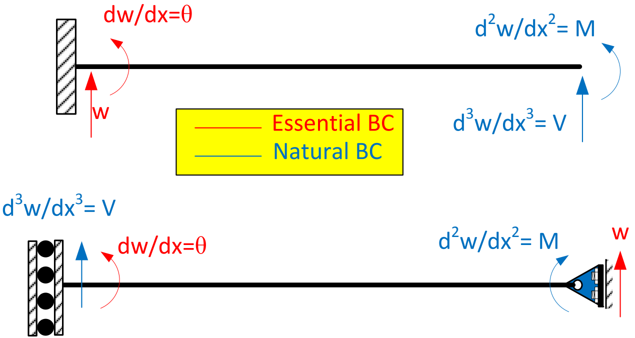
- Generalizing, for a problem with, one field variable, in which the highest derivative in the governing differential equation is of order $2m$ (or simply m in the corresponding functional), then we have

  Essential (or forced, or geometric) boundary conditions, (because it was essential for the derivation of the Euler equation) if $\eta(a)$ or $\eta(b)$ =0. Essential boundary conditions, involve derivatives of order zero (the field variable itself) through m-1. Mathematically, this corresponds to Dirichlet boundary-value problems.

  Natural (or natural or static) if we left η to be arbitrary, then it would be necessary to use $\frac{\partial F}{\partial u'} = 0$ at $x = a$ or b. Natural boundary conditions, involve derivatives of order m and up. This B.C. is implied by the satisfaction of the variational statement but not explicitly stated in the functional itself. Mathematically, this corresponds to Neuman boundary-value problems.

  Mixed Boundary-Value problems, are those in which both essential and natural boundary conditions are specified on complementary portions of the boundary (such as Γ_u and Γ_t).

Problem	Axial Member Distributed load	Flexural Member Distributed load
Differential Equation	$AE\frac{d^2u}{dx^2} - q = 0$	$EI\frac{d^4w}{dx^4} - q = 0$
m	1	2
Essential B.C. $[0, m-1]$	u	w, $\frac{dw}{dx}$
Natural B.C. $[m, 2m-1]$	$\frac{du}{dx}$ or $\sigma_{xx} = Eu_{,x}$	$\frac{d^2w}{dx^2}$ and $\frac{d^3w}{dx^3}$ or $M = EIw_{,xx}$ and $V = EIw_{,xxx}$


dw/dx=θ
w
d²w/dx²= M
d³w/dx³= V
Essential BC
Natural BC
d³w/dx³= V
dw/dx=θ
d²w/dx²= M
w

Potential energy Π of an axial member (L, E, A), fixed at left end and subjected to an axial force P at the right one is given by

$$\Pi = \underbrace{\int_0^L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx}_{\text{Strain Energy}} \underbrace{-Pu(L)}_{\text{Work}} \tag{20}$$

Determine the Euler Equation by requiring that Π be a minimum.

Solution I

- Follow the procedure used for the derivation of the Euler Equations.
- First variation of Π:

$$\delta\Pi = \int_0^L \underbrace{\frac{EA}{2} 2\left(\frac{du}{dx}\right)}_{a} \underbrace{\delta\left(\frac{du}{dx}\right)}_{b'} dx - P\delta u(L)$$

- Integrating by parts

$$\begin{aligned}
\delta\Pi &= +\underbrace{EA\frac{du}{dx}}_{a}\underbrace{\delta u}_{b}\Bigg|_0^L - \int_0^L \underbrace{\frac{d}{dx}\left(EA\frac{du}{dx}\right)}_{a'}\underbrace{\delta u}_{b} dx - P\delta u(L) = 0 \\
&= -\int_0^L \delta u \underbrace{\frac{d}{dx}\left(EA\frac{du}{dx}\right)}_{\text{Euler Eq.}} dx + \underbrace{\left[\left(EA\frac{du}{dx}\right)\Bigg|_{x=L} - P\right]}_{\text{B.C.}}\delta u(L) \\
&\quad \underbrace{-\left(EA\frac{du}{dx}\right)\Bigg|_{x=0}\underbrace{\delta u(0)}_{0}}_{0}
\end{aligned}$$

- Recall that δ in an arbitrary operator which can be assigned any value, we set the coefficients of δu between $(0, L)$ and those for δu at $x = L$ equal to zero separately, and obtain

- Euler Equation:

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = EA\frac{d^2u}{dx^2} = 0 \qquad 0 < x < L \tag{21}$$

Note how the functional was in terms of u' and the Euler equation in terms of u''.

- Natural Boundary Condition:

$$EA\frac{du}{dx} - P = 0 \quad \text{at} \quad x = L \tag{22}$$

Solution II Use results from previous derivation (Eq. 12):

- We have derived:

$$F(x,u,u') = \frac{EA}{2}\left(\frac{du}{dx}\right)^2$$

(note that since P is an applied load at the end of the member, it does not appear as part of $F(x,u,u')$.

- Euler equation: Substituting into Eq. 12

- To evaluate the Euler Equation from Eq. 12

$$
\begin{aligned}
\frac{\partial F}{\partial u} &= 0 \\
\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} &= 0 \\
\Rightarrow -\frac{d}{dx}(EAu') = -EA\frac{d^2u}{dx^2} &= 0 \ \text{Euler Equation}
\end{aligned}
$$

- Boundary Condition From Eq. 18:

$$
\begin{aligned}
\underbrace{\eta(x)}_{\text{Ess.}}\underbrace{\left.\frac{\partial F}{\partial u'}\right|_a^b}_{\text{Nat.}} &= 0 \\
\frac{\partial F}{\partial u'} &= EAu' \\
EA\frac{du}{dx} &= 0
\end{aligned}
$$

The total potential energy of a beam supporting a uniform load p is given by

$$\Pi = \int_0^L \left(\frac{1}{2}M\kappa - pw\right) dx = \int_0^L \underbrace{\left(\frac{1}{2}(EIw'')w'' - pw\right)}_{F} dx \qquad (23)$$

Derive the first variational of Π.

1. Extending Eq. 15, and integrating by part twice

$$\begin{aligned}
\delta\Pi &= \int_0^L \delta F dx = \int_0^L \left(\frac{\partial F}{\partial w''}\delta w'' + \frac{\partial F}{\partial w}\delta w\right) dx \\
&= \int_0^L (EIw''\delta w'' - p\delta w) dx \\
&= (EIw''\delta w')\big|_0^L - \int_0^L \left[(EIw'')'\delta w' - p\delta w\right] dx \\
&= \underbrace{\underbrace{(EIw''}_{\text{Nat.}}\ \underbrace{\delta w')\big|_0^L}_{\text{Ess.}} - \underbrace{[(EIw'')'}_{\text{Nat.}}\ \underbrace{\delta w]\big|_0^L}_{\text{Ess.}}}_{\text{BC}} + \int_0^L \underbrace{\left[(EIw'')'' + p\right]}_{\text{Euler Eq.}} \delta w dx = 0
\end{aligned}$$

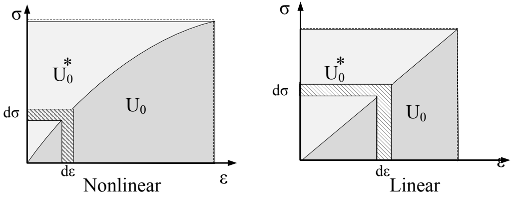
2. Or

$$(EIw'')'' = -p \quad \text{for all x}$$

which is the governing differential equation of beams and

$$\begin{array}{lcl} \text{Essential} & & \text{Natural} \\ \delta w' = 0 & \text{or} & EIw'' = -M = 0 \\ \delta w = 0 & \text{or} & (EIw'')' = -V = 0 \end{array}$$

at $x = 0$ and $x = L$



Strain energy density :

$$U_0 \stackrel{\text{def}}{=} \int_0^{\varepsilon} \sigma d\varepsilon$$

Complementary strain energy density :

$$U_0^* \stackrel{\text{def}}{=} \int_0^{\sigma} \varepsilon d\sigma$$

strain and complementary strain energy :

$$U \stackrel{\text{def}}{=} \int_\Omega U_0 d\Omega$$
$$U^* \stackrel{\text{def}}{=} \int_\Omega U_0^* d\Omega$$

Stress Strain Relation :

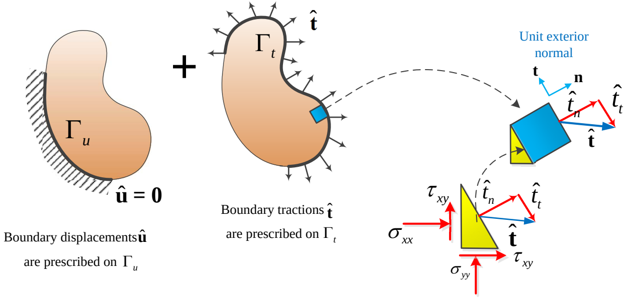
$$\boldsymbol{\sigma} = \mathrm{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) + \boldsymbol{\sigma}_0$$

Strain Energy for Linear Systems :

$$U = \frac{1}{2}\int_\Omega \boldsymbol{\epsilon}^T \mathrm{D}\boldsymbol{\epsilon} d\Omega - \int_\Omega \boldsymbol{\epsilon}^T \mathrm{D}\boldsymbol{\epsilon}_0 d\Omega + \int_\Omega \boldsymbol{\epsilon}^T \boldsymbol{\sigma}_0 d\Omega$$

Only two types of forces:

- Surface traction $\hat{\mathrm{t}}$



Note: Point force related to traction through Dirac function $\delta(z-d) = 0, z \neq d$;
$\int_{\infty}^{\infty} \delta(z-d)dz = 1$, $\int_0^L \delta(z-d)dz = 1$; $\int_0^L f(z)\delta(z-d)dz = f(d)$;

- Body force b

External work $W_e \stackrel{\text{def}}{=} \int_\Omega \mathbf{u}^T \mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{u}^T \hat{\mathbf{t}} d\Gamma$

Point Force/Moment $W_e = \int_0^{\Delta_f} P d\Delta + \int_0^{\theta_f} M d\theta$

Internal Strain Energy/Virtual Work $\delta\overline{U} = -\delta\overline{W}_i \stackrel{\text{def}}{=} \int_\Omega \boldsymbol{\sigma}\delta\overline{\boldsymbol{\varepsilon}} d\Omega$

External Virtual Work $\delta\overline{W}_e \stackrel{\text{def}}{=} \int_{\Gamma_t} \delta\overline{\mathbf{u}}^t \hat{\mathbf{t}} d\Gamma + \int_\Omega \delta\overline{\mathbf{u}}^t \mathbf{b} d\Omega$

Complementary Internal Strain Energy-Internal Virtual Work

$$\delta\overline{U}^* = -\delta\overline{W}_i^* \stackrel{\text{def}}{=} \int_\Omega \boldsymbol{\varepsilon}\delta\overline{\boldsymbol{\sigma}} d\Omega \tag{24}$$

Complementary External Virtual Work

$$\delta\overline{W}_e^* \stackrel{\text{def}}{=} \int_{\Gamma_u} \hat{\mathbf{u}}^t \delta\overline{\mathbf{t}} d\Gamma \tag{25}$$

Potential of external work W

$$W_e \stackrel{\text{def}}{=} \int_\Omega \mathrm{u}^T \mathrm{b} d\Omega + \int_{\Gamma_t} \mathrm{u}^T \hat{\mathrm{t}} d\Gamma + \mathrm{uP}$$

Strictly speaking, we ought to differentiate work from its potential and use two distinct symbols W and $\mathcal{W}$ respectively. For the sake of clarity we will replace $\mathcal{W}$ by W in the notes.

Potential energy

$$\Pi \stackrel{\text{def}}{=} U - W_e = \int_\Omega U_0 d\Omega - \left(\int_\Omega \mathrm{ub} d\Omega + \int_{\Gamma_t} \mathrm{u}\hat{\mathrm{t}} d\Gamma + \mathrm{uP} \right) \tag{26}$$

Complementary potential energy

$$\Pi^* \stackrel{\text{def}}{=} U^* - W_e^* = \int_\Omega U_0^* d\Omega - \left(\int_\Omega \mathrm{ub} d\Omega + \int_{\Gamma_t} \mathrm{u}\hat{\mathrm{t}} d\Gamma + \mathrm{uP} \right)$$

- First Law of Thermodynamics: The time-rate of change of the total energy (i.e., sum of the kinetic energy K and the internal energy U) is equal to the sum of the rate of work done by the external forces W_e and the change of heat content per unit time H: $\frac{d}{dt}(K+U) = W_e + H$
- For an adiabatic system (no heat exchange) and if loads are applied in a quasi static manner (no kinetic energy), the above relation simplifies to: $W_e = U$

- The complementary internal virtual strain energy is expressed in terms of stresses or internal forces ($P(x)$, $M(x)$).
- It will lead to a formulation similar to the one seen in introductory courses in structural analysis (virtual force method)

Axial Members

Stresses and forces constitute the virtual quantities identified by δ.

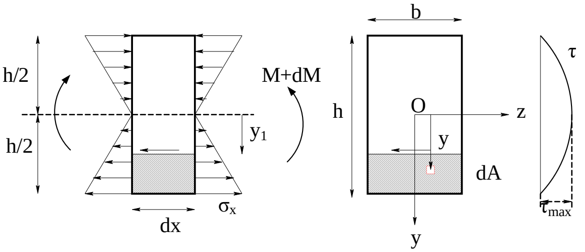
Elastic System

$$\left.\begin{aligned} \delta\overline{U}^* &= \int_\Omega \delta\overline{\sigma}\varepsilon d\Omega \\ d\Omega &= Adx \end{aligned}\right\} \delta\overline{U}^* = A\int_0^L \delta\overline{\sigma}\varepsilon dx$$

Linear Elastic

$$\left.\begin{aligned} \delta\overline{U}^* &= \int_\Omega \varepsilon\delta\overline{\sigma} d\Omega \\ \delta\overline{\sigma} &= \frac{\delta\overline{P}}{A} \\ \varepsilon &= \frac{P}{AE} \\ d\Omega &= Adx \end{aligned}\right\} \delta\overline{U}^* = \int_0^L \underbrace{\delta\overline{P}}_{``\delta\overline{\sigma}''} \underbrace{\frac{P}{AE}}_{``\varepsilon''} dx$$

Flexural Members



Elastic System

$$
\left.\begin{array}{rcl}
\delta\overline{U}^{*} & = & \int_{\Omega} \delta\overline{\sigma}_{xx}\varepsilon_x d\Omega \\
\delta\overline{M}(x) & = & \int_A \delta\overline{\sigma}_x y dA \Rightarrow \dfrac{\delta\overline{M}(x)}{y} = \int_A \delta\overline{\sigma}_x dA \\
\phi & = & \frac{\varepsilon}{y} \Rightarrow \phi y = \varepsilon_x \\
d\Omega & = & \int_0^L \int_A dA dx
\end{array}\right\} \delta\overline{U}^{*} = \int_0^L \delta\overline{M}(x)\phi dx
$$

Linear Elastic

$$
\left.
\begin{array}{rcl}
\delta\overline{U}^{*} & = & \int_{\Omega} \varepsilon \underbrace{E\delta\overline{\varepsilon}}_{\delta\overline{\sigma}} d\Omega \\
\sigma_x & = & \frac{M_z y}{I_z} \\
\varepsilon_x & = & \frac{M_z y}{EI_z} \\
d\Omega & = & dAdx \\
\int_A y^2 dA & = & I_z
\end{array}
\right\}
\delta\overline{U}^{*} = \int_0^L \underbrace{\delta\overline{M}(x)}_{``\delta\overline{\sigma}''} \underbrace{\frac{M(x)}{EI_z}}_{``\varepsilon''} dx
$$

- The internal virtual strain energy is expressed in terms of strain or internal displacements.
- It will lead to the formulation at the root of the finite element method.

Axial Members Strains and displacements constitute the virtual quantities identified by δ.

Elastic System

$$\left.\begin{array}{rcl}\delta\overline{U} & = & \int_\Omega \sigma_x \delta\overline{\varepsilon}_x d\Omega \\ d\Omega & = & Adx\end{array}\right\} \delta\overline{U} = A\int_0^L \sigma_x \delta\overline{\varepsilon}_x dx$$

Linear Elastic

$$\left.\begin{array}{rcl}\delta\overline{U} & = & \int \sigma_x \delta\overline{\varepsilon}_x d\Omega \\ \sigma_x & = & E\varepsilon_x = E\frac{du}{dx} \\ \delta\overline{\varepsilon}_x & = & \frac{d(\delta\overline{u})}{dx} \\ d\Omega & = & Adx\end{array}\right\} \delta\overline{U} = \int_0^L \underbrace{E\frac{du}{dx}}_{"\sigma"} \underbrace{\frac{d(\delta\overline{u})}{dx}}_{"\delta\varepsilon"} \underbrace{Adx}_{d\Omega}$$

Flexural Members

Elastic System

$$\left.\begin{array}{rcl}\delta\overline{U} & = & \int \sigma_x \delta\overline{\varepsilon}_x d\Omega \\ M(x) & = & \int_A \sigma_x y dA \Rightarrow \frac{M(x)}{y} = \int_A \sigma_x dA \\ \delta\overline{\phi} & = & \frac{\delta\overline{\varepsilon}_x}{y} \Rightarrow \delta\overline{\phi} y = \delta\overline{\varepsilon}_x \\ d\Omega & = & \int_0^L \int_A dAdx\end{array}\right\} \delta\overline{U} = \int_0^L M(x)\delta\overline{\phi}dx$$

Linear Elastic

$$\left.\begin{array}{rcl} & & \delta\overline{U} = \int_\Omega \sigma_x \delta\overline{\varepsilon}_x d\Omega \\ \left.\begin{array}{rcl}\sigma_x & = & \frac{M(x)y}{I_z} \\ M(x) & = & \frac{d^2v}{dx^2} EI_z\end{array}\right\} & & \sigma_x = \underbrace{\frac{d^2v}{dx^2}}_{\kappa} Ey \\ & & \delta\overline{\varepsilon}_x = \frac{\delta\overline{\sigma}_x}{E} = \frac{d^2(\delta\overline{v})}{dx^2} y \\ & & d\Omega = Adx\end{array}\right\} \delta\overline{U} = \int_0^L \int_A \frac{d^2v}{dx^2} Ey \frac{d^2(\delta\overline{v})}{dx^2} ydAdx$$

Since $\int_A y^2 dA = I_z \Rightarrow$

$$\delta\overline{U} = \int_0^L \underbrace{EI_z \frac{d^2 v}{dx^2}}_{"\sigma"} \underbrace{\frac{d^2(\delta\overline{v})}{dx^2}}_{"\delta\overline{\varepsilon}"} dx$$

		Virtual Work: $\delta\overline{U}$		Complementary Virtual Work: $\delta\overline{U}^*$	
Continuous System		$-\int_\Omega \delta\overline{\mathbf{u}}^T\left(\mathbf{L}^T\boldsymbol{\sigma}+\mathbf{b}\right)d\Omega + \int_{\Gamma_t}\delta\overline{\mathbf{u}}^T(\mathbf{t}-\hat{\mathbf{t}})d\Gamma = 0$		$\int_\Omega \left(\varepsilon_{ij}-u_{i,j}\right)\delta\overline{\sigma}_{ij}d\Omega - \int_{\Gamma_u}(u_i-\hat{u})\,\delta\overline{t}_i d\Gamma = 0$	
	Strain Energy U	Elastic	Linear Elastic	Elastic	Linear Elastic
Axial	$\frac{1}{2}\int_0^L \frac{P^2}{AE}dx$	$A\int_0^L \sigma\delta\overline{\varepsilon}dx$	$\int_0^L \underbrace{E\frac{du}{dx}}_{\sigma}\underbrace{\frac{d(\delta\overline{u})}{dx}}_{\delta\overline{\varepsilon}}\underbrace{Adx}_{d\Omega}$	$A\int_0^L \delta\overline{\sigma}\varepsilon dx$	$\int_0^L \underbrace{\delta\overline{P}}_{\delta\overline{\sigma}}\underbrace{\frac{P}{AE}}_{\varepsilon}dx$
Flexure	$\frac{1}{2}\int_0^L \frac{M^2}{EI_z}dx$	$\int_0^L M\delta\overline{\phi}dx$	$\int_0^L \underbrace{EI_z\frac{d^2v}{dx^2}}_{\sigma}\underbrace{\frac{d^2(\delta\overline{v})}{dx^2}}_{\delta\overline{\varepsilon}}dx$	$\int_0^L \delta\overline{M}\phi dx$	$\int_0^L \underbrace{\delta\overline{M}}_{\delta\overline{\sigma}}\underbrace{\frac{M}{EI_z}}_{\varepsilon}dx$

	Work	Virtual Work: δW	Complementary Virtual Work: $\delta W*$
P	$\Sigma_i P_i\Delta_i$	$\sum_i^n P_i\delta\overline{\Delta}_i$	$\sum_i^n \delta\overline{P}_i\Delta_i$
M	$\sum_i^n M_i\theta_i$	$\sum_i^n M_i\delta\overline{\theta}_i$	$\sum_i^n \delta\overline{M}_i\theta_i$
w	$\int_0^L w(x)v(x)dx$	$\int_0^L w(x)\delta\overline{v}(x)dx$	$\int_0^L \delta\overline{w}(x)v(x)dx$

Formulation	Potential Energy	Complementary Potential Energy
	Displacement	Force
Axial	$\frac{1}{2}\int_0^L E\left(\frac{du}{dx}\right)^2 dx$	$\frac{1}{2}\int_0^L \frac{P^2}{AE}dx$
Flexural	$\frac{1}{2}\int_0^L EI_z(v'')^2dx - \int_0^L q(x)vdx$	$\frac{1}{2}\int_0^L \frac{M^2}{EI_z}dx$

Strong/Weak We will refer to a strong form a derivation stemming from a differential equation, and one which is exactly satisfied.
The weak form will be only satisfied in an average sense over a volume Ω.

Boundary Conditions A more detailed coverage of B.C. entails calculus of variation, and derivation of the Euler equation associated with a potential.

Γ	Traction	Displ.	Math.	Structural Mechanics			DOF
Γ_t	$\mathtt{t}\surd$	$\mathtt{u}^?$	Dirichlet	Essential	Primary	Kinematic	Free
Γ_u	$\mathtt{t}^?$	$\mathtt{u}\surd$	Neuman	Natural	Secondary	Static	Fixed/Constrained

Simply put, the Gauss' integral theorem relates a vector field on the surface to the scalar response inside the corresponding volume.

$$\int_\Gamma \mathbf{v}.\mathbf{n}\, d\Gamma = \int_\Omega \text{div}\mathbf{v}\, d\Omega$$

or

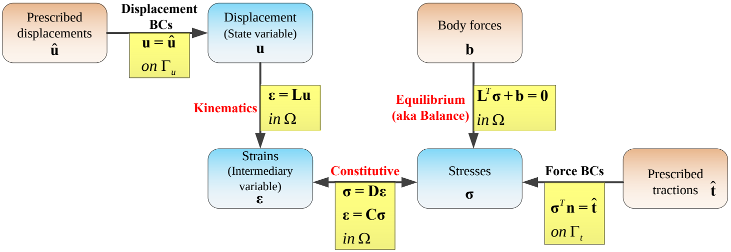
$$\int_\Gamma v_i n_i d\Gamma = \int_\Omega v_{i,i} d\Omega$$

Note if we apply Gauss' theorem to an expression such as work ($u_i t_i$) where the traction t_i is related to the stress through $t_i = \sigma_{ij} n_j$ then

$$\begin{aligned} \int_\Gamma t_i u_i d\Gamma &= \int_\Gamma \sigma_{ij} n_j u_i d\Gamma = \int_\Gamma (\sigma_{ij} u_i) n_j d\Gamma \\ &= \int_\Omega (\sigma_{ij} u_i)_{,j} d\Omega = \int_\Omega (\sigma_{ij,j} u_i + \sigma_{ij} u_{i,j}) d\Omega \end{aligned} \tag{27}$$

Just in case:

$$
\begin{aligned}
\text{grad } A = \boldsymbol{\nabla} A &= \left(\mathtt{i}\frac{\partial}{\partial x} + \mathtt{j}\frac{\partial}{\partial y} + \mathtt{k}\frac{\partial}{\partial z}\right) A \\
&= \mathtt{i}\frac{\partial A}{\partial x} + \mathtt{j}\frac{\partial A}{\partial y} + \mathtt{k}\frac{\partial A}{\partial z} \\
\text{div } \mathtt{A} = \boldsymbol{\nabla}\cdot\mathtt{A} &= \left(\mathtt{i}\frac{\partial}{\partial x} + \mathtt{j}\frac{\partial}{\partial y} + \mathtt{k}\frac{\partial}{\partial z}\right)\cdot(\mathtt{i}A_x + \mathtt{j}A_y + \mathtt{k}A_z) \\
&= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\text{Laplacian } \boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} &= \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}
\end{aligned}
$$



Principle	Virtual			Real		
	Starting with	Satisfying Strongly	Satisfying B.C.	Seek	Satisfying Weakly	Satisfying B.C.
VW $\delta\overline{U} = \delta\overline{W}_e$	Displ.	Compatibility	Essential Γ_u	Forces	Equilibrium	Natural Γ_t
CVW $\delta\overline{U}^* = \delta\overline{W}_e^*$	Forces	Equilibrium	Natural Γ_t	Displ.	Compatibility	Essential Γ_u

Virtual

Strong Equation
Equilibrium

Primary Variable
δ Forces

Corresponding B.C.
Natural

REAL

Weak Equation
Compatibility

Variable
Displacement

Corresponding B.C.
Essential

Galerkin
Weak
∫(Compatibility δF) dΩ -∫Essential δF) dΓ =0

Gauss
Strong: Equilibrium
on Natural

PCVW

Virtual

Strong Equation
Compatibility

Primary Variable
δ Displacements

Corresponding B.C.
Essential

REAL

Weak Equation
Equilibrium

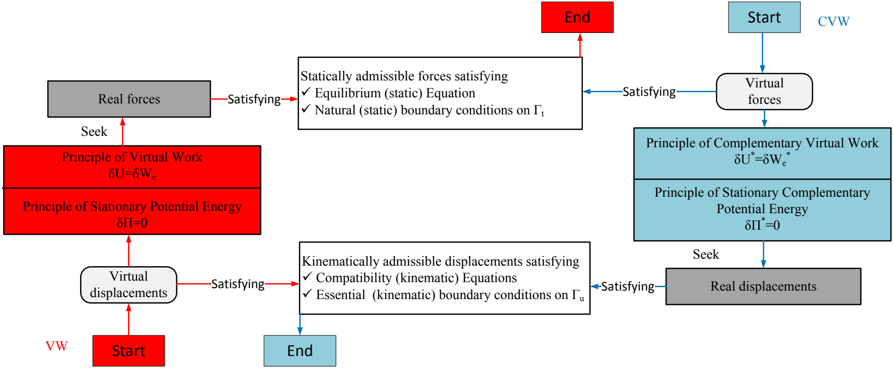
Variable
Forces

Corresponding B.C.
Natural

Galerkin
Weak
∫(Equilibrium δ Disp.) dΩ -∫(Natural δ Disp.) dΓ =0

Gauss
Strong: Compatibility
On Essential

PVW


End
Start
CVW
Real forces
Satisfying
Statically admissible forces satisfying
✓ Equilibrium (static) Equation
✓ Natural (static) boundary conditions on Γ_t
Satisfying
Virtual forces
Seek
Principle of Virtual Work
$\delta U = \delta W_e$
Principle of Stationary Potential Energy
$\delta \Pi = 0$
Principle of Complementary Virtual Work
$\delta U^* = \delta W_e^*$
Principle of Stationary Complementary Potential Energy
$\delta \Pi^* = 0$
Seek
Virtual displacements
Satisfying
Kinematically admissible displacements satisfying
✓ Compatibility (kinematic) Equations
✓ Essential (kinematic) boundary conditions on Γ_u
Satisfying
Real displacements
VW
Start
End

- The principles of Virtual Work and Complementary Virtual Work relate

  force systems which satisfy the requirements of equilibrium
  deformation systems which satisfy the requirement of kinematic.

	Force		Deformation		IVW	Formulation
	External	Internal	External	Internal		
1	$\delta\overline{\mathrm{p}}$	$\delta\overline{\sigma}$	$d\mathrm{u}$	$d\varepsilon$	$\delta\overline{U}^*$	CVW/Flexibility
2	$d\mathrm{p}$	$d\sigma$	$\delta\overline{\mathrm{u}}$	$\delta\overline{\varepsilon}$	$\delta\overline{U}$	VW/Stiffness

- The principle of Complementary Virtual Work (of Principle of Virtual Force) is what we have already seen previously (unit force method).
- The Principle of Virtual work is new, and is at the basis of the finite element method.

Intermediary Structural Analysis

Finite Element Formulation

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Table of Contents I

- So far we have considered continuous systems, in this chapter we seek to apply the previously derived relations to discretized systems.
- Primary solutions only at the nodes only (as opposed to a continuous solution inside Ω).
- Application of the Principle of Virtual Displacement requires an assumed displacement field. This displacement field can be approximated by interpolation functions written in terms of:
  1. Unknown polynomial coefficients, most appropriate for continuous systems,. For example: and the Rayleigh-Ritz method

$$v(x) = a_1 \underbrace{x(L-x)}_{\Phi_1} + a_2 \underbrace{x^2(L-x)^2}_{\Phi_2} + \ldots$$

     A major drawback of this approach, is that the coefficients have no physical meaning.
  2. Unknown nodal deformations, most appropriate for discrete systems and Potential Energy based formulations

$$v(\overline{\mathbf{u}}_i) = u = N_1\overline{\mathbf{u}}_1 + N_2\overline{\mathbf{u}}_2 + \ldots + N_n\overline{\mathbf{u}}_n$$

     where $\overline{\mathbf{u}}_i$ is the known displacement at dof i.

Expression for the generalized known displacement (translation or rotation), u at any degree of freedom in terms of all its known nodal ones, $\overline{\mathbf{u}}$.

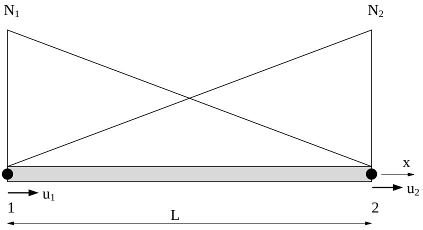
$$u(x) = \sum_{i=1}^{n} N_i(x)\overline{\mathbf{u}}_i = \lfloor \mathbf{N}(x) \rfloor \{\overline{\mathbf{u}}\}$$

$\overline{\mathbf{u}}_i$ is the (generalized) nodal displacement corresponding to d.o.f i

1. N_i is an interpolation function, or shape function which has the following characteristics: $N_i = 1$ at dof i and $N_i = 0$ at dof j where $i \neq j$.
2. Summation of N at any point is equal to unity $\Sigma N = 1$.
3. $\mathbf{N}$ can be derived on the bases of:
   1. Assumed deformation state defined in terms of polynomial series.
   2. Interpolation function (Lagrangian or Hermitian).
4. As with the Rayleigh-Ritz method, polynomial functions should
   1. Be continuous, of the type required by the variational principle.
   2. Exhibit rigid body motion (i.e. $v = a_1 + \ldots$)
   3. Exhibit constant strain.

- Shape functions should be complete, and meet the same requirements as the coefficients of the Rayleigh Ritz method.

- Shape functions can often be written in non-dimensional coordinates (i.e. $\xi = \frac{x}{l}$). This will be exploited later by the so-called isoparametric elements.



- Let $u(x) = N_1(x)\overline{u}_1 + N_2(x)\overline{u}_2$ or $\theta_x = N_1\overline{\theta}_{x1} + N_2\overline{\theta}_{x2}$
- We have 2 d.o.f's, we will assume a linear deformation state $u(x) = a_1 x + a_2$ where u can be either u or θ, and the essential B.C.'s are given by: $u = \overline{u}_1$ at $x = 0$, and $u = \overline{u}_2$ at $x = L$. Thus we have:

$$\overline{u}_1 = a_2; \qquad \overline{u}_2 = a_1 L + a_2$$

- Solving for a_1 and a_2 in terms of $\overline{u}_1$ and $\overline{u}_2$ we obtain:

$$a_1 = \frac{\overline{u}_2}{L} - \frac{\overline{u}_1}{L}; \qquad a_2 = \overline{u}_1$$

- Substituting and rearranging those expressions we obtain

$$\begin{aligned} u(x) &= \left(\frac{\overline{u}_2}{L} - \frac{\overline{u}_1}{L}\right)x + \overline{u}_1 \\ &= \underbrace{\left(1 - \frac{x}{L}\right)}_{N_1(x)}\overline{u}_1 + \underbrace{\frac{x}{L}}_{N_2(x)}\overline{u}_2 \end{aligned}$$

Note that $N1(x) + N2(x) = 1 \quad \forall x \in [0 \;\; L]$

- The previous derivation can be generalized by writing:

$$u(x) = a_1 x + a_2 = \underbrace{\lfloor\; x \quad 1 \;\rfloor}_{\lfloor \mathbf{p}(x) \rfloor} \underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\}}_{\{\mathbf{a}\}}$$

where $\lfloor \mathbf{p}(x) \rfloor$ corresponds to the polynomial approximation, and $\{\mathbf{a}\}$ is the coefficient vector.

- Apply the boundary conditions:

$$\underbrace{\left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \end{array} \right\}}_{\{\overline{\mathbf{u}}\}} = \underbrace{\left[\begin{array}{cc} 0 & 1 \\ L & 1 \end{array} \right]}_{[\mathcal{L}]} \underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\}}_{\{\mathbf{a}\}}$$

- Following inversion of $[\mathcal{L}]$, this leads to

$$\underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\}}_{\{\mathbf{a}\}} = \underbrace{\frac{1}{L} \left[\begin{array}{cc} -1 & 1 \\ L & 0 \end{array} \right]}_{[\mathcal{L}]^{-1}} \underbrace{\left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \end{array} \right\}}_{\{\overline{\mathbf{u}}\}}$$

- Substituting this last equation, we obtain:

$$u(x) = \underbrace{\underbrace{\lfloor\; (1 - \tfrac{x}{L}) \quad \tfrac{x}{L} \;\rfloor}_{\lfloor \mathbf{p}(x) \rfloor [\mathcal{L}]^{-1}}}_{[\mathbf{N}(x)]} \underbrace{\left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \end{array} \right\}}_{\{\overline{\mathbf{u}}\}}$$

- Hence, the shape functions $[\mathbf{N}]$ can be directly obtained from

$$[\mathbf{N}(x)] = \lfloor \mathbf{p}(x) \rfloor [\mathcal{L}]^{-1}$$

- We have 4 d.o.f.'s, $\{\overline{\mathbf{u}}\}_{4\times 1}$:and hence will need 4 shape functions, N_1 to N_4, and those will be obtained through 4 boundary conditions.
- With four essential boundary conditions (two on each node), we must assume a polynomial with four coefficients

$$
\begin{aligned}
v(x) &= a_1 x^3 + a_2 x^2 + a_3 x + a_4 \\
\theta(x) &= \frac{dv}{dx} = 3a_1 x^2 + 2a_2 x + a_3
\end{aligned}
$$

- Note that v can be rewritten as:

$$\left\{\begin{array}{c} v(x) \end{array}\right\} = \underbrace{\lfloor\begin{array}{cccc} x^3 & x^2 & x & 1 \end{array}\rfloor}_{\lfloor\mathbf{p}(x)\rfloor} \underbrace{\left\{\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array}\right\}}_{\{\mathbf{a}\}}$$

- We now apply the boundary conditions:

$$\begin{array}{lll} v = \overline{v}_1 & \text{at} & x = 0 \\ v = \overline{v}_2 & \text{at} & x = L \\ \theta = \overline{\theta}_1 = \frac{dv}{dx} & \text{at} & x = 0 \\ \theta = \overline{\theta}_2 = \frac{dv}{dx} & \text{at} & x = L \end{array}$$

or:

$$\underbrace{\left\{\begin{array}{c} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{array}\right\}}_{\{\overline{\mathbf{u}}\}} = \underbrace{\left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3L^2 & 2L & 1 & 0 \end{array}\right]}_{[\mathcal{L}]} \underbrace{\left\{\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array}\right\}}_{\{\mathbf{a}\}}$$

- Inverting

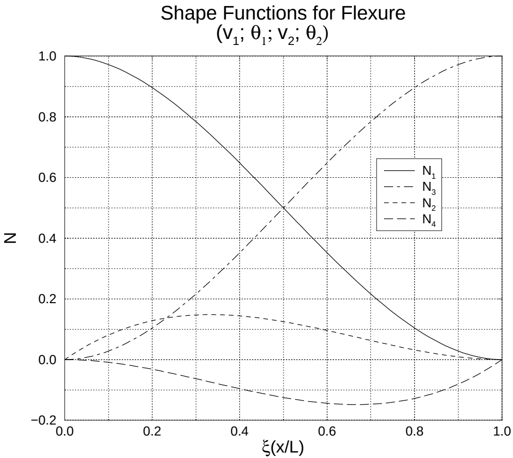
$$\underbrace{\left\{\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array}\right\}}_{\{\mathbf{a}\}} = \underbrace{\frac{1}{L^3}\left[\begin{array}{cccc} 2 & L & -2 & L \\ -3L & -2L^2 & 3L & -L^2 \\ 0 & L^3 & 0 & 0 \\ L^3 & 0 & 0 & 0 \end{array}\right]}_{[\mathcal{L}]^{-1}} \underbrace{\left\{\begin{array}{c} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{array}\right\}}_{\{\overline{\mathbf{u}}\}}$$

- Combining, and substituting $\xi = \frac{x}{L}$

$$\begin{aligned} \mathbf{u}(x) &= \underbrace{\lfloor\; x^3 \quad x^2 \quad x \quad 1 \;\rfloor}_{\lfloor \mathbf{p}(x) \rfloor} \underbrace{\frac{1}{L^3}\left[\begin{array}{cccc} 2 & L & -2 & L \\ -3L & -2L^2 & 3L & -L^2 \\ 0 & L^3 & 0 & 0 \\ L^3 & 0 & 0 & 0 \end{array}\right]}_{[\mathcal{L}]^{-1}} \underbrace{\left\{\begin{array}{c} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{array}\right\}}_{\{\overline{\mathbf{u}}\}} \\ &= \underbrace{\underbrace{\lfloor\; \underbrace{(1+2\xi^3-3\xi^2)}_{N_1} \quad \underbrace{x(1-\xi)^2}_{N_2} \quad \underbrace{(3\xi^2-2\xi^3)}_{N_3} \quad \underbrace{x(\xi^2-\xi)}_{N_4} \;\rfloor}_{[\mathbf{p}][\mathcal{L}]^{-1}}}_{[\mathbf{N}]} \underbrace{\left\{\begin{array}{c} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{array}\right\}}_{\{\overline{\mathbf{u}}\}} \end{aligned}$$

- Hence, the shape functions for the flexural element are given by:

$$\begin{array}{rclrcl} N_1 & = & (1+2\xi^3-3\xi^2); & N_2 & = & x(1-\xi)^2 \\ N_3 & = & (3\xi^2-2\xi^3); & N_4 & = & x(\xi^2-\xi) \end{array}$$



- Note that Shape function associated with dof 1 is equal to one a $\xi = 0$, equal to zero at $\xi = 1$, and its slopes at those two points is equal to zero. Similarly, shape function 2 is zero at the two end points, slope equal to 1 at $\xi = 0$, and zero at $\xi = 1$.
- Summary

Function	$\xi = 0$		$\xi = 1$	
	N_i	$N_{i,x}$	N_i	$N_{i,x}$
$N_1 = (1 + 2\xi^3 - 3\xi^2)$	1	0	0	0
$N_2 = \xi(1 - \xi)^2$	0	1	0	0
$N_3 = (3\xi^2 - 2\xi^3)$	0	0	1	0
$N_4 = \xi(\xi^2 - \xi)$	0	0	0	1

- Since the transverse displacements and the rotations are uncoupled, we can write

$$\left\{ \begin{array}{c} v \\ \theta \end{array} \right\} = \left[\begin{array}{cccc} N_1 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_4 \end{array} \right] \left\{ \begin{array}{c} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{array} \right\}$$

- Earlier in the semester, we derived the stiffness matrices of one dimensional rod elements, the approach used could not be generalized to general finite element. Alternatively, the derivation of this chapter will be applicable to both one dimensional rod (or nearly continuum) elements or contnuum (2D or 3D) elements.
- It is important to note that whereas the previously presented method to derive the stiffness matrix yielded an exact solution, it can not be generalized to continuum (2D/3D elements). On the other hands, the method presented here is an approximate method, which happens to result in an exact stiffness matrix for flexural one dimensional elements. Despite its approximation, this so-called finite element method will yield excellent results if enough elements are used.

- The displacement u at any point inside an element can be written in terms of the shape functions $\lfloor \mathbf{N} \rfloor$ and the nodal displacements $\{\overline{\mathbf{u}}\}$ as

$$\mathbf{u}(x) \stackrel{\text{def}}{=} \lfloor \mathbf{N}(x) \rfloor \{\overline{\mathbf{u}}\} \tag{1}$$

- The strain is then defined as:

$$\varepsilon(x) \stackrel{\text{def}}{=} [\mathbf{B}(x)] \{\overline{\mathbf{u}}\} \tag{2}$$

where $[\mathbf{B}]$ is the matrix which relates nodal displacements to strain field and is clearly expressed in terms of derivatives of $\mathbf{N}$.

$$u(x) = \underbrace{\lfloor \underbrace{(1-\frac{x}{L})}_{N_1} \quad \underbrace{\frac{x}{L}}_{N_2} \rfloor}_{\lfloor \mathbf{N} \rfloor} \underbrace{\left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \end{array} \right\}}_{\{\overline{\mathbf{u}}\}}$$

$$\varepsilon(x) = \varepsilon_{xx} = \frac{du}{dx} = \underbrace{\left\lfloor \underbrace{-\frac{1}{L}}_{\frac{\partial N_1}{\partial x}} \quad \underbrace{\frac{1}{L}}_{\frac{\partial N_2}{\partial x}} \right\rfloor}_{[\mathbf{B}]} \underbrace{\left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \end{array} \right\}}_{\{\overline{\mathbf{u}}\}}$$

Using the shape functions for flexural elements previously derived in

$$
\begin{aligned}
\varepsilon &= \frac{y}{\rho} = y\frac{d^2 v}{dx^2} = y\frac{d^2 N}{dx^2}\overline{\mathbf{v}} \\
&= \underbrace{y \left\lfloor \underbrace{\frac{6}{L^2}(2\xi - 1)}_{\frac{\partial^2 N_1}{\partial x^2}} \quad \underbrace{-\frac{2}{L}(3\xi - 2)}_{\frac{\partial^2 N_2}{\partial x^2}} \quad \underbrace{\frac{6}{L^2}(-2\xi + 1)}_{\frac{\partial^2 N_3}{\partial x^2}} \quad \underbrace{-\frac{2}{L}(3\xi - 1)}_{\frac{\partial^2 N_4}{\partial x^2}} \right\rfloor}_{[\mathbf{B}]} \underbrace{\left\{ \begin{array}{c} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{array} \right\}}_{\{\overline{\mathbf{u}}\}}
\end{aligned}
$$

- In anticipation of the application of the principle of virtual displacement, we define the vectors of virtual displacements and strain in terms of nodal displacements and shape functions:

$$\delta \mathbf{u}(x) = [\mathbf{N}(x)]\{\delta \overline{\mathbf{u}}\} \tag{3}$$

$$\delta \varepsilon(x) = [\mathbf{B}(x)]\{\delta \overline{\mathbf{u}}\} \tag{4}$$

- Let us now apply the principle of virtual displacement and restate some known relations (careful with matrices):

$$\begin{aligned}
\delta U &= \delta W && (5)\\
\delta U &= \int_\Omega \lfloor \delta \boldsymbol{\varepsilon} \rfloor \{\boldsymbol{\sigma}\} d\Omega && (6)\\
\{\boldsymbol{\sigma}\} &= [\mathbf{D}]\{\boldsymbol{\varepsilon}\} - [\mathbf{D}]\{\boldsymbol{\varepsilon}^0\} && (7)\\
\{\boldsymbol{\varepsilon}\} &= [\mathbf{B}]\{\overline{\mathbf{u}}\} && (8)\\
\{\delta\boldsymbol{\varepsilon}\} &= [\mathbf{B}]\{\delta\overline{\mathbf{u}}\} && (9)\\
\lfloor \delta\boldsymbol{\varepsilon} \rfloor &= \lfloor \delta\overline{\mathbf{u}} \rfloor [\mathbf{B}]^T && (10)
\end{aligned}$$

- Combining Eqns. 5, 6, 7, 10, and 8, the internal virtual strain energy is given by:

$$\begin{aligned}
\delta U &= \int_\Omega \underbrace{\lfloor \delta\overline{\mathbf{u}} \rfloor [\mathbf{B}]^T}_{\lfloor \delta \boldsymbol{\varepsilon} \rfloor} \underbrace{[\mathbf{D}][\mathbf{B}]\{\overline{\mathbf{u}}\}}_{\{\boldsymbol{\sigma}\}} d\Omega - \int_\Omega \underbrace{\lfloor \delta\overline{\mathbf{u}} \rfloor [\mathbf{B}]^T}_{\lfloor \delta \boldsymbol{\varepsilon} \rfloor} \underbrace{[\mathbf{D}]\{\boldsymbol{\varepsilon}^0\}}_{\{\boldsymbol{\sigma}^0\}} d\Omega && (11)\\
&= \lfloor \delta\overline{\mathbf{u}} \rfloor \int_\Omega [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}]\, d\Omega \{\overline{\mathbf{u}}\} - \lfloor \delta\overline{\mathbf{u}} \rfloor \int_\Omega [\mathbf{B}]^T [\mathbf{D}]\{\boldsymbol{\varepsilon}^0\} d\Omega
\end{aligned}$$

- The virtual external work in turn is given by:

$$\delta W = \underbrace{\lfloor \delta \overline{\mathbf{u}} \rfloor}_{\text{Virt. Nodal Displ.}} \underbrace{\{\overline{\mathbf{F}}\}}_{\text{Nodal Force}} + \int_l \lfloor \delta \mathbf{u} \rfloor q(x) dx \tag{12}$$

- Combining this equation with $\{\delta \mathbf{u}\} = [\mathbf{N}]\{\delta \overline{\mathbf{u}}\}$ yields:

$$\delta W = \lfloor \delta \overline{\mathbf{u}} \rfloor \{\overline{\mathbf{F}}\} + \lfloor \delta \overline{\mathbf{u}} \rfloor \int_0^l [\mathbf{N}]^T q(x)\, dx \tag{13}$$

- Equating the internal strain energy Eqn. 11 with the external work Eqn. 13, we obtain:

$$\underbrace{\lfloor \delta \overline{\mathbf{u}} \rfloor \underbrace{\int_\Omega [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}]\, d\Omega}_{[\mathbf{k}]} \{\overline{\mathbf{u}}\} - \lfloor \delta \overline{\mathbf{u}} \rfloor \underbrace{\int_\Omega [\mathbf{B}]^T [\mathbf{D}]\{\varepsilon^0\} d\Omega}_{\{\overline{\mathbf{F}}^0\}}}_{\delta U} = \underbrace{\lfloor \delta \overline{\mathbf{u}} \rfloor \{\overline{\mathbf{F}}\} + \lfloor \delta \overline{\mathbf{u}} \rfloor \underbrace{\int_0^l [\mathbf{N}]^T q(x)\, dx}_{\{\overline{\mathbf{F}}^e\}}}_{\delta W} \tag{14}$$

or

$$\boxed{[\mathbf{k}]\{\overline{\mathbf{u}}\} - \{\overline{\mathbf{F}}^{o}\} = \{\overline{\mathbf{F}}\} + \{\overline{\mathbf{F}}^{e}\}} \tag{15}$$

- Canceling out the $\lfloor \delta \overline{\mathbf{u}} \rfloor$ term, this is the same equation of equilibrium as the one written earlier on. It relates the (unknown) nodal displacement $\{\overline{\mathbf{u}}\}$, the structure stiffness matrix $[\mathbf{k}]$, the external nodal force vector $\{\overline{\mathbf{F}}\}$, the distributed element force $\left\{\overline{\mathbf{F}}^{e}\right\}$, and the vector of initial displacement.
- From this relation we define:

  The element stiffness matrix:

$$\boxed{[\mathbf{k}] = \int_{\Omega} [\mathbf{B}]^{T}[\mathbf{D}][\mathbf{B}]d\Omega} \tag{16}$$

  Element initial force vector:

$$\boxed{\{\overline{\mathbf{F}}^{0}\} = \int_{\Omega} [\mathbf{B}]^{T}[\mathbf{D}]\{\varepsilon^{0}\}d\Omega} \tag{17}$$

Element equivalent load vector:

$$\boxed{\{\overline{\mathbf{F}}^e\} = \int_0^L [\mathbf{N}]\, q(x)\, \mathrm{d}x} \tag{18}$$

- The general equation of equilibrium can be written as:

$$\boxed{\underbrace{[\mathbf{k}]\{\overline{\mathbf{u}}\} - \{\overline{\mathbf{F}}^0\}}_{F_{int}} - \underbrace{\{\overline{\mathbf{F}}\} + \{\overline{\mathbf{F}}^e\}}_{F_{ext}} = 0} \tag{19}$$

- This is the discretized Euler equation (equilibrium equation) associated with the variational defined by the principle of virtual work.

- Whereas from the preceding section, we derived a general relationship in which the nodal displacements are the primary unknowns, we next seek to determine the internal (generalized) stresses which are most often needed for design.
- Recalling that we have:

$$\{\boldsymbol{\sigma}\} = [\mathbf{D}]\{\boldsymbol{\varepsilon}\} \quad (20)$$

$$\{\boldsymbol{\varepsilon}\} = [\mathbf{B}]\{\overline{\mathbf{u}}\} \quad (21)$$

- With the vector of nodal displacement $\{\overline{\mathbf{u}}\}$ known, those two equations would yield:

$$\boxed{\{\boldsymbol{\sigma}\} = [\mathbf{D}] \cdot [\mathbf{B}]\{\overline{\mathbf{u}}\}} \quad (22)$$

- We note that the secondary variables (strain and stresses) are derivatives of the primary variables (displacement), and as such may not always be determined with the same accuracy.

- The shape functions of the truss element were derived earlier:

$$\begin{aligned} N_1 &= 1-\frac{x}{L} \\ N_2 &= \frac{x}{L} \end{aligned}$$

- The corresponding strain displacement relation $[\mathbf{B}]$ is given by:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\mathbf{du}}{\mathrm{d}x} \\ &= \left[\begin{array}{cc} \frac{dN_1}{\mathrm{d}x} & \frac{dN_2}{\mathrm{d}x} \end{array}\right] \\ &= \underbrace{\left[\begin{array}{cc} -\frac{1}{L} & \frac{1}{L} \end{array}\right]}_{[\mathbf{B}]} \end{aligned}$$

- For the truss element, the constitutive matrix $[\mathbf{D}]$ reduces to the scalar E; Hence, substituting into Eq. 16, with $\mathrm{d}\Omega = \mathrm{d}A\mathrm{d}x$: $[\mathbf{k}] = \int_\Omega [\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]\mathrm{d}\Omega$
- But $\mathrm{d}\Omega = A\mathrm{d}x$ and for element with constant cross sectional area we obtain:

$$[\mathbf{k}] = A\int_0^L \left\{\begin{array}{c} -\frac{1}{L} \\ \frac{1}{L} \end{array}\right\} \cdot E \cdot \lfloor \begin{array}{cc} -\frac{1}{L} & \frac{1}{L} \end{array} \rfloor \mathrm{d}x$$

$$\boxed{\begin{aligned} [\mathbf{k}] &= \frac{AE}{L^2}\int_0^L \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] \mathrm{d}x \\ &= \frac{AE}{L}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] \end{aligned}}$$

- For a beam element, for which we have previously derived the shape functions and the $[\mathbf{B}]$ matrix. Substituting in Eq. 16:

$$[\mathbf{k}] = \int_0^L \int_A [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}]\, y^2 \, dA \, dx$$

- Noting that $\int_A y^2 \, dA = I_z$ Eq. 16 reduces to

$$[\mathbf{k}] = \int_0^L [\mathbf{B}]^T [\mathbf{D}][\mathbf{B}] I_z \, dx$$

- For this simple case, we have: $[\mathbf{D}] = E$, thus:

$$[\mathbf{k}] = EI_z \int_0^l [\mathbf{B}]^T [\mathbf{B}] \, dx$$

- Using the shape function for the beam element, and noting the change of integration variable from dx to dξ, we obtain

$$[\mathbf{k}] = EI_z \int_0^1 \left\{ \begin{array}{c} \frac{6}{L^2}(2\xi - 1) \\ -\frac{2}{L}(3\xi - 2) \\ \frac{6}{L^2}(-2\xi + 1) \\ -\frac{2}{L}(3\xi - 1) \end{array} \right\} \left\lfloor \frac{6}{L^2}(2\xi - 1) \quad -\frac{2}{L}(3\xi - 2) \quad \frac{6}{L^2}(-2\xi + 1) \quad -\frac{2}{L}(3\xi - 1) \right\rfloor \underbrace{L d\xi}_{dx}$$

or

$$\boxed{[\mathbf{k}] = \begin{array}{c c} & \begin{array}{cccc} \overline{v}_1 & \overline{\theta}_1 & \overline{v}_2 & \overline{\theta}_2 \end{array} \\ \begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} & \left[\begin{array}{cccc} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{array} \right] \end{array}}$$

Identical to the matrix previously derived earlier in the semester ☺

Intermediary Structural Analysis

Isoparameteric Elements

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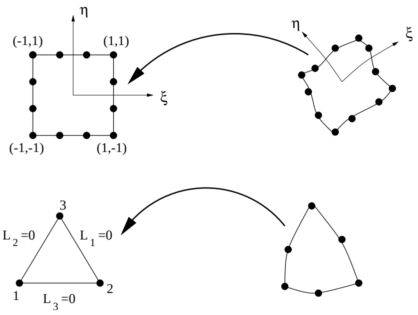
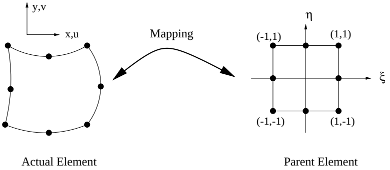


- In the isoparametric formulation, displacements are expressed in terms of natural coordinates.
- Must be differentiated with respect to cartesian coordinates x, y, z. This is accomplished through a transformation matrix (Jacobian) $\mathbf{J}$, and integration can no longer be performed analytically but must be done numerically.

- Natural coordinates range from -1 to +1



- Nodal displacements at any point inside the element can be written in terms of the nodal known displacements and the shape functions

$$
\begin{aligned}
u &= N_1\overline{u}_1 + N_2\overline{u}_2 + \cdots &= \mathrm{N}\left\{\begin{array}{c}\overline{u}_1\\ \overline{u}_2\\ \vdots\end{array}\right\} &= \mathrm{N}\overline{\mathrm{u}}_e\\
v &= N_1\overline{v}_1 + N_2\overline{v}_2 + \cdots &= \mathrm{N}\left\{\begin{array}{c}\overline{v}_1\\ \overline{v}_2\\ \vdots\end{array}\right\} &= \mathrm{N}\overline{\mathrm{v}}_e\\
w &= N_1\overline{w}_1 + N_2\overline{w}_2 + \cdots &= \mathrm{N}\left\{\begin{array}{c}\overline{w}_1\\ \overline{w}_2\\ \vdots\end{array}\right\} &= \mathrm{N}\overline{\mathrm{w}}_e
\end{aligned} \tag{1}
$$

or

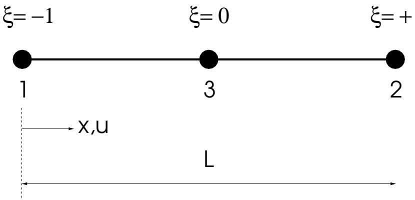
$$\mathrm{u} = \lfloor\ u \quad v \quad w\ \rfloor^T = [\mathrm{N}]\overline{\mathrm{u}}_e$$

- When elements are also distorted, the coordinates of any point can **also** be expressed in terms of nodal coordinates

$$
\begin{aligned}
x &= \tilde{N}_1\overline{x}_1 + \tilde{N}_2\overline{x}_2 + \cdots &= \tilde{\mathrm{N}}\left\{\begin{array}{c}\overline{x}_1\\ \overline{x}_2\\ \vdots\end{array}\right\} &= \tilde{\mathrm{N}}\overline{\mathrm{x}} \\
y &= \tilde{N}_1\overline{y}_1 + \tilde{N}_2\overline{y}_2 + \cdots &= \tilde{\mathrm{N}}\left\{\begin{array}{c}\overline{y}_1\\ \overline{y}_2\\ \vdots\end{array}\right\} &= \tilde{\mathrm{N}}\overline{\mathrm{y}}
\end{aligned} \tag{2}
$$

or

$$c = \lfloor \; x \quad y \quad z \; \rfloor^T = [\tilde{\mathrm{N}}]\overline{\mathrm{c}} \tag{3}$$



- The simplest introduction to isoparamteric elements is through a straight three noded quadratic elements.
- The shape functions for the element can be obtained from the Lagrangian interpolation function used earlier, and in which we substitute x by ξ. The kth term in a polynomial of order $n-1$ would be

$$N_k^n = \frac{\prod_{i=1,i\neq k}^{n}(\xi-\xi_i)}{\prod_{i=1,i\neq k}^{n}(\xi_k-\xi_i)} \tag{4}$$

$$= \frac{(\xi-\xi_1)(\xi-\xi_2)\cdots(\xi-\xi_{k-1})(\xi-\xi_{k+1})\cdots(\xi-\xi_n)}{(\xi_k-\xi_1)(\xi_k-\xi_2)\cdots(\xi_k-\xi_{k-1})(\xi_k-\xi_{k+1})\cdots(\xi_k-\xi_n)} \tag{5}$$

- For a three noded quadratic element $\xi_1 = -1$, $\xi_2 = +1$, and $\xi_3 = 0$. Substituting, we obtain the three shape functions

$$\begin{aligned} N_1(\xi) &= \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} &= \frac{(\xi-1)(\xi-0)}{(-1-1)(-1-0)} &= \frac{1}{2}(\xi^2-\xi) \\ N_2(\xi) &= \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} &= \frac{(\xi+1)(\xi-0)}{(1+1)(1-0)} &= \frac{1}{2}(\xi^2+\xi) \\ N_3(\xi) &= \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} &= \frac{(\xi+1)(\xi-1)}{(0+1)(0-1)} &= 1-\xi^2 \end{aligned} \tag{6}$$

Hence,

$$x(\xi) = \lfloor \mathrm{N} \rfloor \lfloor\ \overline{x}_1 \quad \overline{x}_2 \quad \overline{x}_3\ \rfloor^T \quad \text{and} \quad u(\xi) = \lfloor \mathrm{N} \rfloor \lfloor\ \overline{u}_1 \quad \overline{u}_2 \quad \overline{u}_3\ \rfloor^T \tag{7}$$

where

$$\lfloor \mathrm{N} \rfloor = \lfloor\ \frac{1}{2}(\xi^2-\xi) \quad \frac{1}{2}(\xi^2+\xi) \quad 1-\xi^2\ \rfloor \tag{8}$$

- The strain displacement relation is given by, $\varepsilon = \mathrm{L}\mathrm{u} = \mathrm{L}\mathrm{N}\overline{\mathrm{u}}_e = \mathrm{B}\overline{\mathrm{u}}_e$, and the differential operator L is equal to $\frac{\mathrm{d}}{\mathrm{d}x}$. For this one dimensional case, this reduces to

$$\varepsilon_x = \frac{du}{\mathrm{d}x} = \underbrace{\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}}_{L} \lfloor \mathrm{N} \rfloor}_{\mathrm{B}} \left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \\ \overline{u}_3 \end{array} \right\} \tag{9}$$

- We invoke the chain rule since the shape functions are expressed in terms of natural coordinates:
$$\mathrm{B} = \frac{d\mathrm{N}}{dx} = \frac{d\mathrm{N}}{d\xi}\frac{d\xi}{dx} \tag{10}$$
The first term may be readily available from the shape functions, Eq. **??**, however the second one is not.
- Whereas, $\frac{d\xi}{dx}$ is not available, we may determine its inverse $\frac{dx}{d\xi}$, from Eq. **??**, which we shall denote by J or Jacobian.
- The Jacobian operator J is a scale factor which relates cartesian to natural coordinates $dx = J d\xi$.

$$J(\xi) = \frac{dx}{d\xi} = \frac{d}{d\xi}\lfloor \mathrm{N} \rfloor \left\{ \begin{array}{c} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{array} \right\} = \underbrace{\underbrace{\lfloor \ \tfrac{1}{2}(2\xi - 1) \quad \tfrac{1}{2}(2\xi + 1) \quad -2\xi \ \rfloor}_{\frac{d\mathrm{N}}{d\xi}} \left\{ \begin{array}{c} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{array} \right\}}_{\frac{dx}{d\xi}} \tag{11}$$

- We can rewrite Eq. **??** as

$$\mathrm{B} = \frac{\mathrm{dN}}{\mathrm{d}x} = \underbrace{\frac{\mathrm{d}\xi}{\mathrm{d}x}}_{J^{-1}} \frac{\mathrm{dN}}{\mathrm{d}\xi} \tag{12}$$

and the B matrix is thus obtained by substituting into Eq. **??**

$$\lfloor \mathrm{B}(\xi) \rfloor = \frac{1}{J}\frac{\mathrm{d}}{\mathrm{d}\xi}\lfloor \mathrm{N} \rfloor = \frac{1}{J}\lfloor \begin{array}{ccc} \frac{1}{2}(2\xi - 1) & \frac{1}{2}(2\xi + 1) & -2\xi \end{array} \rfloor \tag{13}$$

- The differential area is

$$d\Omega = A\mathrm{d}x = AJd\xi \tag{14}$$

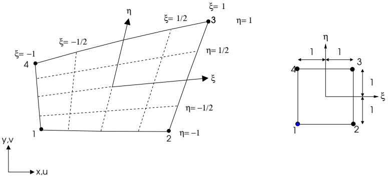
- Substituting, the element stiffness matrix is finally obtained from Eq. **??**

$$\mathrm{K}_e(\xi) = \int_0^L \mathrm{B}^T(\xi)AE\mathrm{B}(\xi)\mathrm{d}x = \int_{-1}^{+1} \mathrm{B}^T(\xi)AE\mathrm{B}(\xi)J(\xi)d\xi$$

- We observe that B, in general, contains ξ terms in both the numerator and denominator, and hence the expression can not be analytically inverted. Furthermore, the limits of integration are now from -1 to +1, and we shall see later on how to numerically integrate it.

- A simple Mathematica code to generate the stiffness matrix of three noded (quadratic) element:

- We have previously derived the stiffness matrix of a rectangular element (aligned with the coordinate axis), this formulation will generalize it to an arbitrary quadrilateral shape.



- For the two-dimensional case

$$u(\xi,\eta) = \sum N_{ij}\overline{u}_k = \sum_{i=1}^{n}\sum_{j=1}^{m} N_j(\xi)N_i(\eta)\overline{u}_k \tag{15}$$

where $k = (i-1)m + j$. For a bilinear element, $n = m = 2$, this can be rewritten as

$$\begin{aligned}
u(\xi,\eta) &= \lfloor\ N_1(\xi) \quad N_2(\xi)\ \rfloor \begin{bmatrix} \overline{u}_1 & \overline{u}_3 \\ \overline{u}_2 & \overline{u}_4 \end{bmatrix} \left\{ \begin{array}{c} N_1(\eta) \\ N_2(\eta) \end{array} \right\} = \mathrm{N}_\xi^T \overline{\mathrm{u}} \mathrm{N}_\eta \\
&= N_1(\xi)N_1(\eta)\overline{u}_1 + N_2(\xi)N_1(\eta)\overline{u}_2 + N_1(\xi)N_2(\eta)\overline{u}_4 + N_2(\xi)N_2(\eta)\overline{u}_3 \\
&= N_1(\xi,\eta)\overline{u}_1 + N_2(\xi,\eta)\overline{u}_2 + N_3(\xi,\eta)\overline{u}_3 + N_4(\xi,\eta)\overline{u}_4 \qquad (16) \\
&= \sum_{i=1}^{4} N_i \overline{u}_i \qquad (17)
\end{aligned}$$

- Applying the Lagrangian interpolation equation, Eq. **??** we obtain

$$N_1(\xi) = \frac{(\xi-\xi_2)}{(\xi_1-\xi_2)} = \frac{(\xi-1)}{(-1-1)} = \frac{1}{2}(1-\xi) \tag{18}$$

$$N_2(\xi) = \frac{(\xi-\xi_1)}{(\xi_2-\xi_1)} = \frac{(\xi+1)}{(1+1)} = \frac{1}{2}(1+\xi) \tag{19}$$

$$N_1(\eta) = \frac{(\eta-\eta_2)}{(\eta_1-\eta_2)} = \frac{(\eta-1)}{(-1-1)} = \frac{1}{2}(1-\eta) \tag{20}$$

$$N_2(\eta) = \frac{(\eta-\eta_1)}{(\eta_2-\eta_1)} = \frac{(\eta+1)}{(1+1)} = \frac{1}{2}(1+\eta) \tag{21}$$

Substituting into Eq. **??**

$$\boxed{\begin{array}{rclrcl} N_1(\xi,\eta) & = & \frac{1}{4}(1-\xi)(1-\eta); & N_2(\xi,\eta) & = & \frac{1}{4}(1+\xi)(1-\eta); \\ N_3(\xi,\eta) & = & \frac{1}{4}(1+\xi)(1+\eta); & N_4(\xi,\eta) & = & \frac{1}{4}(1-\xi)(1+\eta); \end{array}} \quad (22)$$

It should be noted that for this simple case, the shape functions could have been determined by mere inspction.

- Coordinates and displacements are given by

$$\begin{array}{rclrcl} x & = & \sum N_i(\xi,\eta)\overline{x}_i; & y & = & \sum N_i(\xi,\eta)\overline{y}_i \\ u & = & \sum N_i(\xi,\eta)\overline{u}_i; & v & = & \sum N_i(\xi,\eta)\overline{v}_i \end{array} \quad (23)$$

- The strain displacement relation is given by Eq. ??

$$\{\varepsilon\} = \left\{ \begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{array} \right\} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{\mathbf{L}} \underbrace{\left\{ \begin{array}{c} u \\ v \end{array} \right\}}_{\mathbf{u}} \tag{24}$$

However the displacements can be obtained from Eq. ??

$$\underbrace{\left\{ \begin{array}{c} u \\ v \end{array} \right\}}_{\mathbf{u}} = \underbrace{\left[\begin{array}{cc|cc|cc|cc} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) \end{array} \right]}_{\mathbf{N}} \tag{25}$$

- Combining Eq. **??** and **??** yields

$$
\begin{aligned}
\varepsilon &= \mathbf{L}\mathbf{N}\overline{\mathbf{u}} = \mathbf{B}\overline{\mathbf{u}} \\
\left\{ \begin{array}{c} \varepsilon_{xx}(\xi,\eta) \\ \varepsilon_{yy}(\xi,\eta) \\ \gamma_{xy}(\xi,\eta) \end{array} \right\} &= \underbrace{\sum_{i=1}^{4} \left[\begin{array}{cc} \frac{\partial N_i(\xi,\eta)}{\partial x} & 0 \\ 0 & \frac{\partial N_i(\xi,\eta)}{\partial y} \\ \frac{\partial N_i(\xi,\eta)}{\partial y} & \frac{\partial N_i(\xi,\eta)}{\partial x} \end{array} \right]}_{\mathbf{B}=\mathbf{L}\mathbf{N}} \underbrace{\left\{ \begin{array}{c} \overline{u}_i \\ \overline{v}_i \end{array} \right\}}_{\overline{\mathbf{u}}} \\
&= \underbrace{\left[\begin{array}{cc|cc|cc|cc} N_{1,x} & 0 & N_{2,x} & 0 & N_{3,x} & 0 & N_{4,x} & 0 \\ 0 & N_{1,y} & 0 & N_{2,y} & 0 & N_{3,y} & 0 & N_{4,y} \\ N_{1,y} & N_{1,x} & N_{2,y} & N_{2,x} & N_{3,y} & N_{3,x} & N_{4,y} & N_{4,x} \end{array} \right]}_{\mathbf{B}} \left\{ \begin{array}{c} \overline{U} \\ \overline{V} \\ \hline \overline{U} \\ \overline{V} \\ \hline \overline{U} \\ \overline{V} \\ \hline \overline{U} \\ \overline{V} \end{array} \right. \\
&= \sum_{i=1}^{4} \left[\begin{array}{cc} \frac{\partial N_i(\xi,\eta)}{\partial \xi} & 0 \\ 0 & \frac{\partial N_i(\xi,\eta)}{\partial \eta} \\ \frac{\partial N_i(\xi,\eta)}{\partial \eta} & \frac{\partial N_i(\xi,\eta)}{\partial \xi} \end{array} \right] \underbrace{\left[\begin{array}{cc} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial y} \end{array} \right]}_{[\mathbf{J}]^{-1}}
\end{aligned}
$$

- Considering the local set of coordinates ξ, η and the corresponding global one x, y, the chain rules would give

$$\left\{ \begin{array}{c} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{array} \right\} = \underbrace{\left[\begin{array}{cc} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{array} \right]}_{\mathbf{J}} \left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{array} \right\} \quad (28)$$

$$\left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{array} \right\} = [\mathrm{J}]^{-1} \left\{ \begin{array}{c} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{array} \right\} \quad (29)$$

This last equation is the key to get all the components which will go inside the B matrix.

- Expanding the definition of the Jacobian

$$\left\{\begin{array}{c}\frac{\partial N_i(\xi,\eta)}{\partial \xi}\\ \frac{\partial N_i(\xi,\eta)}{\partial \eta}\end{array}\right\} = \underbrace{\left[\begin{array}{cc}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{array}\right]}_{\mathbf{J}}\left\{\begin{array}{c}\frac{\partial N_i}{\partial x}\\ \frac{\partial N_i}{\partial y}\end{array}\right\} = \sum_{i=1}^{4}\left[\begin{array}{cc}\frac{\partial N_i}{\partial \xi}\overline{x}_i & \frac{\partial N_i}{\partial \xi}\overline{y}_i\\ \frac{\partial N_i}{\partial \eta}\overline{x}_i & \frac{\partial N_i}{\partial \eta}\overline{y}_i\end{array}\right]\left\{\begin{array}{c}\frac{\partial N_i}{\partial x}\\ \frac{\partial N_i}{\partial y}\end{array}\right\}$$

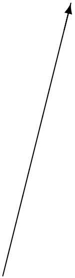
$$= \left[\begin{array}{cccc}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi}\\ \hline \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta}\end{array}\right]\left[\begin{array}{c|c}\overline{x}_1 & \overline{y}_1\\ \overline{x}_2 & \overline{y}_2\\ \overline{x}_3 & \overline{y}_3\\ \overline{x}_4 & \overline{y}_4\end{array}\right]\left\{\begin{array}{c}\frac{\partial N_i}{\partial x}\\ \frac{\partial N_i}{\partial y}\end{array}\right\}$$

$$= \underbrace{\frac{1}{4}\left[\begin{array}{cccc}-(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta)\\ \hline -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi)\end{array}\right]\left[\begin{array}{c|c}\overline{x}_1 & \overline{y}_1\\ \overline{x}_2 & \overline{y}_2\\ \overline{x}_3 & \overline{y}_3\\ \overline{x}_4 & \overline{y}_4\end{array}\right]}_{\mathbf{J}}$$

- Back to the Jacobian

$$[J]^{-1} \stackrel{\text{def}}{=} \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} = \frac{1}{\det J} \sum_{i=1}^{4} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \overline{y}_i & -\frac{\partial N_i}{\partial \eta} \overline{y}_i \\ -\frac{\partial N_i}{\partial \eta} \overline{x}_i & \frac{\partial N_i}{\partial \xi} \overline{x}_i \end{bmatrix} \tag{33}$$

- From calculus, if ξ and η are some arbitrary curvilinear coordinates

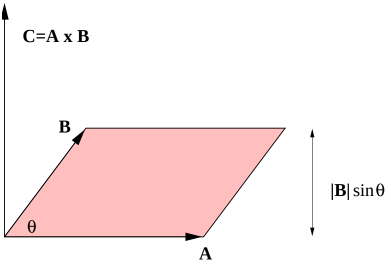


then

$$d\mathbf{r} = \left\{ \begin{array}{c} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \end{array} \right\} d\xi \quad \text{and} \quad d\mathbf{s} = \left\{ \begin{array}{c} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \end{array} \right\} d\eta \tag{34}$$

are vectors directed tangentially to $\xi =$ constant, and $\eta =$ constant respectively.

- From vector algebra, the cross product of two vectors lying in the x-y plane.



is

$$
\begin{aligned}
\mathrm{C} &= \mathrm{A} \times \mathrm{B} && (35)\\
&= |\mathrm{A}||\mathrm{B}| \sin\theta \mathrm{k} && (36)\\
&= \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ A_x & A_y & 0 \\ B_x & B_y & 0 \end{vmatrix} = \underbrace{(A_x B_y - B_x A_y)}_{\text{Area}} \mathrm{k} && (37)\\
|\mathrm{C}| &= |\mathrm{A}||\mathrm{B}| \sin\theta && (38)
\end{aligned}
$$

hence, the differential area dxdy is then equal to the length of the vector resulting from the cross product of $d\mathrm{r}d\mathrm{s}$ and is equal to

$$\boxed{\mathrm{d}(area) = \mathrm{d}x\mathrm{d}y = \det \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\mathbf{J}} \mathrm{d}\xi\mathrm{d}\eta} \tag{39}$$

- Finally determine the element stiffness matrix from

$$\boxed{[\mathrm{k}]_{8\times 8} = \int\int [\mathrm{B}]^T_{8\times 3}[\mathrm{D}]_{3x3}[\mathrm{B}]_{3\times 8} t\mathrm{d}x\mathrm{d}y = [\mathrm{k}] = \int_{-1}^{1}\int_{-1}^{1} [\mathrm{B}]^T[\mathrm{D}][\mathrm{B}]t|\mathrm{J}|\mathrm{d}\xi\mathrm{d}\eta} \tag{40}$$

- The evaluation of the element stiffness matrix involves *dA*. If we consider an infinitesimal element, of length *dr* and *ds*, at the vertex of an element, it has the boundaries of the element as its sides. Then, from Eq. **??**

$$dA = \mathrm{d}x.\mathrm{d}y.\sin\theta \tag{41}$$

however, from Eq. **??** we have $dA = \det \mathrm{J} \mathrm{d}\xi.\mathrm{d}\eta$, thus

$$\boxed{\det \mathrm{J} = \frac{\mathrm{d}x.\mathrm{d}y}{\mathrm{d}\xi.\mathrm{d}\eta}\sin\theta} \tag{42}$$

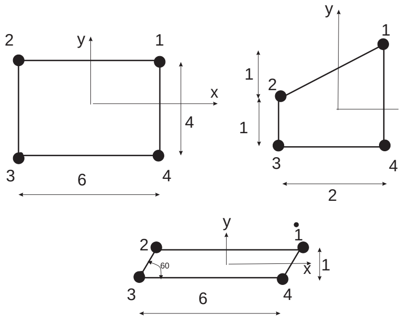
Thus we observe that if θ is small or close to 180^o, then $\det \mathrm{J}$ will be very small, if the angle is greater than 180 ^{o}, the determinant is negative (implying a negative stiffness which will usually trigger an error/stop in a FE analysis).

- In general it is recommended that $30^o < \theta < 150^o$.
- The inverse of the jacobian exists as long as the element is not much distorted or folds back upon itself.

  in those cases there is no unique relation between the coordinates.

- It can be easily shown that for parallelograms, the Jacobian is constant, whereas for nonparallelograms it is not.
- In general $\mathtt{J}$ is an indicator of the amount of element distorsion with respect to a 2x2 square one. Some times it is constant, others it varies within the element.

Determine the Jacobian operators J for the following 2 dimensional elements.



The coordinates are given by Eq. ??, the shape functions by Eq. ??, and the Jacobian by Eq. ??.
Element 1:

$$\begin{aligned} x &= \frac{1}{4}(1-\xi)(1-\eta)\overline{x}_3 + \frac{1}{4}(1+\xi)(1-\eta)\overline{x}_4 \\ &\quad + \frac{1}{4}(1+\xi)(1+\eta)\overline{x}_1 + \frac{1}{4}(1-\xi)(1+\eta)\overline{x}_2 \end{aligned} \tag{43}$$

$$\begin{aligned} &= \frac{1}{4}(1-\xi)(1-\eta)(-3) + \frac{1}{4}(1+\xi)(1-\eta)(3) \\ &\quad + \frac{1}{4}(1+\xi)(1+\eta)(3) + \frac{1}{4}(1-\xi)(1+\eta)(-3) \end{aligned} \tag{44}$$

$$\begin{aligned} y &= \frac{1}{4}(1-\xi)(1-\eta)\overline{y}_3 + \frac{1}{4}(1+\xi)(1-\eta)\overline{y}_4 \\ &\quad + \frac{1}{4}(1+\xi)(1+\eta)\overline{y}_1 + \frac{1}{4}(1-\xi)(1+\eta)\overline{y}_2 \end{aligned} \tag{45}$$

$$\begin{aligned} &= \frac{1}{4}(1-\xi)(1-\eta)(-2) + \frac{1}{4}(1+\xi)(1-\eta)(-2) \\ &\quad + \frac{1}{4}(1+\xi)(1+\eta)(2) + \frac{1}{4}(1-\xi)(1+\eta)(2) \end{aligned} \tag{46}$$

$$[\mathrm{J}] = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \tag{47}$$

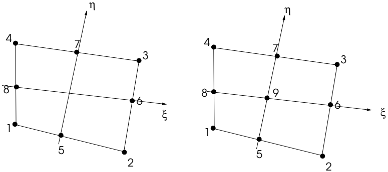
We note that $A = 24 = \det[J](2 \times 2) = 6 \times 4$
Element 2:

$$
\begin{aligned}
x &= \frac{1}{4}(1-\xi)(1-\eta)(-(3+1/(2\sqrt{3}))) + \frac{1}{4}(1+\xi)(1-\eta)(3-1/2\sqrt{3})) \\
&\quad + \frac{1}{4}(1+\xi)(1+\eta)(3+1/2\sqrt{3})) + \frac{1}{4}(1-\xi)(1+\eta)(-(3-1/2\sqrt{3}))) \quad (48) \\
y &= \frac{1}{4}(1-\xi)(1-\eta)(-2) + \frac{1}{4}(1+\xi)(1-\eta)(-2) \\
&\quad + \frac{1}{4}(1+\xi)(1+\eta)(2) + \frac{1}{4}(1-\xi)(1+\eta)(2) \quad (49) \\
[\mathrm{J}] &= \begin{bmatrix} 3 & 0 \\ \frac{1}{2\sqrt{3}} & \frac{1}{2} \end{bmatrix} \quad (50)
\end{aligned}
$$

Element 3:

$$
\begin{aligned}
x &= \frac{1}{4}(1-\xi)(1-\eta)(-1)+\frac{1}{4}(1+\xi)(1) \\
&\quad +\frac{1}{4}(1+\xi)(1+\eta)(1)+\frac{1}{4}(1-\xi)(-1) && (51)\\
y &= \frac{1}{4}(1-\xi)(1-\eta)(-3/4)+\frac{1}{4}(1+\xi)(1-\eta)(-3/4) \\
&\quad +\frac{1}{4}(1+\xi)(1+\eta)(5/4)+\frac{1}{4}(1-\xi)(1+\eta)(1/4) && (52)\\
[\mathrm{J}] &= \frac{1}{4}\begin{bmatrix} 4 & (1+\eta) \\ 0 & (3+\xi) \end{bmatrix} && (53)
\end{aligned}
$$

- For a quadratic quadrilateral element, there are two possibilities,



- The Pascal triangle, will be later used to justify the choice of terms in the displacement field of isoparameteric elements.

Constant term					1					
Linear terms				ξ		η				Linear elem
Quadratic terms			ξ^2		$\xi\eta$		η^2			Quadratic
Cubic terms		ξ^3		$\xi^2\eta$		$\xi\eta^2$		η^3		Cubic
Quartic terms			$\xi^3\eta$				$\xi\eta^3$			

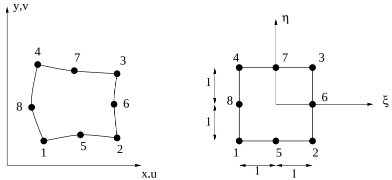
Constant term				1				
Linear terms			ξ		η			
Quadratic terms		ξ^2		$\xi\eta$		η^2		
Cubic terms	ξ^3		$\xi^2\eta$		$\xi\eta^2$		η^3	
Quartic terms		$\xi^3\eta$		$\xi^2\eta^2$		$\xi\eta^3$		
Quintic terms			$\xi^3\eta^2$		$\xi^2\eta^3$			

Constant term				1					
Linear terms			ξ		η				Linear elem
Quadratic terms		ξ^2		$\xi\eta$		η^2			Quadrati
Cubic term	ξ^3		$\xi^2\eta$		$\xi\eta^2$		η^3		Cubic

- Based on Pascal's triangle, the displacement field is given by

$$u = \underbrace{a_1}_{\boxed{0}} + \underbrace{a_2 x + a_3 y}_{\boxed{1}} + \underbrace{a_4 x^2 + a_5 xy + a_6 y^2}_{\boxed{2}} + \underbrace{a_8 x^2 y + a_9 xy^2}_{\boxed{3}} \tag{54}$$

- In this formulation, the $x^2 y^2$ term is missing and the 8 terms in the assumed polynomial expansion correspond the 8 nodes (4 corner and 4 midside).



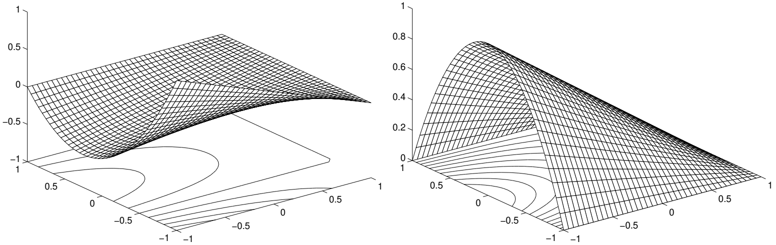
- The shape functions may be obtained by mere inspection (i.e. **serependitiously**),

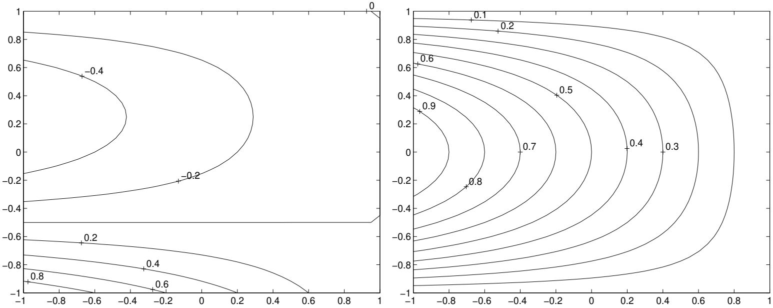
$$
\begin{array}{rclr}
N_i & = & \frac{1}{4}\left(1+\xi\xi_i\right)\left(1+\eta\eta_i\right)\left(\xi\xi_i+\eta\eta_i-1\right) & i=1,2,3,4 \\
N_i & = & \frac{1}{2}\left(1-\xi^2\right)\left(1+\eta\eta_i\right) & i=5,7 \\
N_i & = & \frac{1}{2}\left(1+\xi\xi_i\right)\left(1+\eta^2\right) & i=6,8
\end{array}
\tag{55}
$$

and are tabulated

i	N_i	$N_{i,\xi}$	$N_{i,\eta}$
1	$\frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1)$	$\frac{1}{4}(2\xi+\eta)(1-\eta)$	$\frac{1}{4}(1-\xi)(2\eta+\xi)$
2	$\frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$	$\frac{1}{4}(2\xi-\eta)(1-\eta)$	$\frac{1}{4}(1+\xi)(2\eta-\xi)$
3	$\frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)$	$\frac{1}{4}(2\xi+\eta)(1+\eta)$	$\frac{1}{4}(1+\xi)(2\eta+\xi)$
4	$\frac{1}{4}(1-\xi)(1+\eta)(-\xi-\eta-1)$	$\frac{1}{4}(2\xi-\eta)(1+\eta)$	$\frac{1}{4}(1-\xi)(2\eta-\xi)$
5	$\frac{1}{2}(1-\xi^2)(1-\eta)$	$-\xi(1-\eta)$	$-\frac{1}{2}(1-\xi^2)$
6	$\frac{1}{2}(1+\xi)(1-\eta^2)$	$\frac{1}{2}(1-\eta^2)$	$-(1+\xi)\eta$
7	$\frac{1}{2}(1-\xi^2)(1+\eta)$	$-\xi(1+\eta)$	$\frac{1}{2}(1-\xi^2)$
8	$\frac{1}{2}(1-\xi)(1-\eta^2)$	$-\frac{1}{2}(1-\eta^2)$	$-(1-\xi)\eta$

- The shape functions for the corner and midisde nodes are





0
−0.4
−0.2
0.2
0.4
0.6
0.8
0.1
0.2
0.6
0.5
0.9
0.7
0.4
0.3
0.8

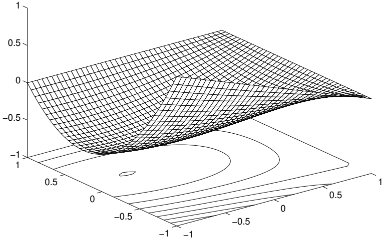
- If we were to follow a similar procedure to the one adopted to extract the bilinear shape fuctions, we would obtain 9 shape functions, which must in turn correspond to 9 (rather than 8) nodes.
- In this element, the displacement field is given by

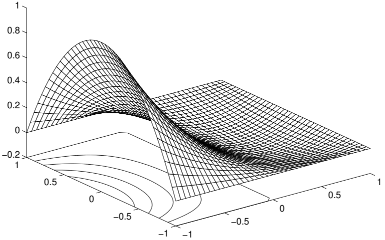
$$u = \underbrace{\underbrace{a_1}_{\boxed{0}} + \underbrace{a_2 x + a_3 y}_{\boxed{1}} + \underbrace{a_4 x^2 + a_5 xy + a_6 y^2}_{\boxed{2}} + \underbrace{a_8 x^2 y + a_9 xy^2}_{\boxed{3}}}_{\text{Serendipity}} + \underbrace{a_{12} x^2 y^2}_{\boxed{4}} \tag{56}$$

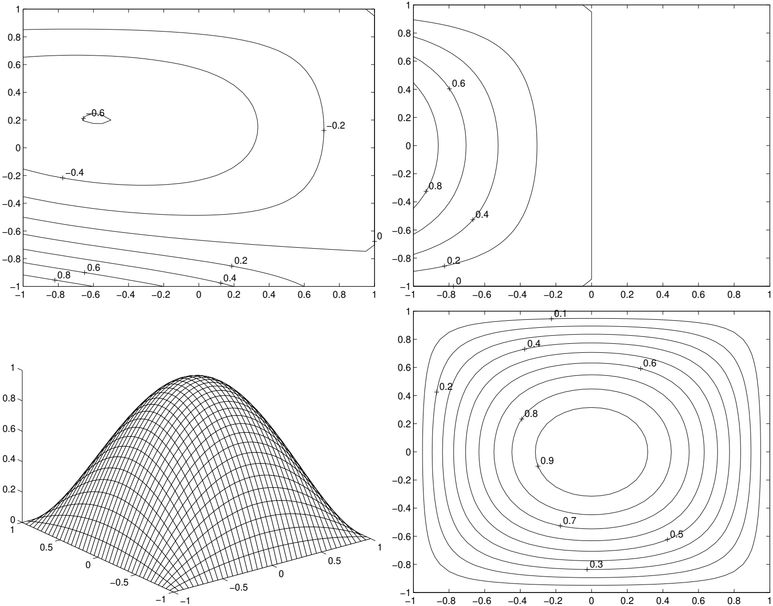
- All the quadratic terms are present, hence there are 9 terms in the polynomial expansion, and the 9th node will correspond to an internal node.
- The shape functions in this case can be directly obtained from the Lagrangian interpolation functions, yielding

$$\begin{aligned}
N_i &= \tfrac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1) && i = 1,2,3,4 \\
N_i &= \tfrac{1}{2}\left(1-\xi^2\right)(1+\eta\eta_i) && i = 5,7 \\
N_i &= \tfrac{1}{2}(1+\xi\xi_i)\left(1-\eta^2\right) && i = 6,8 \\
N_9 &= (1-\xi^2)(1-\eta^2) && i = 9
\end{aligned} \tag{57}$$

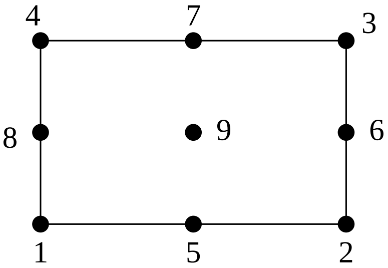
- The last shape function is often called **bubble function**
- Those shape function differ slightly from those of the serendipity element.
- Q9 elements perform much better than the Q8 if edges are not parallel or slightly curved.
- The shape functions for the corner and midisde nodes are






−0.6
−0.2
−0.4
0
0.2
0.6
0.4
0.8
0.6
0.8
0.4
0.2
0
0.1
0.4
0.6
0.2
0.8
0.9
0.7
0.5
0.3

- Based on the above, we can generalize the formulation to one of a quadrilateral element with variable number of nodes.
- This element may have different order of variation along different edges, and is quite useful to facilitate the grading of a finite element mesh.
- In its simplest formulation, it has four nodes, and has a linear variation along all sides, and in the most general case it is a full quadratic element.
- The shape functions may than be obtained from the table. Note that these shape functions are for the hierarchical element in which the corner nodes are numbered first, and midside ones after.



		Only if node i is present				
		$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
N_1	$\frac{1}{4}(1-\xi)(1-\eta)$	$-\frac{1}{2}N_5$			$-\frac{1}{2}N_8$	$\frac{1}{4}N_9$
N_2	$\frac{1}{4}(1+\xi)(1-\eta)$	$-\frac{1}{2}N_5$	$-\frac{1}{2}N_6$			$\frac{1}{4}N_9$
N_3	$\frac{1}{4}(1+\xi)(1+\eta)$		$-\frac{1}{2}N_6$	$-\frac{1}{2}N_7$		$\frac{1}{4}N_9$
N_4	$\frac{1}{4}(1-\xi)(1+\eta)$			$-\frac{1}{2}N_7$	$-\frac{1}{2}N_8$	$\frac{1}{4}N_9$
N_5	$\frac{1}{2}(1-\xi^2)(1-\eta)$					$-\frac{1}{2}N_9$
N_6	$\frac{1}{2}(1+\xi)(1-\eta^2)$					$-\frac{1}{2}N_9$
N_7	$\frac{1}{2}(1-\xi^2)(1+\eta)$					$-\frac{1}{2}N_9$
N_8	$\frac{1}{2}(1-\xi)(1-\eta^2)$					$-\frac{1}{2}N_9$
N_9	$(1-\xi^2)(1-\eta^2)$					

- For the six noded triangle, the partial derivatives of a variable ϕ with respect to x and y can be expressed as, [**?**]

$$\left\{ \begin{array}{c} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{array} \right\} = \left[\begin{array}{ccc} \frac{\partial L_1}{\partial x} & \frac{\partial L_2}{\partial x} & \frac{\partial L_3}{\partial x} \\ \frac{\partial L_1}{\partial y} & \frac{\partial L_2}{\partial y} & \frac{\partial L_3}{\partial y} \end{array} \right] \left\{ \begin{array}{c} \sum_{i=1}^{6} \phi_i \frac{\partial N_i}{\partial L_1} \\ \sum_{i=1}^{6} \phi_i \frac{\partial N_i}{\partial L_2} \\ \sum_{i=1}^{6} \phi_i \frac{\partial N_i}{\partial L_3} \end{array} \right\} \tag{58}$$

- Transposing both sides

$$\left\lfloor \sum_{i=1}^{6} \phi_i \frac{\partial N_i}{\partial L_1} \quad \sum_{i=1}^{6} \phi_i \frac{\partial N_i}{\partial L_2} \quad \sum_{i=1}^{6} \phi_i \frac{\partial N_i}{\partial L_3} \right\rfloor \left[\begin{array}{cc} \frac{\partial L_1}{\partial x} & \frac{\partial L_1}{\partial y} \\ \frac{\partial L_2}{\partial x} & \frac{\partial L_2}{\partial y} \\ \frac{\partial L_3}{\partial x} & \frac{\partial L_3}{\partial y} \end{array} \right] = \left\lfloor \frac{\partial\phi}{\partial x} \quad \frac{\partial\phi}{\partial y} \right\rfloor \tag{59}$$

- We now make $\phi \equiv 1$, x, and y:

$$\left[\begin{array}{ccc} \sum_{i=1}^{6} \frac{\partial N_i}{\partial L_1} & \sum_{i=1}^{6} \frac{\partial N_i}{\partial L_2} & \sum_{i=1}^{6} \frac{\partial N_i}{\partial L_3} \\ \sum_{i=1}^{6} x_i \frac{\partial N_i}{\partial L_1} & \sum_{i=1}^{6} x_i \frac{\partial N_i}{\partial L_2} & \sum_{i=1}^{6} x_i \frac{\partial N_i}{\partial L_3} \\ \sum_{i=1}^{6} y_i \frac{\partial N_i}{\partial L_1} & \sum_{i=1}^{6} y_i \frac{\partial N_i}{\partial L_2} & \sum_{i=1}^{6} y_i \frac{\partial N_i}{\partial L_3} \end{array} \right] \left[\begin{array}{cc} \frac{\partial L_1}{\partial x} & \frac{\partial L_1}{\partial y} \\ \frac{\partial L_2}{\partial x} & \frac{\partial L_2}{\partial y} \\ \frac{\partial L_3}{\partial x} & \frac{\partial L_3}{\partial y} \end{array} \right] = \left[\begin{array}{cc} \frac{\partial 1}{\partial x} & \frac{\partial 1}{\partial y} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{array} \right] \tag{60}$$

- But $\frac{\partial x}{\partial x} = \frac{\partial y}{\partial y} = 1$, and $\frac{\partial 1}{\partial x} = \frac{\partial 1}{\partial y} = \frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = 0$ since x and y are independent coordinates. Furthermore all entries in the first row are equal to a constant (3 for the T6 element), and since the corresponding right hand side, this row can be normalized, yielding the Jacobian matrix for this element

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ \sum_{i=1}^{6} x_i \frac{\partial N_i}{\partial L_1} & \sum_{i=1}^{6} x_i \frac{\partial N_i}{\partial L_2} & \sum_{i=1}^{6} x_i \frac{\partial N_i}{\partial L_3} \\ \sum_{i=1}^{6} y_i \frac{\partial N_i}{\partial L_1} & \sum_{i=1}^{6} y_i \frac{\partial N_i}{\partial L_2} & \sum_{i=1}^{6} y_i \frac{\partial N_i}{\partial L_3} \end{bmatrix}}_{\mathbf{J}} \begin{bmatrix} \frac{\partial L_1}{\partial x} & \frac{\partial L_1}{\partial y} \\ \frac{\partial L_2}{\partial x} & \frac{\partial L_2}{\partial y} \\ \frac{\partial L_3}{\partial x} & \frac{\partial L_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (61)$$

- Substituting

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ x_1(4L_1-1)+4x_4L_2+4x_6L_3 & x_2(4L_2-1)+4x_5L_3+4x_4L_1 & x_3(4L_3-1)+4x_6 \\ y_1(4L_1-1)+4y_4L_2+4y_6L_3 & y_2(4L_2-1)+4y_5L_3+4y_4L_1 & y_3(4L_3-1)+4y_6 \end{bmatrix}}_{\mathbf{J}}$$

$$\begin{bmatrix} \frac{\partial L_1}{\partial x} & \frac{\partial L_1}{\partial y} \\ \frac{\partial L_2}{\partial x} & \frac{\partial L_2}{\partial y} \\ \frac{\partial L_3}{\partial x} & \frac{\partial L_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (62)$$

- Next we invert the matrix and solve for the six triangular coordinates partials and substitute in Eq. **??** which in turn will enable us to determine the B matrix in Eq. **??**

$$\left\{\begin{array}{c}\varepsilon_{xx}(L_1,L_2,L_3)\\ \varepsilon_{yy}(L_1,L_2,L_3)\\ \gamma_{xy}(L_1,L_2,L_3)\end{array}\right\} = \underbrace{\sum_{i=1}^{6}\left[\begin{array}{cc}\frac{\partial N_i(L_1,L_2,L_3)}{\partial x} & 0\\ 0 & \frac{\partial N_i(L_1,L_2,L_3)}{\partial y}\\ \frac{\partial N_i(L_1,L_2,L_3)}{\partial y} & \frac{\partial N_i(L_1,L_2,L_3)}{\partial x}\end{array}\right]}_{\mathrm{B}=\mathrm{LN}} \underbrace{\left\{\begin{array}{c}\overline{u}_i\\ \overline{v}_i\end{array}\right\}}_{\overline{\mathrm{u}}} \tag{63}$$

- Understanding numerical integration is not only essential for a proper integration of the isoparameteric family of elements, but also helpful in understanding the Weighted Residual methods (Chapter **??**),
- A crucial aspect of isoparametric element formulation is the numerical integration which can be expressed as

$$\int F(\xi)d\xi \quad \text{or} \quad \int\int F(\xi,\eta)d\xi d\eta \tag{64}$$

- In practice we perform the integration numerically using

$$\int F(\xi)d\xi = \sum_i W_i F(\xi_i) + R_n \quad \text{or} \quad \int\int F(\xi,\eta)d\xi d\eta = \sum_i \sum_j W_{ij} F(\xi_i,\eta_j) + R_n \tag{65}$$

  where the summations extend over all i and j, and W_i, W_{ij} are weighting factors, and $F(\xi_i)$ and $F(\xi_i,\eta_j)$ are the matrices evaluated at the points specified in the arguments.
- The matrices R_n are error matrices, which are in general not computed.

- As shown above, $\mathrm{F} = \mathrm{B}^T.\mathrm{D}.\mathrm{B}$ for finite element stiffness matrix evaluation, and each element is integrated individually.
- The integration of $\int_a^b F(\xi)d\xi$ is essentially based on passing a polynomial $P(\xi)$ through given values of $F(\xi)$ and then use $\int_a^b P(\xi)d\xi$ as an approximation.

$$\boxed{\int_a^b F(\xi)d\xi \approx \int_a^b P(\xi)d\xi} \tag{66}$$

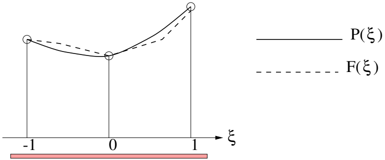
- Using $P(\xi) = F(\xi)$ at n points, and recalling the properties of Lagrangian interpolation functions, we obtain

$$P(\xi) = l_1(\xi)F(\xi_1) + l_2(\xi)F(\xi_2) + \cdots + l_n(\xi)F(\xi_n) \tag{67}$$

$$= \sum_{i=1}^{n} l_i(\xi)F(\xi_i) \tag{68}$$

and note that at $\xi = \xi_i$ $l_i = 1$, while all other $l_i = 0$.

- In Newton-Cotes integration, it is assumed that the sampling points are **equally spaced**.



thus we define

$$\int_a^b P(\xi)\mathrm{d}\xi = \int_a^b \sum_{i=1}^{n} l_i(\xi)F(\xi_i)\mathrm{d}\xi = \sum_{i=1}^{n}\int_a^b l_i(\xi)\mathrm{d}\xi F(\xi_i) \tag{69}$$

or

$$\boxed{\begin{array}{r|rcl} \text{Approximation} & \int_a^b P(\xi)\mathrm{d}\xi & = & \sum_{i=1}^{n} W_i^{(n)} F(\xi_i) \\ \text{Weights} & W_i^{(n)} & = & \int_a^b l_i(\xi)\mathrm{d}\xi = (b-a)C_i^{(n)} \end{array}} \tag{70}$$

where $C_i^{(n)}$ are the "weights" of the **Newton-Cotes quadrature** for numerical integration with n equally spaced sampling points.

- Newton-Cotes constants, and corresponding reminder are shown

n	$C_0^{(n)}$	$C_1^{(n)}$	$C_2^{(n)}$	$C_3^{(n)}$	$C_4^{(n)}$	Error
2	$\frac{1}{2}$	$\frac{1}{2}$				$10^{-1}(b-a)^3F^{II}(\xi)$
3	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$			$10^{-3}(b-a)^5F^{IV}(\xi)$
4	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$		$10^{-3}(b-a)^5F^{IV}(\xi)$
5	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$	$10^{-6}(b-a)^7F^{VI}(\xi)$

- It can be shown that this method permits exact integration of polynomial of order $n-1$, and that if n is odd, then we can exactly integrate polynomials of order n. Hence we use in general odd values of n,

- For $n = 2$ over $[-1, 1]$, we select equally spaced points at $\xi_1 = -1$ and $\xi_2 = 1$ to evaluate $\int_{-1}^{1} P(\xi) \mathrm{d}\xi$

$$
\begin{aligned}
P(\xi) &= \sum_{i=1}^{2} l_i(\xi) F(\xi_i) && (71)\\
l_1(\xi) &= \frac{\xi - \xi_2}{\xi_1 - \xi_2} = \frac{1}{2}(1-\xi) && (72)\\
l_2(\xi) &= \frac{\xi - \xi_1}{\xi_2 - \xi_1} = \frac{1}{2}(1+\xi) && (73)\\
W_1^{(2)} &= \int_{-1}^{1} l_1(\xi)\mathrm{d}\xi = \frac{1}{2}\int_{-1}^{1}(1-\xi)\mathrm{d}\xi = 1 && (74)\\
W_2^{(2)} &= \int_{-1}^{1} l_2(\xi)\mathrm{d}\xi = \frac{1}{2}\int_{-1}^{1}(1+\xi)\mathrm{d}\xi = 1 && (75)\\
\int_{-1}^{1} F(\xi)\mathrm{d}\xi &\approx \int_{-1}^{1} P(\xi)\mathrm{d}\xi = \sum_{i=1}^{2} W_i^{(2)} F(\xi_i) = F(-1) + F(1) && (76)
\end{aligned}
$$

which is the **trapezoidal rule**

- For $n = 3$ over $[-1, 1]$, we select equally spaced points at $\xi_1 = -1$ $\xi_2 = 0$, and $\xi_3 = 1$, to evaluate $\int_{-1}^{1} P(\xi)d\xi$

$$
\begin{aligned}
P(\xi) &= \sum_{i=1}^{3} l_i(\xi)F(\xi_i) && (77)\\
l_1(\xi) &= \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} = \frac{1}{2}\xi(\xi-1) && (78)\\
l_2(\xi) &= \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} = -(1+\xi)(\xi-1) && (79)\\
l_3(\xi) &= \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} = \frac{1}{2}\xi(1+\xi) && (80)\\
W_1^{(3)} &= \int_{-1}^{1} l_1(\xi)d\xi = \frac{1}{2}\int_{-1}^{1} \xi(\xi-1)d\xi = \frac{1}{3} && (81)\\
W_2^{(3)} &= \int_{-1}^{1} l_2(\xi)d\xi = \int_{-1}^{1} -(1+\xi)(\xi-1)d\xi = \frac{4}{3} && (82)\\
W_3^{(3)} &= \int_{-1}^{1} l_3(\xi)d\xi = \frac{1}{2}\int_{-1}^{1} \xi(1+\xi)d\xi = \frac{1}{3} && (83)\\
\int_{-1}^{1} F(\xi)d\xi &\approx \int_{-1}^{1} P(\xi)d\xi = \sum_{i=1}^{3} W_i^{(3)}F(\xi_i) = \frac{1}{3}[F(-1)+4F(0)+F(1)] && (84)
\end{aligned}
$$

- In Gauss-Legendre quadrature, the points are not fixed and equally spaced, but are selected to achieve **best accuracy**.
- Again we start with

$$\boxed{\int_{-1}^{1} F(\xi)d\xi \approx \int_{-1}^{1} P(\xi)d\xi = \sum_{i=1}^{n} W_i^{(n)} F(\xi_i)} \tag{85}$$

however in this formulation **both** $W_i^{(n)}$ and ξ_i are unknowns to be yet determined. Thus, we have a total of $2n$ unknowns.

- At the integration points $P(\xi_i) = F(\xi_i)$, however at intermediary points the difference can be expressed as

$$F(\xi) = P(\xi) + \underbrace{\chi(\xi)(\beta_0 + \beta_1\xi + \beta_2\xi^2 + \cdots)}_{0 \text{ at } \xi=\xi_i; i=1,2,\cdots,n} \tag{86}$$

since we want the left side to be exactly equal to $P(\xi)$ at the integration points, we define

$$\chi(\xi) = (\xi - \xi_1)(\xi - \xi_2)...(\xi - \xi_n) \tag{87}$$

as a polynomial of order n, to be equal to zero at the integration points ξ_i.

- β_i should be appropriately selected in orer to eliminate the gap between $F(\xi)$ and $P(\xi)$ at intermediary points.
- Integrating

$$\int_{-1}^{1} F(\xi)\mathrm{d}\xi = \int_{-1}^{1} P(\xi) + \sum_{j=0}^{\infty} \beta_j \int_{-1}^{1} \chi(\xi)\xi^j \mathrm{d}\xi \tag{88}$$

We split the last term

$$\sum_{j=0}^{\infty} \beta_j \int_{-1}^{1} \chi(\xi)\xi^j \mathrm{d}\xi = \sum_{j=0}^{n-1} \beta_j \int_{-1}^{1} \chi(\xi)\xi^j \mathrm{d}\xi + \sum_{j=n}^{\infty} \beta_j \int_{-1}^{1} \chi(\xi)\xi^j \mathrm{d}\xi \tag{89}$$

- Truncating the last terms of the expansion

$$\int_{-1}^{1} F(\xi)\mathrm{d}\xi \approx \int_{-1}^{1} P(\xi) + \sum_{j=0}^{n-1} \beta_j \int_{-1}^{1} \chi(\xi)\xi^j \mathrm{d}\xi \tag{90}$$

we observe that the first integral on the right-hand side involves a polynomial of order $n-1$, and the second integral a polynomial of order $2n-1$. Thus we set

$$\int_{-1}^{1} \chi(\xi)\xi^j \mathrm{d}\xi = 0 \qquad j = 0, 1, \cdots, n-1 \tag{91}$$

which would result in a set of n simultaneous equations of order n in terms of the unknowns ξ_i, $i = 0, 1, \cdots, n-1$.

- Back to Eq. **??**

$$\int_{-1}^{1} F(\xi)d\xi \approx \int_{-1}^{1} P(\xi) = \sum_{i=1}^{n} F(\xi_i) \int_{-1}^{1} l_i(\xi)d\xi = \sum_{i=1}^{n} W_i^{(n)} F(\xi_i) \qquad (92)$$

or

$$\begin{array}{|r|ccl|}\hline \text{Approximation} & \int_{-1}^{1} F(\xi)d\xi & \approx & \sum_{i=1}^{n} W_i^{(n)} F(\xi_i) \\ \text{Weights} & W_i^{(n)} & = & \int_{-1}^{1} l_i(\xi)d\xi \\ \text{Gauss Points} & \int_{-1}^{1} \chi(\xi)\xi^j d\xi & = & 0 \qquad j = 0, 1, \cdots, n-1 \\ \hline \end{array} \qquad (93)$$

- Integration points ξ_i and weight coefficients $W_i^{(n)}$ are

n	ξ_i	$W_i^{(n)}$	Error
1	0	2	$\frac{1}{3}F^{(2)}(\xi)$
2	$-1/\sqrt{3} \quad 1/\sqrt{3}$	$1 \quad 1$	$\frac{1}{135}F^{(4)}(\xi)$
3	$-\sqrt{3/5} \quad 0 \quad \sqrt{3/5}$	$5/9 \quad 8/9 \quad 5/9$	$\frac{1}{15,750}F^{(6)}(\xi)$

- The solutions (Gauss integration points) are equal to the roots of a Legendre polynomial defined by

$$\boxed{\begin{array}{rcl} L_0(\xi) & = & 1 \\ L_1(\xi) & = & \xi \\ L_k(\xi) & = & \frac{2k-1}{k}\xi L_{k-1}(\xi) - \frac{k-1}{k}\xi L_{k-2}(\xi) \qquad 2 \leq k \leq n \end{array}} \tag{94}$$

  and the n Gauss integration points are determined by solving $L_n(\xi) = 0$ for its roots ξ_i, $i = 0, 1, \cdots, n-1$.

- The weighting functions are then given by

$$\boxed{W_i^{(n)} = \frac{2(1-\xi_i^2)}{[nL_{n-1}(\xi_i)]^2}} \tag{95}$$

- First we seek the integration points for $n = 2$, $\chi(\xi) = (\xi - \xi_1)(\xi - \xi_2)$, and the resulting equations are

$$\int_{-1}^{1} (\xi - \xi_1)(\xi - \xi_2)\xi^0 d\xi = 0 \tag{96}$$

$$\int_{-1}^{1} (\xi - \xi_1)(\xi - \xi_2)\xi^1 d\xi = 0 \tag{97}$$

(98)

Upon integration, we obtain

$$\xi_1 \xi_2 = -\frac{1}{3} \quad \text{and} \quad \xi_1 + \xi_2 = 0 \tag{99}$$

hence

$$\xi_1 = -\frac{1}{\sqrt{3}} \quad \text{and} \quad \xi_2 = \frac{1}{\sqrt{3}} \tag{100}$$

The weight coefficients are

$$W_i^{(n)} = \int_{-1}^{1} l_i(\xi)\mathrm{d}\xi \tag{101}$$

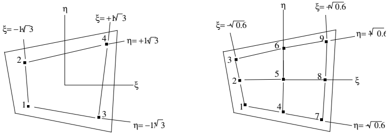
$$W_1^{(2)} = \int_{-1}^{1} \frac{\xi - \xi_2}{\xi_1 - \xi_2}\mathrm{d}\xi = \frac{-2\xi_2}{\xi_1 - \xi_2} = 1.0 \tag{102}$$

$$W_2^{(2)} = \int_{-1}^{1} \frac{\xi - \xi_1}{\xi_2 - \xi_1}\mathrm{d}\xi = \frac{-2\xi_1}{\xi_1 - \xi_2} = 1.0 \tag{103}$$

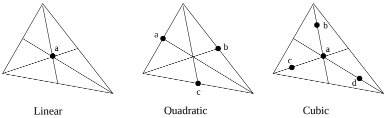
- Numerical integration of $F(\xi, \eta)$ over a rectangular region $-1 \leq \xi \leq 1$, and $-1 \leq \eta \leq 1$, is accomplished by selecting m and n (not to be confused with the order of the polynomial) integration points in the ξ and η directions

$$\boxed{\int_{-1}^{1}\int_{-1}^{1} F(\xi.\eta)d\xi d\eta \approx \int_{-1}^{1}\int_{-1}^{1} P(\xi,\eta)d\xi d\eta = \sum_{i=1}^{n}\sum_{j=1}^{m} W_i^{(m)} W_j^{(n)} F(\xi_i, \eta_j)} \tag{104}$$

and the total number of integration points will thus be $m \times n$, Fig. ??.



- For the numerical integration over a triangle, the Gauss points are shown in Fig. ??, and the corresponding triangular coordinates are given by Table ??.



Order	Error	Points	Triang. Coord.	Weights
Linear	$R = O(h^2)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	1
Quadratic		a	$\frac{1}{2}, \frac{1}{2}, 0$	$\frac{1}{3}$
	$R = O(h^3)$	b	$0, \frac{1}{2}, \frac{1}{2},$	$\frac{1}{3}$
		c	$\frac{1}{2}, 0, \frac{1}{2}$	$\frac{1}{3}$
Cubic		a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$-\frac{27}{48}$
	$R = O(h^4)$	b	$0.6, 0.2, 0.2$	$\frac{25}{48}$
		c	$0.2, 0.6, 0.2$	$\frac{25}{48}$
		d	$0.2, 0.2, 0.6$	$\frac{25}{48}$

- Stresses are evaluated from

$$\sigma = \mathbf{D}\mathbf{B}\overline{\mathbf{u}} \tag{105}$$

  in general, it is desirable to have them evaluated at the elements nodal points. However, it should be kept in mind that stresses computed at a given nodes from different elements need not be the same (since stresses are not required to be continuous in displacement based finite element formulations).

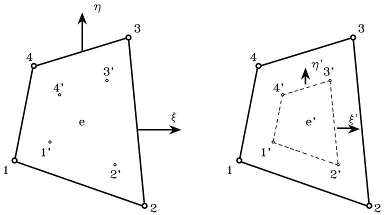
- Hence some sort of stress averaging at nodal points may be desirable.

- In the isoparametric formulation, nodal stresses are very poor, and best results are obtained at the Gauss points.

- To evaluate nodal stresses, two approaches:
  1. Evaluate σ directly at the nodes ($\xi = \eta = \pm 1$)
  2. Evaluate the stresses at the Gauss points and then extrapolate.

  The second approach yields far better results.

- Extrapolation from Gauss points will be illustrated through Fig. **??** for the 4 noded isoparametric quadrilateral.



We specify an "internal" element with its own nodes and natural coordinates ξ' and η' which are related to ξ and η through Table **??**

Corner Node	ξ	η	ξ'	η'	Gauss Node	ξ	η	ξ'	η'
1	-1	-1	$-\sqrt{3}$	$-\sqrt{3}$	1'	$-1/\sqrt{3}$	$-1/\sqrt{3}$	-1	-1
2	$+1$	-1	$+\sqrt{3}$	$-\sqrt{3}$	2'	$+1/\sqrt{3}$	$-1/\sqrt{3}$	$+1$	-1
3	$+1$	$+1$	$+\sqrt{3}$	$+\sqrt{3}$	3'	$+1/\sqrt{3}$	$+1/\sqrt{3}$	$+1$	$+1$
4	-1	$+1$	$-\sqrt{3}$	$+\sqrt{3}$	4'	$-1/\sqrt{3}$	$+1/\sqrt{3}$	-1	$+1$

or

$$\xi' = \sqrt{3}\xi \qquad \eta' = \sqrt{3}\eta \tag{106}$$

hence any scalar quantity σ (such as σ_{xx}) whose values σ'_i is known at the Gauss element corners can be interpolated through the usual bilinear shape functions now expressed in terms of ξ' and η'

$$\sigma(\xi', \eta') = \lfloor \; N_1^{e'} \quad N_2^{e'} \quad N_3^{e'} \quad N_4^{e'} \; \rfloor \left\{ \begin{array}{c} \sigma'_1 \\ \sigma'_2 \\ \sigma'_3 \\ \sigma'_4 \end{array} \right\} \tag{107}$$

where

$$N_1^{e'} = \frac{1}{4}(1-\xi')(1-\eta') \tag{108}$$

$$N_2^{e'} = \frac{1}{4}(1+\xi')(1-\eta') \tag{109}$$

$$N_3^{e'} = \frac{1}{4}(1+\xi')(1+\eta') \tag{110}$$

$$N_4^{e'} = \frac{1}{4}(1-\xi')(1+\eta') \tag{111}$$

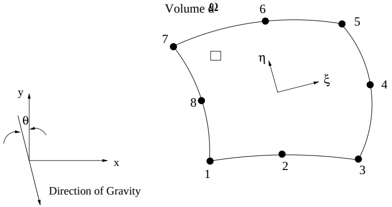
Similarly, we can **extrapolate** σ to the corners of the element. For corner 1, for instance, we replace ξ' and η' in the preceding equations by $-\sqrt{3}$. Doing that for the four corners, we obtain

$$\left\{ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} \right\} = \begin{bmatrix} 1+\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 1-\frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & 1+\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 1-\frac{1}{2}\sqrt{3} \\ 1-\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 1+\frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 1+\frac{1}{2}\sqrt{3} \end{bmatrix} \left\{ \begin{array}{c} \sigma'_1 \\ \sigma'_2 \\ \sigma'_3 \\ \sigma'_4 \end{array} \right\} \tag{112}$$

As expected, the sum of each row is equal to one, and for stresses we replace σ by σ_{xx}, σ_{yy}, and τ_{xy}

- As we know, different nodal stresses will be obtained from adjacent elements. To obtain a single value we can either take
  1. Unweighted average of all the nodal stresses.
  2. Weighted average of nodal stresses based on the relative sizes of the elements as determined from their area through $\det(J)$.

- In the finite element formulation, all loads must be replaced by an "energy equivalent" nodal load.
- We shall consider the following cases: nodal load, gravity, tractions, and thermal.
- Gravity forces are equivalent to a body force/unit volume acting within the solid in the direction of the gravity axis, Fig. **??**,



(which need not be coincident with either of the coordinate axis).

$$dG_x = \rho g d\Omega \sin\theta \tag{113}$$
$$dG_y = -\rho g d\Omega \cos\theta \tag{114}$$

where g is the gravitational acceleration and ρ is the mass density.

- Recalling from Eq. **??** that

$$\mathrm{f}_e = \int_{\Omega_e} \mathrm{N}^T \mathrm{b} d\Omega \tag{115}$$

we obtain

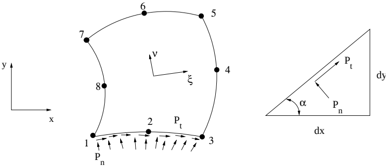
$$\left\{ \begin{array}{c} P_{xi} \\ P_{yi} \end{array} \right\} = \int_{\Omega_e} N_i \rho g \left\{ \begin{array}{c} \sin\theta \\ -\cos\theta \end{array} \right\} d\Omega \tag{116}$$

or the energy equivalent nodal forces for node i are

$$\boxed{\left\{ \begin{array}{c} P_{xi} \\ P_{yi} \end{array} \right\} = \sum_{j=1}^{ngaus} \sum_{k=1}^{ngaus} \rho g t \left\{ \begin{array}{c} \sin\theta \\ -\cos\theta \end{array} \right\} N_i(\xi_j \eta_k) W_j W_k \det \mathrm{J}(\xi_j, \eta_k)} \tag{117}$$

- The angle θ is to be measured counter-clockwise from the positive y axis.
- Any element edge can have a distributed load per unit length in a normal and tangential direction prescribed.
- The variation of the distributed load is polynomial and its order can not exceed the order of the element.
- For the sake of consistency, loaded nodes are listed also counterclockwise.

- First we determine the components of the distributed loads in the x and y directions by considering the forces acting on an incremental length dS of the loaded edge, Fig. **??**:



$$\begin{array}{rcccc} dP_x & = & (p_t dS \cos\theta - p_n dS \sin\theta) & = & (p_t \mathrm{d}x - p_n \mathrm{d}y) \\ dP_y & = & (p_n dS \cos\theta - p_t dS \sin\theta) & = & (p_n \mathrm{d}x - p_t \mathrm{d}y) \end{array} \quad (118)$$

- But since integration is to be carried on in terms of natural coordinates:

$$\mathrm{d}x = \frac{\partial x}{\partial \xi}\mathrm{d}\xi \qquad \mathrm{d}y = \frac{\partial y}{\partial \xi}\mathrm{d}\xi \tag{119}$$

Substituting

$$dP_x = \left(p_t\frac{\partial x}{\partial \xi} - p_n\frac{\partial y}{\partial \xi}\right)\mathrm{d}\xi \tag{120}$$

$$dP_y = \left(p_n\frac{\partial x}{\partial \xi} + p_t\frac{\partial y}{\partial \xi}\right)\mathrm{d}\xi \tag{121}$$

$$\tag{122}$$

- From Eq. **??** we have

$$\mathrm{f}_e = \int_{\Gamma_t} \mathrm{N}^T\hat{\mathrm{t}}\mathrm{d}\Gamma \tag{123}$$

or

$$P_{xi} = \int_{\Gamma_t} N_i\left(p_t\frac{\partial x}{\partial \xi} - p_n\frac{\partial y}{\partial \xi}\right)\mathrm{d}\xi \tag{124}$$

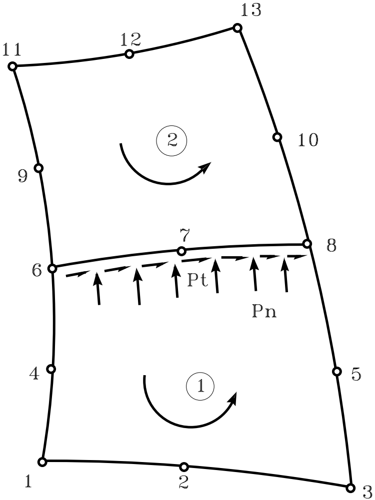
$$P_{yi} = \int_{\Gamma_t} N_i\left(p_n\frac{\partial x}{\partial \xi} + p_t\frac{\partial y}{\partial \xi}\right)\mathrm{d}\xi \tag{125}$$

The integration is again carried on numerically (along the edge) and the energy equivalent nodal forces for node *i* are

$$\boxed{\begin{array}{lcl} P_{xi} & = & \sum_{j=1}^{ngaus} N_i \left(p_t \frac{\partial x}{\partial \xi} - p_n \frac{\partial y}{\partial \xi} \right) W_j \\ P_{yi} & = & \sum_{j=1}^{ngaus} N_i \left(p_n \frac{\partial x}{\partial \xi} + p_t \frac{\partial y}{\partial \xi} \right) W_j \end{array}} \tag{126}$$

where the integration is carried on numerically along the edge. and note that $\frac{\partial x}{\partial \xi}$ and $\frac{\partial y}{\partial \xi}$ are taken straight out of the Jacobian matrix.

- Note that since integration is to be carried along the edge, we have used ξ.
- For adjacent elements


13
12
11
10
9
2
7
8
6
Pt
Pn
4
5
1
1
2
3

- We distinguish between two cases:

  Plane Stress which is the simplest

$$\varepsilon_{xx}^0 = \alpha\Delta T \quad (127)$$
$$\varepsilon_{yy}^0 = \alpha\Delta T \quad (128)$$
$$\gamma_{xy}^0 = 0 \quad (129)$$

  Plane Strain we have

$$\varepsilon_{xx}^0 = -\frac{\nu\sigma_{zz}^0}{E} + \alpha\Delta T \quad (130)$$
$$\varepsilon_{yy}^0 = -\frac{\nu\sigma_{zz}^0}{E} + \alpha\Delta T \quad (131)$$
$$\gamma_{xy}^0 = 0 \quad (132)$$
$$\varepsilon_{zz}^0 = \frac{\sigma_{zz}^0}{E} + \alpha\Delta T = 0 \quad (133)$$

Using the last expression to eliminate σ^0_{zz}, we obtain

$$\begin{aligned}
\varepsilon^0_{xx} &= (1+\nu)\alpha\Delta T && (134)\\
\varepsilon^0_{yy} &= (1+\nu)\alpha\Delta T && (135)\\
\gamma^0_{xy} &= 0 && (136)\\
\sigma^0_{zz} &= -E\alpha\Delta T && (137)
\end{aligned}$$

- Those expressions are then substituted into Eq. **??**

$$\mathrm{f}_{0_e} = \int_{\Omega_e} \mathrm{B}^T \mathrm{D} \boldsymbol{\epsilon}_0 d\Omega - \int_{\Omega_t} \mathrm{B}^T \boldsymbol{\sigma}_0 d\Omega \qquad (138)$$

and integrated numerically.

- The computer implementation of a numerically integrated isoparametric element is summarized as follows.

  But first, it is assumed that this operation is to be performed in a function called `stiff` and it takes as input arguments `elcod, young, poiss, type, ndime, ndofn, ngaus`. In turn it will compute the stiffness matrix `KELEM` of element `ielem`.

  1. Retrieve element geometry and material properties for the current element
  2. Zero the stiffness matrix
  3. Call function `dmat` to set the constitutive matrix D^e of the element
  4. Enter (nested) loop covering all integration points
     1. Look up the sampling position of the current point (ξ_p, η_q) (`s, t`) and their weights (`weigp`)
     2. Call shape function routine `sfr` given ξ_p, η_q which will return the shape function N_i^e (`sfr`) and the derivatives $\frac{\partial N_i^e}{\partial \xi}$ and $\frac{\partial N_i^e}{\partial \eta}$ (`deriv`) at the point ξ_p, η_q
     3. Call another subroutine (`jacob`), given N_i^e, $\frac{\partial N_i^e}{\partial \xi}$ and $\frac{\partial N_i^e}{\partial \eta}$ at point ξ_p, η_q will return cartesian shape function derivatives $\frac{\partial N_i^e}{\partial x}$ and $\frac{\partial N_i^e}{\partial y}$ (`cartd`), the Jacobian matrix J^e (`jacm`), its inverse J^{e-1} (`jaci`), its determinant $\det \mathrm{J}^e$ (`djac`) and the x and y coordinates all at the point ξ_p, η_q

4. Call strain matrix (`bmatps`) routine, given N_i^e, $\frac{\partial N_i^e}{\partial x}$, $\frac{\partial N_i^e}{\partial y}$, at ξ_p, η_q will return the strain matrix B_i^e (`bmat`)
5. Call a routine (`dbmat`) to evaluate $\mathrm{D}^e\mathrm{B}_i^e$ (`dbmat`)
6. Evaluate ${\mathrm{B}_i^e}^T\mathrm{D}^e\mathrm{B}_j^e\mathrm{detJ}^e\times$ integration weights and assemble them into the element stiffness matrix K_{ij}^e
7. Assemble $\mathrm{D}^e\mathrm{B}^e$ (`smat` into a stress array for later evaluation of stresses from the nodal displacements.

5. Write Stiffness matrix

Suggested list of variables:

`idime, ndime`	Index, Number of dimensions (2 for 2D)
`idofn,ndofn`	Index, Number of degree of freedom per node
`ielem,nelem`	Index, number of elements
`igaus,jgaus,ngaus`	Index, Index, Number of Gauss rule adopted
`inode, nnode`	Index, number of nodes per element
`kgasp,ngasp`	Kounter, number of Gauss points used
`type`	1 for plane stress; 2 for plane strain
`poiss`	Poisson's ratio
`young`	Young's modulus
`elcod(ndime,nnode)`	Local array of nodal cartesian coordinates of the element ation $\begin{bmatrix} x(\xi_1,\eta_1) & \cdots & x(\xi_8,\eta_8) \\ y(\xi_1,\eta_1) & \cdots & y(\xi_8,\eta_8) \end{bmatrix}$
`s`	ξ coordinate of sampling point
`t`	η coordinate of sampling point
`gpcod(ndime,ngasp)`	Local array of cartesian coordinates of the Gauss points fo consideration $\begin{bmatrix} x(\xi_{G_1},\eta_{G_1} & \cdots & x(\xi_{G_5},\eta_{G_5}) & \cdots \\ y(\xi_{G_1},\eta_{G_1} & \cdots & y(\xi_{G_5},\eta_{G_5}) & \cdots \end{bmatrix}$
`posgp(mgaus)`	ξ coordinates of Gauss point
`weigp(mgaus)`	Weight factor for Gauss point

`shape(nnode)`	Shape function associated with each node of curre $\begin{bmatrix} N_1(\xi_p,\eta_p) \\ \vdots N_8(\xi_p,\eta_p) \end{bmatrix}$
`deriv(ndime,nnode)`	Shape function derivative at sampling point $(\xi_p,\eta$ $\begin{bmatrix} \frac{\partial N_1}{\partial \xi}(\xi_p,\eta_p) & \cdots & \frac{\partial N_8}{\partial \xi}(\xi_p,\eta_p) \\ \frac{\partial N_1}{\partial \eta}(\xi_p,\eta_p) & \cdots & \frac{\partial N_8}{\partial \eta}(\xi_p,\eta_p) \end{bmatrix}$
`cartd(ndime,nnode)`	Cartesian shape function derivatives associated the current element sampled at any point $(\xi_p,\eta$ $\begin{bmatrix} \frac{\partial N_1}{\partial x}(\xi_p,\eta_p) & \cdots & \frac{\partial N_8}{\partial x}(\xi_p,\eta_p) \\ \frac{\partial N_1}{\partial y}(\xi_p,\eta_p) & \cdots & \frac{\partial N_8}{\partial y}(\xi_p,\eta_p) \end{bmatrix}$
`djacb`	Determinant of the Jacobian matrix sampled at any point (
`jacm(ndime,ndime)`	Jacobian matrix at sampling point
`jaci(ndime,ndime)`	Inverse of Jacobian matrix at sampling point
`bmat(nstre,nevab)`	Element strain matrix at any point within the element B where $\mathrm{B}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$
`dbmat(istre,ievab)`	stores DB

```
function KELEM = stiff(ngaus,posgp,weigp,type,nelem,lnods,coord,nnode)
%------------------------------------------------------------------------
% The purpose of this function is calculate element stiffness matricies for
% bilinear and biquadratic isoparemtric elements using Gaussian integration.
% Functions called by this function are: dmat,sfr,jacob,bmatps
%------------------------------------------------------------------------
%                                  VARIABLES
%------------------------------------------------------------------------
% nelem          Global variable NELEM
% nnode          Global variable NNODES
% posgp          Global variable POSGP
% weigp          Global variable WEIGP
% ngaus          Global variable NGAUS
% type           Global variable TYPE
% lnods          Global variable LNODS
% coord          Global variable COORD
%
% stifsize       Number of columns in the element stiffness matrix
% nrowcount      Position indicator for element stiffness matrix
% elmt           Current element for formation of stiffness matrix
% young          Modulus of elasticity for current element
% poiss          Poisson's ration for current element
% D              Constitutive matix
% elcod          Matrix of element coordinates
% row            Counter
% kelem          Element stiffness matrix
% KELEM          Element stiffness matricies for all elements returned by function
% s              Current integration position
% t              Current integration position
% shape          Shape function at current point
% deriv          Derivative of shape function at current point
% cartd          Cartesian shape function derivatives
```

```
% jacm          Jacobian matrix
% jaci          Jacobian matrix inverse
% djac          Determinant of Jacobian matrix
% xy            x and y coordinates at the current point in the element
% bmat          Strain matrix [B]
% dbmat         Strain matrix * constitutive matrix [B]*[D]
%-----------------------------------------------------------------------
tic
fprintf('Calculating ELEMENT STIFFNESS matricies\n')
stifsize = 2*nnode;
nrowcount = stifsize-1;
for ielem = 1:nelem
    %--------------------------------------
    % Extract material constants from lnods
    %--------------------------------------
    elmt  = lnods(ielem,1);
    young = lnods(ielem,2);
    poiss = lnods(ielem,3);
    %--------------------------------------
    % Extract element coordinates
    %
    % elcod = [ X1 X2 X3 . . . Xn]
    %         [ Y1 Y2 Y3 . . . Yn]
    %--------------------------------------
    elcod = zeros(2,nnode);
    for inode = 1:nnode
        row = find(coord(:,1:1)==lnods(ielem,inode+3));
        elcod(:,inode:inode) = coord(row:row,2:3)';
    end
    %--------------------------------------
    % Constitutive matrix
    %--------------------------------------
```

```
    D = dmat(young,poiss,type);
    %----------------------------------------
    % Element stiffness matrix - element ielem
    %----------------------------------------
    kelem = zeros(stifsize);
    for igaus = 1:ngaus
        for jgaus = 1:ngaus
            s = posgp(igaus);
            t = posgp(jgaus);
            W = weigp(igaus)*weigp(jgaus);
            [shape,deriv] = sfr(s,t,nnode);
            [cartd,jacm,jaci,djac,xy] = jacob(shape,deriv,elcod);
            [bmat,dbmat] = bmatps(shape,cartd,D);
            kelem = kelem + bmat'*dbmat*djac;
        end
    end
    %------------------------------------------------------------------
    % Store element stiffness matricies in as a stack in a single matrix :
    %           kelem(1)
    % KELEM  =      :
    %           kelem(nelem)
    %------------------------------------------------------------------
    startrow = stifsize*ielem - nrowcount;
    endrow   = ielem*stifsize;
    KELEM(startrow:endrow,:) = kelem;
end
t = toc;
fprintf(1,'Elapsed time for this operation =%3.4fsec\n\n',t)
```

```
function D = dmat(young,poiss,type)
%------------------------------------------------------------------------
% This function calculates the constitutive matrix for an element
%------------------------------------------------------------------------
%                                   VARIABLES
%------------------------------------------------------------------------
% young     Young's modulus/Modulus of elasticity
% poiss     Poisson's ratio
% E         Modulus of elasticity
% type      type of problem - plain stress = 1, plain strain = 2
% D         Constitutive matrix returned by function
%------------------------------------------------------------------------
tic
%fprintf('Calculating element constitutive matrix\n')
E = young;
v = poiss;
%-------------------------
% Plain stress
%-------------------------
if type == 1.0
    D = (E/(1-v^2))*[   1      v       0;
                        v      1       0;
                        0      0    (1-v)/2];
%-------------------------
% Plain strain
%-------------------------
else
    D = E/((1+v)*(1-2*v))*[ 1-v      v       0      ;
                            v      1-v       0      ;
                            0        0    (1-2*v)/2];
end
t = toc;
```

```
%fprintf(1,'Elapsed time for this operation =%3.4fsec\n\n',t)
```

```
function [shape,deriv] = sfr(s,t,nnode)
%-------------------------------------------------------------------------
% This function calculates the shape function and derivative for the current node
%-------------------------------------------------------------------------
%                                 VARIABLES
%-------------------------------------------------------------------------
% shape     Shape function returned by function
% deriv     Derivative of shape function returned by function
% nnode     Number of nodes per element
% s         Natural coordinate (xi)  of sampling point - horizontal
% t         Natural coordinate (eta) of sampling point - vertical
%-------------------------------------------------------------------------
tic
%fprintf('Calculating shape functions and derivatives\n')
%------------------
% Q9 Element
%------------------
if nnode == 9
    N9 = (1-s^2)*(1-t^2);
    N8 = .5*(1-s)*(1-t^2)  -.5*N9;
    N7 = .5*(1-s^2)*(1+t)  -.5*N9;
    N6 = .5*(1+s)*(1-t^2)  -.5*N9;
    N5 = .5*(1-s^2)*(1-t)  -.5*N9;
    N4 = .25*(1-s)*(1+t)   -.5*N7 - .5*N8 - .25*N9;
    N3 = .25*(1+s)*(1+t)   -.5*N6 - .5*N7 - .25*N9;
    N2 = .25*(1+s)*(1-t)   -.5*N5 - .5*N6 - .25*N9;
    N1 = .25*(1-s)*(1-t)   -.5*N5 - .5*N8 - .25*N9;
    shape = [N1 N2 N3 N4 N5 N6 N7 N8 N9]';
    dN9ds = -2*s*(1-t^2);
    dN9dt = -2*t*(1-s^2);
    dN8ds = -.5*(1-t^2)  -.5*dN9ds;
    dN8dt = -t*(1-s)     -.5*dN9dt;
```

```
    dN7ds = -s*(1+t)      -.5*dN9ds;
    dN7dt = .5*(1-s^2)    -.5*dN9dt;
    dN6ds = .5*(1-t^2)    -.5*dN9ds;
    dN6dt = -t*(1+s)      -.5*dN9dt;
    dN5ds = -s*(1-t)      -.5*dN9ds;
    dN5dt = -.5*(1-s^2)   -.5*dN9dt;
    dN4ds = -.25*(1+t)    -.5*dN7ds - .5*dN8ds - .25*dN9ds;
    dN4dt = .25*(1-s)     -.5*dN7dt - .5*dN8dt - .25*dN9dt;
    dN3ds = .25*(1+t)     -.5*dN6ds - .5*dN7ds - .25*dN9ds;
    dN3dt = .25*(1+s)     -.5*dN6dt - .5*dN7dt - .25*dN9dt;
    dN2ds = .25*(1-t)     -.5*dN5ds - .5*dN6ds - .25*dN9ds;
    dN2dt = -.25*(1+s)    -.5*dN5dt - .5*dN6dt - .25*dN9dt;
    dN1ds = -.25*(1-t)    -.5*dN5ds - .5*dN8ds - .25*dN9ds;
    dN1dt = -.25*(1-s)    -.5*dN5dt - .5*dN8dt - .25*dN9dt;
    deriv = [dN1ds dN2ds dN3ds dN4ds dN5ds dN6ds dN7ds dN8ds dN9ds;
             dN1dt dN2dt dN3dt dN4dt dN5dt dN6dt dN7dt dN8dt dN9dt];
%-------------------
% Q8 Element
%-------------------
elseif nnode == 8
    N8 = .5*(1-s)*(1-t^2);
    N7 = .5*(1-s^2)*(1+t);
    N6 = .5*(1+s)*(1-t^2);
    N5 = .5*(1-s^2)*(1-t);
    N4 = .25*(1-s)*(1+t)   -.5*N7 - .5*N8;
    N3 = .25*(1+s)*(1+t)   -.5*N6 - .5*N7;
    N2 = .25*(1+s)*(1-t)   -.5*N5 - .5*N6;
    N1 = .25*(1-s)*(1-t)   -.5*N5 - .5*N8;
    shape = [N1 N2 N3 N4 N5 N6 N7 N8]';
    dN8ds = -.5*(1-t^2);
    dN8dt = -t*(1-s);
    dN7ds = -s*(1+t);
```

```
    dN7dt = .5*(1-s^2);
    dN6ds = .5*(1-t^2);
    dN6dt = -t*(1+s);
    dN5ds = -s*(1-t);
    dN5dt = -.5*(1-s^2);
    dN4ds = -.25*(1+t)  -.5*dN7ds - .5*dN8ds;
    dN4dt = .25*(1-s)   -.5*dN7dt - .5*dN8dt;
    dN3ds = .25*(1+t)   -.5*dN6ds - .5*dN7ds;
    dN3dt = .25*(1+s)   -.5*dN6dt - .5*dN7dt;
    dN2ds = .25*(1-t)   -.5*dN5ds - .5*dN6ds;
    dN2dt = -.25*(1+s)  -.5*dN5dt - .5*dN6dt;
    dN1ds = -.25*(1-t)  -.5*dN5ds - .5*dN8ds;
    dN1dt = -.25*(1-s)  -.5*dN5dt - .5*dN8dt;
    deriv = [dN1ds dN2ds dN3ds dN4ds dN5ds dN6ds dN7ds dN8ds;
             dN1dt dN2dt dN3dt dN4dt dN5dt dN6dt dN7dt dN8dt];
%------------------
% Q4 Element
%------------------
else
    N4 = .25*(1-s)*(1+t);
    N3 = .25*(1+s)*(1+t);
    N2 = .25*(1+s)*(1-t);
    N1 = .25*(1-s)*(1-t);
    shape = [N1 N2 N3 N4]';
    dN4ds = -.25*(1+t);
    dN4dt = .25*(1-s);
    dN3ds = .25*(1+t);
    dN3dt = .25*(1+s);
    dN2ds = .25*(1-t);
    dN2dt = -.25*(1+s);
    dN1ds = -.25*(1-t);
    dN1dt = -.25*(1-s);
```

```
    deriv = [dN1ds dN2ds dN3ds dN4ds;
             dN1dt dN2dt dN3dt dN4dt];
end
t = toc;
%fprintf(1,'Elapsed time for this operation =%3.4fsec\n\n',t)
```

```
function [cartd,jacm,jaci,djac,xy] = jacob(shape,deriv,elcod)
%--------------------------------------------------------------------------
% This function calculates the cartesian shape function derivatives
% the Jacobian matrix, Jacobian inverse and Jacobian determinant
%--------------------------------------------------------------------------
%                                  VARIABLES
%--------------------------------------------------------------------------
% shape     Shape function at current point
% deriv     Derivative of shape function at current point
% cartd     Cartesian shape function derivatives returned by function
% jacm      Jacobian matrix returned by function
% jaci      Jacobian matrix inverse returned by function
% djac      Determinant of Jacobian matrix returned by function
% xy        x and y coordinates at the current point in the element
%--------------------------------------------------------------------------
tic
%fprintf('Calculating Jacobian matrix\n')
%---------------------------------------------------------
% The cartesian shape function derivatives {cartd} are given by:
%
%            {dN/dx}      -1  {dN/ds}
% {cartd} = {       } = [J]   {      }
%            {dN/dy}          {dN/dt}
% Start by calculating Jacobian [J] = jacm:
%                              T          T
%        [dX/ds  dY/ds]      [{dN/ds}*{x} {dN/ds}*{y}]
% [J] = [              ]  =  [      T           T    ]
%        [dX/dt  dY/dt]      [{dN/dt}*{x} {dN/dt}*{y}]
%---------------------------------------------------------
jacm = deriv*elcod';
jaci = inv(jacm);
djac = det(jacm);
```

```
cartd = jaci*deriv;
xy = elcod*shape;
t = toc;
%fprintf(1,'Elapsed time for this operation =%3.4fsec\n\n',t)
```

```
function [bmat,dbmat] = bmatps(shape,cartd,D)
%-------------------------------------------------------------------------
% This function calculates the strain matrix B
%-------------------------------------------------------------------------
%                                    VARIABLES
%-------------------------------------------------------------------------
% shape      Shape function at current point
% cartd      Cartesian shape function derivatives
% bmat       Strain matrix returned by function
% dbmat      Strain matrix * constitutive matrix D
%-------------------------------------------------------------------------
tic
%fprintf('Calculating strain matrix [B]\n')
numcols = 2*length(cartd);
bmat = zeros(3,numcols);
cartdcol = 0;
for ibmatcol = 1:2:numcols
    cartdcol = cartdcol+1;
    bmat(:,ibmatcol:ibmatcol+1) = [cartd(1,cartdcol)          0          ;
                                        0               cartd(2,cartdcol);
                                   cartd(2,cartdcol)    cartd(1,cartdcol)];
end
dbmat = D*bmat;
t = toc;
%fprintf(1,'Elapsed time for this operation =%3.4fsec\n\n',t)
```

```
X=-1:1/20:1;
Y=X;
YT=Y';
XT=X';
N9=(1-YT.*YT)*(1-X.*X);
N8=0.5*(1-YT.*YT)*(1-X);
N7=0.5*(1-XT.*XT)*(1+Y);
N6=0.5*(1-YT.*YT)*(1+X);
N5=0.5*(1-XT.*XT)*(1-Y);
N4=0.25*(1-XT)*(1+Y)-0.5*(N7+N8);
N3=0.25*(1+XT)*(1+Y)-0.5*(N6+N7);
N2=0.25*(1+XT)*(1-Y)-0.5*(N5+N6);
N1=0.25*(1-XT)*(1-Y)-0.5*(N8+N5);
meshc(X,Y,N1)
print -deps2 shap8-1.eps
c=contour(X,Y,N1);
clabel(c)
print -deps2 shap8-1-c.eps
meshc(X,Y,N8)
print -deps2 shap8-8.eps
c=contour(X,Y,N8);
clabel(c)
print -deps2 shap8-8-c.eps
N1=N1-0.25*N9;
meshc(X,Y,N1)
print -deps2 shap9-1.eps
c=contour(X,Y,N1);
clabel(c)
print -deps2 shap9-1-c.eps
N8=N8-0.5*N9;
meshc(X,Y,N8)
print -deps2 shap9-8.eps
```

```
c=contour(X,Y,N8);
clabel(c)
print -deps2 shap9-8-c.eps
meshc(X,Y,N9)
print -deps2 shap9-9.eps
c=contour(X,Y,N9);
clabel(c)
print -deps2 shap9-9-c.eps
```

Intermediary Structural Analysis

Introduction to Nonlinear Analysis

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Fall 2019

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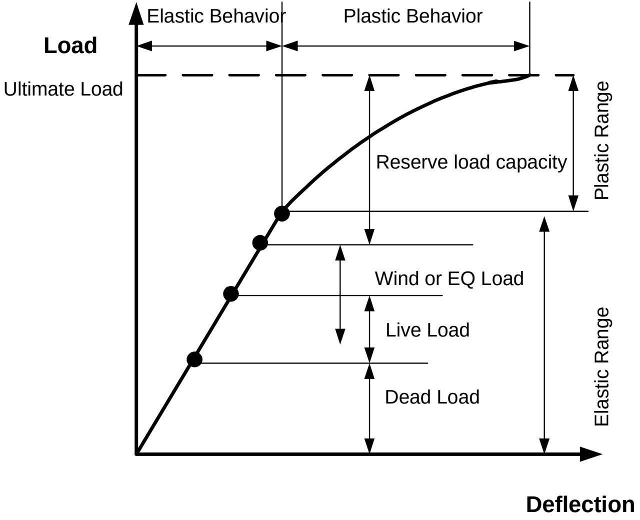
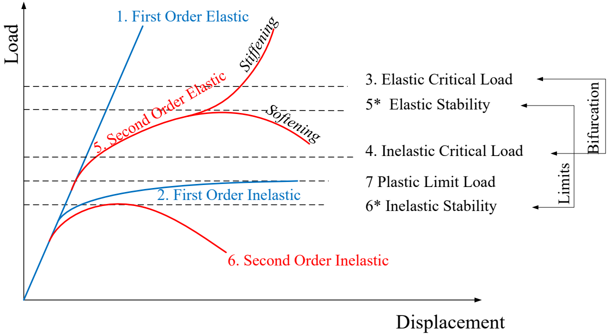

Elastic Behavior
Plastic Behavior
Load
Ultimate Load
Reserve load capacity
Plastic Range
Wind or EQ Load
Live Load
Dead Load
Elastic Range
Deflection

- Constitutive model (non-linear stress strain curve of steel, concrete), $\mathbf{k} = \int_\Omega \mathbf{B}^T\mathbf{D}\mathbf{B}d\Omega$
- large strains:

$$\varepsilon_{xx} = \underbrace{u_{,x}}_{\text{First Order}} + \underbrace{\frac{1}{2}(u_{,x}^2 + v_{,x}^2 + w_{,x}^2)}_{\text{Second Order}}$$

$$\mathbf{k} = \int_\Omega \mathbf{B}^T\mathbf{D}\mathbf{B}d\Omega$$



		Constitutive Equations			
		Undeformed Shape		Deformed Shape	
		Elastic (Linear)	Inelastic (Non Linear)	Elastic (Linear)	Inelastic (Non Linear)
Kinematic Eq.	1^{st} Order (Linear)	1 (C:L-K:L)	2 (C:NL-K:L)	Critical Load: 3 Elastic	Critical Load: 4 Inelastic
	2^{nd} Order (Non Linear)	5 (C:L-K:NL)	6 (C:NL-K:NL)	-	-

First Order Elastic excludes any nonlinearities. If the equilibrium equation is written in terms of

- **1 (C:L-K:L); Undeformed Shape** This is the most common case, linear elastic. It is usually acceptable for service loads. For time dependent cases, we must consider visco-elastic models.
- **3 Bifurcation; Deformed shape** (or ‘zero order”) an eigenvalue analysis which would lead to the Elastic Critical Load. Note that we do not have a corresponding load-displacement curve, but rather “buckling modes”.

First Order Inelastic Accounts for material non-linearity. In such an analysis, the inelastic region (plastic zone) develops gradually, and it will provide a good estimate of the elasto-plastic response (note that instability is not addressed). We consider

- Non-linear Elasticity: reversible non-linear stress-strain (upon unloading, the strain goes back to zero).
- Plasticity, non reversible non-linear stress-strain.
- Damage

If the equilibrium equation is written in terms of

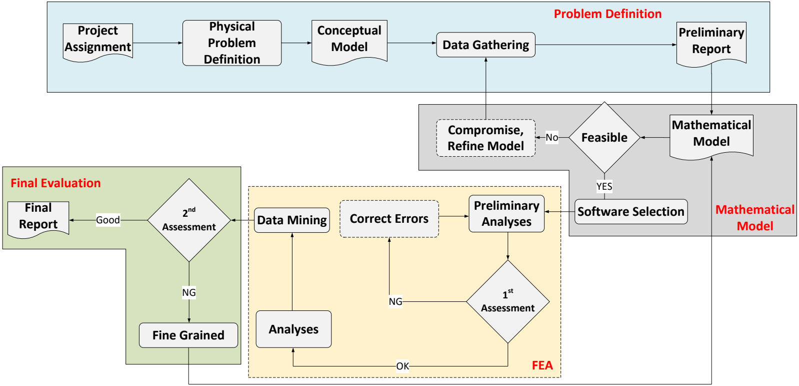
2 (C:NL-K:L); Undeformed Shape Second most common form of analysis, typically conducted for ultimate/unusual loads.

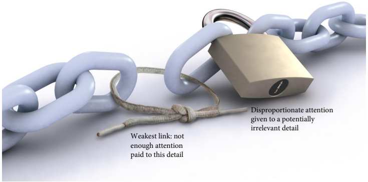
4 Bifurcation; Deformed shape an eigenvalue analysis which would lead to the Inelastic Critical Load. Note that we do not have a corresponding load-displacement curve, but rather “buckling modes”. This inelastic critical load will be smaller than the elastic one.

For time dependent cases, we consider visco-plasticity, or fatigue, or continuous damage models.

5 (C:L-K:NL); Second-order elastic accounts for the effects of finite deformation and displacements, equilibrium equations are written in terms of the geometry of the deformed shape (Eulerian), does not account for material non-linearities, may be able to detect bifurcation and or increased stiffness (when a member is subjected to a tensile axial load). Best for the analysis of cables, nets, catenary structures.

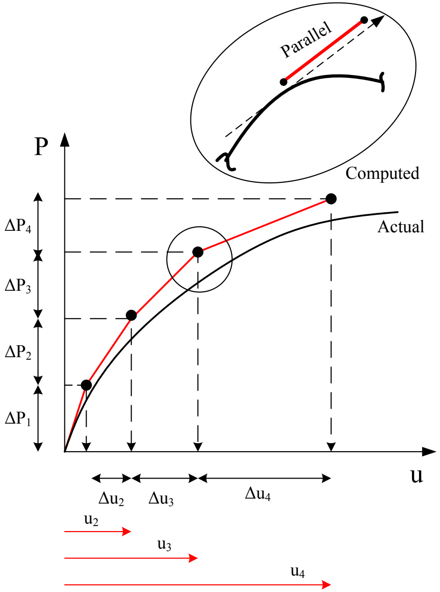
6 (C:NL-K:NL); Second-order inelastic equations of equilibrium written in terms of the geometry of the deformed shape, can account for both geometric and material nonlinearities. Most suitable to determine failure or ultimate loads. By far the most complex form of analysis, used in Metal Forming simulation, fragmentation of structures (missile impact).


Problem Definition
Project Assignment
Physical Problem Definition
Conceptual Model
Data Gathering
Preliminary Report
Compromise, Refine Model
No
Feasible
Mathematical Model
YES
Software Selection
Mathematical Model
Final Evaluation
Final Report
Good
2nd Assessment
NG
Fine Grained
Data Mining
Correct Errors
Preliminary Analyses
NG
1st Assessment
Analyses
OK
FEA


Disproportionate attention given to a potentially irrelevant detail
Weakest link: not enough attention paid to this detail

- EC8 and PBE require the completion of
  - Nonlinear Static Procedure or Nonlinear Pushover (NPO)
  - Nonlinear Dynamic Procedure or Nonlinear Time History (NTH)

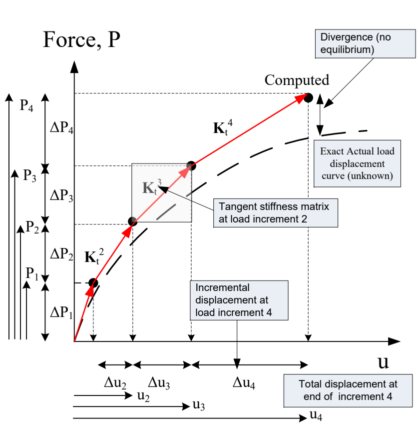
Two classes of solutions



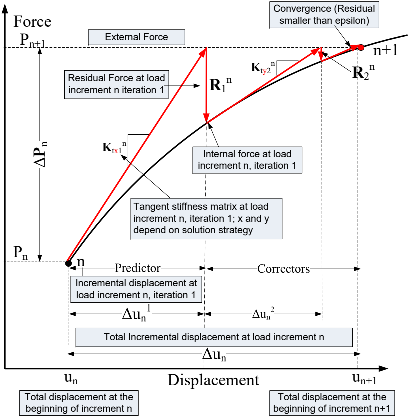
- **Event to Event:** No iterations, small increments, easy to implement, no check for convergence. Explicit method

$$
\begin{aligned}
\{\Delta\bar{\mathbf{u}}\}_i &= \mathbf{K}_i^{-1}\{\Delta\overline{\mathbf{P}}\}_i \\
u_i &= u_{i-1} + \Delta u_i
\end{aligned}
$$

- **Newton's Method**; Numerical aspects will first be introduced from a conceptual point of view first, connection to structural analysis will be made at the end. Implicit method



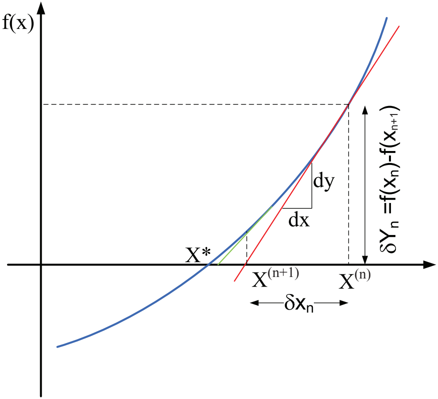
Force, P
Divergence (no equilibrium)
Computed
Exact Actual load displacement curve (unknown)
Tangent stiffness matrix at load increment 2
Incremental displacement at load increment 4
Total displacement at end of increment 4
P_4
P_3
P_2
P_1
ΔP_4
ΔP_3
ΔP_2
ΔP_1
K_t^4
K_t^3
K_t^2
u
Δu_2
Δu_3
Δu_4
u_2
u_3
u_4



Convergence (Residual smaller than epsilon)
Force
External Force
P_{n+1}
n+1
Residual Force at load increment n iteration 1
R_1^n
K_{ty2}^n
R_2^n
K_{tx1}^n
Internal force at load increment n, iteration 1
ΔP_n
Tangent stiffness matrix at load increment n, iteration 1; x and y depend on solution strategy
P_n
n
Predictor
Correctors
Incremental displacement at load increment n, iteration 1
Δu_n^1
Δu_n^2
Total Incremental displacement at load increment n
Δu_n
u_n
Displacement
u_{n+1}
Total displacement at the beginning of increment n
Total displacement at the beginning of increment n+1

- Linear problems: unique solution; Nonlinear problems: can not ensure the existence of a solution, nor ensure the uniqueness of one.
- At best we can say that an approximate numerical solution of the problem is given, or that an approximation does not exist (typically this implies local or global failure).
- Most widely used class of numerical solution: "Newton Methods", or "Quasi Newton". Other methods may include the bisection method (only linearly convergent).
- Essence of the method which seeks to solve $f(x) = 0$, is to linearize the equation about the current approximation x_n and solve for the resulting linear equation for the next approximation x_{n+1}

- If we set $f(x) = 0 \Rightarrow x \simeq \overline{x} - \frac{f(\overline{x})}{f'(\overline{x})}$
- This is an approximate solution, at $\overline{x}$, which presumes that we also have $f'(x)$.
- In an iterative procedure, this equation can be rewritten as



$$
\begin{aligned}
\frac{dy}{dx} &= f'(x_n) \\
\Rightarrow dx &= \frac{dy}{f'(x_n)} = \frac{\overbrace{f(x_{n+1})}^{0} - f(x_n)}{f'(x_n)} \\
x_{n+1} &\simeq x_n \underbrace{- \frac{f(x_n)}{f'(x_n)}}_{\delta x_n}
\end{aligned}
$$

- Convergence will be ensured when $|\delta x_n| \leq \varepsilon_\delta$ or $|f(x_{n+1})| \leq \varepsilon_f$

Solve $f(x) = Tan(x) - x = 0$

```
clear
xn = 4.3;
n = 0;
epsi=1e 4;
maxiter = 20;
disp("                ")
disp("     n            xn                          norm")
xn_m1=0.;
for i = 1:maxiter
    f_x=tan(xn) xn;df_dx=sec(xn)^2 1;
    xn = xn   f_x/df_dx;
    my_norm = abs(xn xn_m1);
    disp(sprintf( "%5i   %16.15e   %16.15e",i, xn,my_norm))
    if my_norm <epsi
        break
    end
    xn_m1=xn;
end
```

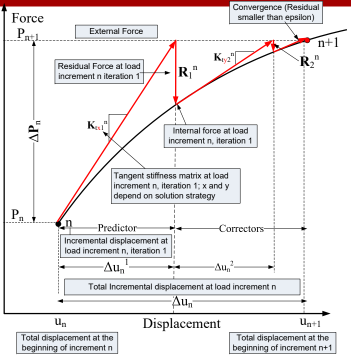
Note that this is a particularly sensitive problem, because `tan` x is discontinuous, a small change in the initial guess may yield to divergence of the solution.

- Given an initial $\mathbf{x}$, a required tolerance $\varepsilon > 0$
  Repeat
  1. Evaluate $g = f(\mathbf{x})$ and $H = J(\mathbf{x})$
  2. If $\| g \| \leq \varepsilon$, **return** $\mathbf{x}$
  3. $\mathbf{v} = \mathbf{x}_n - \mathbf{x}_{n-1} = \frac{f(\mathbf{x})}{J(\mathbf{x})}$
  4. Solve $H\mathbf{v} = -g$
  5. $\mathbf{x} := \mathbf{x} + \mathbf{v}$

  until maximum number of iterations is exceeded
- Each iteration requires the evaluation of $(\mathbf{x})$ (n scalar functions evaluation in terms of $\mathbf{x}$) and $J(\mathbf{x})$ (n^2 derivatives).

$$\text{Solve } f(\mathbf{x}) = \left\{ \begin{array}{c} x_1^2 + x_2^2 + x_3^2 - 9 \\ x_3 - x_2 \sin(x_1) \\ 3x_2 + 4x_3 \end{array} \right\} \Rightarrow J(\mathbf{x}) = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \\ -x_2\cos(x_1) & -\sin(x_1) & 1 \\ 0 & 3 & 4 \end{bmatrix}$$

```
f = @(x) [x(1)^2 + x(2)^2 + x(3)^2   9
          x(3) x(2)*sin(x(1))
          3*x(2)+4*x(3)];
% The Jacobian matrix:
J = @(x) [2*x(1)              2*x(2)     2*x(3)
           x(2)*cos(x(1))    sin(x(1))   1
           0                    3        4];
% initial guess:
x = [ 1;  2; 1];
maxiter = 10;
tol = 1e 12;
disp("  ")
disp("iteration     x(1)           x(2)            x(3)            norm(delta)")
for n=1:maxiter
   delta =  J(x) \ f(x);
   x = x + delta;
   disp(sprintf("%5i     %10.5e    %10.5e    %10.5e       %8.3e", ...
                n,x(1),x(2),x(3),norm(delta,inf)));
    if norm(delta,inf) < tol
      break
      end
    end
if n==maxiter
   disp("*** Warning: may not have converged    tolerance not satisfied")
end
end
```



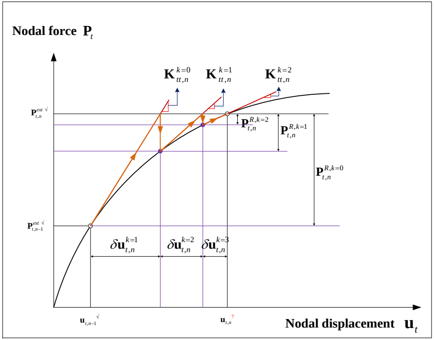
- Objective go from n to $n+1$.
- Jacobian corresponds to the tangent stiffness matrix of the structure which in turn depends on the tangent of the constitutive matrix ($\mathbf{D}_T$).

- So far: $\mathbf{f}(x) = 0$, in structural analysis $\mathbf{P}^R_{t,n} = \mathbf{P}^{ext}_{t,n} - \mathbf{P}^{int}_{t,n} = 0$, superscript R refers to the residual, and both $\mathbf{P}^{ext}_{t,n}$ and $\mathbf{P}^{int}_{t,n}$ are determined from the principle of virtual displacement. Internal nodal force vector $\mathbf{P}^{int}_{t,n}$ is a function of nodal displacements $\mathbf{u}_{t,n}$, thus we have a nonlinear problem. (Recall $\mathbf{P}^{int} = \int \mathbf{B}^T \sigma d\Omega$ or $\mathbf{K}\boldsymbol{\Delta}$)
- Within each iteration we determine the residual nodal force vector, and this would yield an incremental nodal displacement vector. The iterations continue until the residual nodal force vector or the incremental nodal displacement vector, is sufficiently small.

- Newton's methods hinge on our ability to linearize (through a truncated Taylor series) the problem as follows $\mathbf{P}^{R,k}_{t,n} = \mathbf{P}^{ext}_{t,n} - \mathbf{P}^{int,k}_{t,n}$; $\delta\mathbf{u}^k_{t,n} = [\mathbf{K}^{k-1}_{tt,n}]^{-1} \cdot \mathbf{P}^{R,k}_{t,n}$; and

$\mathbf{u}_{t,n}^{k} = \mathbf{u}_{t,n}^{k-1} + \delta\mathbf{u}_{t,n}^{k}$ where, $\mathbf{u}_{t,n}^{k=0} = \mathbf{u}_{t,n-1}$ and $\mathbf{P}_{t,n}^{int,k=0} = \mathbf{P}_{t,n-1}^{int}$ and subscript n refers to the load increment, and subscript k to the iteration number within a load increment.

- Assume equilibrium to have been reached at increment n, we then apply an increment of external force $\Delta\mathbf{P}^{ext}$, and we seek to determine the corresponding incremental displacement $\Delta\mathbf{u}_{n+1}$.
- The internal forces and corresponding displacements will then be in (near) equilibrium.
- We distinguish between load increment, and iterations within an increment to reach equilibrium.
- At each iteration, we determine the residual $\mathbf{R}_i^{(n+1)}$ which corresponds to $\mathbf{P}_{ext} - \mathbf{P}_{int}$, and seek to minimize this residual. At each iteration, we update (in the Newton method) the tangent stiffness matrix which corresponds to the jacobian.
- At the heart of all of them, is the determination of the internal nodal force vector $\mathbf{P}_{t,n}^{int,k}$, and the tangent stiffness matrix $\mathbf{K}_{tt,n}^{k-1}$.

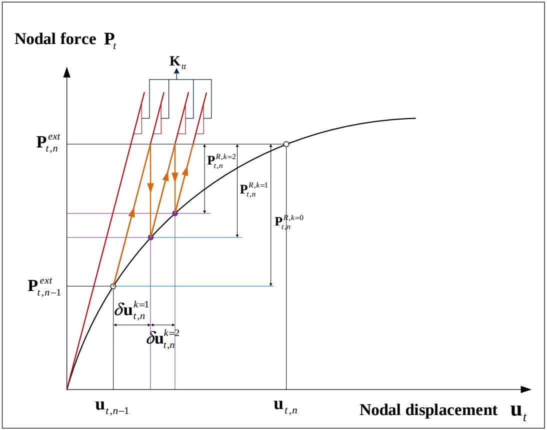


- Need to solve $\mathbf{f}(\mathbf{u}^*) = \mathbf{P}^{ext}_{t,n}(\mathbf{u}^*) - \mathbf{P}^{int}_{t,n}(\mathbf{u}^*) = 0$ and $\mathbf{f}(\cdot)$ is the function of internal state value $(\cdot)$. In the preceding equation it is often, but not exclusively, the vector of nodal displacement $\mathbf{u}$.

- Assuming that $\mathbf{u}^{k-1}_{t,n}$ is known, then a Taylor series expansion gives $\mathbf{f}(\mathbf{u}^*) = f(\mathbf{u}^{k-1}_{t,n}) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}|_{\mathbf{u}^{k-1}_{t,n}} \cdot (\mathbf{u}^* - \mathbf{u}^{k-1}_{t,n}) +$ High-order terms
  Substituting we obtain
  $\frac{\partial \mathbf{P}^{int}_t}{\partial \mathbf{u}}|_{\mathbf{u}^{k-1}_{t,n}} \cdot (\mathbf{u}^* - \mathbf{u}^{k-1}_{t,n}) +$ High-order terms $=$
  $\mathbf{P}^{ext}_{t,n} - \mathbf{P}^{int,k-1}_{t,n} = \mathbf{P}^{R,k}_{t,n}$ where we assume that the external nodal forces are displacement-independent.

- Since an incremental analysis is driven by external force steps (or time steps Δt), the initial conditions are given by $\mathbf{K}^{k=0}_{tt,n} = \mathbf{K}_{tt,n-1}$, $\mathbf{u}^{k=0}_{t,n} = \mathbf{u}_{t,n-1}$, $\mathbf{P}^{int,k=0}_{t,n} = \mathbf{P}^{int}_{t,n-1}$. Again, the iterations continue until an appropriate convergence criteria is satisfied.

- A characteristic of this iterative method is that an updated tangent stiffness matrix must be determined at each iteration, as such this method is often referred to as the full Newton-Raphson iterative method.



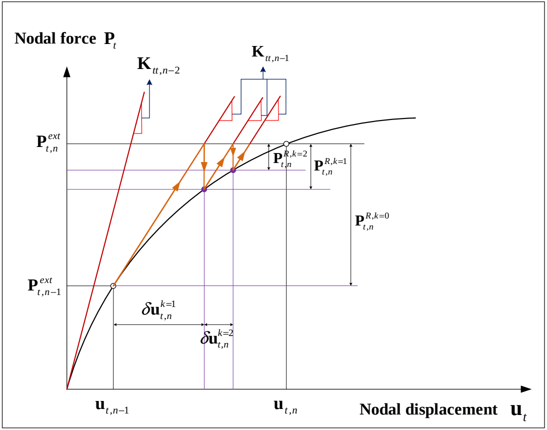
- In the Newton-Raphson iterative method most of the computational effort is associated with the factorization of the tangent stiffness matrix. For large systems, it is often more convenient to modify the approach by reducing the number of such factorizations albeit at the cost of increased number of iterations to reach proper convergence.
- Initial stiffness algorithm

$$\delta \mathbf{u}^{k}_{t,n} = [\mathbf{K}_{tt}]^{-1} \cdot \mathbf{P}^{R,k}_{t,n}$$

with the initial conditions defined by

$$\mathbf{u}^{k=0}_{t,n} = \mathbf{u}_{t,n-1}$$
$$\mathbf{P}^{int,k=0}_{t,n} = \mathbf{P}^{int}_{t,n-1}$$

In this method, only the initial $\mathbf{K}^{k=0}_{tt,n=0}$ needs to be factorized, thus avoiding the expense of recalculating and factorizing many times the tangent stiffness matrix. This initial stiffness iterative method corresponds to a linearization of the response about the initial configuration of the finite element system and will converge very slowly and may even diverge.



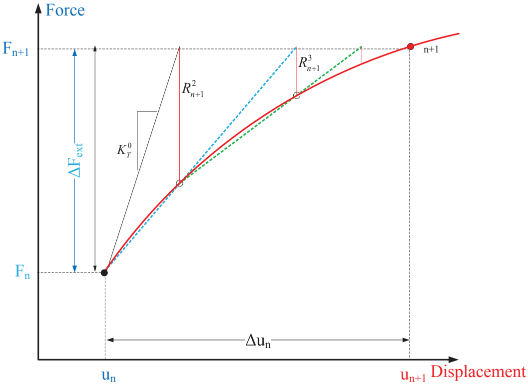
- Modified Newton-Raphson iterative method is an approach somewhat in between Newton-Raphson iterative method and the initial stiffness iterative method.

$$\delta\mathbf{u}^{k}_{t,n} = [\mathbf{K}_{tt,n-1}]^{-1} \cdot \mathbf{P}^{R,k}_{t,n}$$

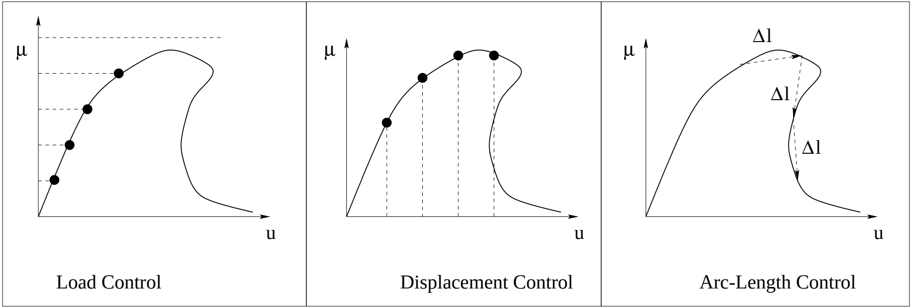
with the initial conditions

$$\mathbf{u}^{k=0}_{t,n} = \mathbf{u}_{t,n-1}$$
$$\mathbf{P}^{int,k=0}_{t,n} = \mathbf{P}^{int}_{t,n-1}$$

- The modified Newton-Raphson iterative method involves fewer stiffness decompositions than the Newton-Raphson iterative method. The choice of external force steps or time steps when the stiffness matrix should be updated depends on the degree of nonlinearity in the system response; i.e. the more nonlinear the response, the more often the updating should be performed.



we do not explicitly invert the Jacobian (or need to invert $\mathbf{K}_T$), but rather compute $\mathbf{K}_T$ through finite difference.



- Displacement control should be used when softening is present; arc length should be used if snap-back is anticipated.
- Arch-length method hinges on our ability to define an arc length in terms of both displacement and force, and then seek a multiplier.

- An appropriate termination criteria of the iteration should be adopted for any incremental solution strategy based on iterative methods. At the end of each iteration, the solution obtained should be checked to see whether it has converged within defined tolerances or whether the iteration may be diverging.
- If the convergence tolerances are too loose, inaccurate results are obtained, and if the tolerances are too tight, much computational effort is spent to obtain needless accuracy.

Some commonly used convergence criteria include:

Displacement criteria $\|\delta \mathbf{u}_n^k\| < \epsilon_D$ where ϵ_D is a displacement convergence tolerance and $\|\cdot\|$ is the Euclidian norm defined as the square root of the sum of the vector components squared.

Force criteria $\mathbf{P}_{t,n}^{R,k}$ and $\|\mathbf{P}_{t,n}^{R,k}\| < \epsilon_F$ where ϵ_F is a force convergence tolerance.

Energy criteria A difficulty with the force criterion is that the displacement solution does not introduce the termination criterion. As an illustration, consider an elasto-plastic truss with a very small strain-hardening modulus entering the plastic region. In this case, the residual force vector may be very small while the displacements may still be much in error. Hence, the convergence criteria may have to be used with very small values of ϵ_D and ϵ_F. Also, the expressions must be modified appropriately when quantities of different units are measured. In order to provide some indication of when both the displacements and the forces are near their equilibrium values, the energy criteria can be used

$$\left|\frac{1}{2} \cdot \mathbf{P}_{t,n}^{R,k} \cdot \delta \mathbf{u}_n^k\right| < \epsilon_E$$

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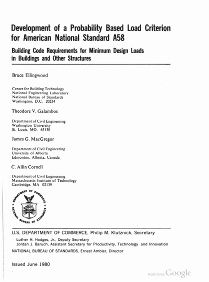
Development of a Probability Based Load Criterion for American National Standard A58

Building Code Requirements for Minimum Design Loads in Buildings and Other Structures

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U.S. DEPARTMENT OF COMMERCE, Philip M. Klutznick, Secretary
Luther H. Hodges, Jr., Deputy Secretary
Jordan J. Baruch, Assistant Secretary for Productivity, Technology and Innovation
NATIONAL BUREAU OF STANDARDS, Ernest Ambler, Director

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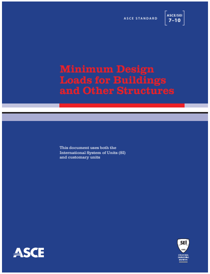
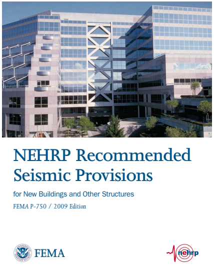
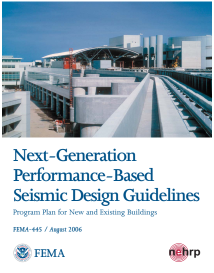






- Load and resistance are not deterministic quantities (as in the allowable stress design, ASD), but are random variables with their own probability distribution functions.
- There is a probability of failure.
- Load will be multiplied by a factor α, (ASCE-7-10) and we shall consider the ultimate resistance (reduced by Φ)
- We will assign α and Φ such that the probability of failure does not exceed a certain value.
- LRFD is generally expressed as

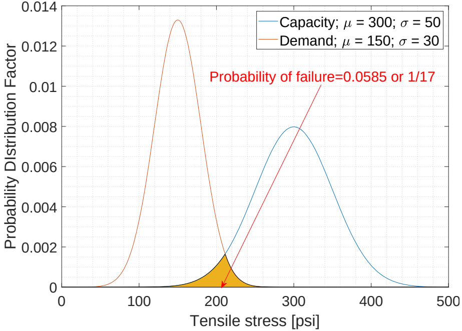
$$\Phi C_n \geq \Sigma \alpha_i D_i \tag{1}$$

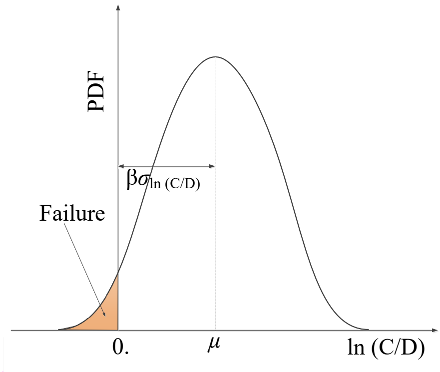
where C_n and D are the nominal capacity and demands (or nominal resistance and load).

- Limit state is generally determined from Plastic capacity without a nonlinear analysis.

- LRFD seeks to have a Reliability Index above ~ 3.5. The Reliability Index is a "universal" indicator on the adequacy of a structure, and can be used as a metric to 1) assess the health of a structure, and 2) compare different structures targeted for possible remediation.

- Capacity *C* and demand *D* are both random variables.





- We define the reliability index as the distance between mean performance value and the limit state normalized with respect to the standard deviation
- $X = \ln \frac{C}{D}$ Failure would occur for negative values of X

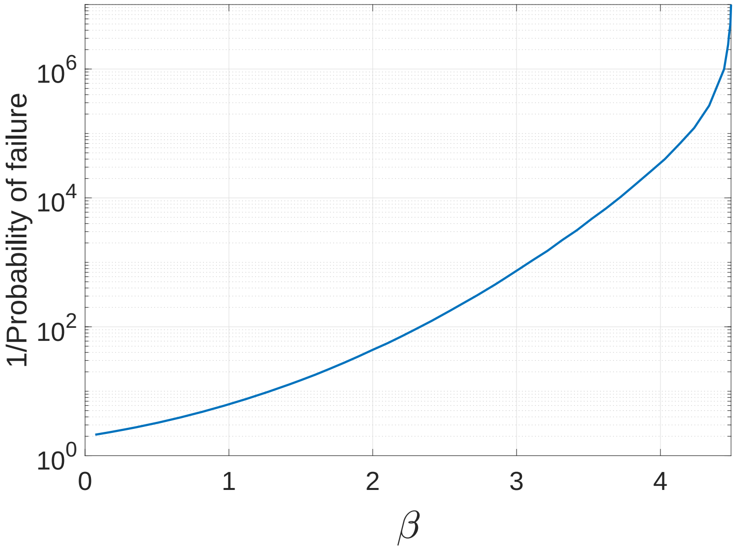
- Reliability Index $\beta = \frac{\ln \frac{\mu_C}{\mu_D}}{\sqrt{\sigma_C^2+\sigma_D^2}}$
- β is selected to reflect failure consequences

Type of Load/Member	β
AISC	
DL + LL; Members	3.0
DL + LL; Connections	4.5
DL + LL + WL; Members	3.5
DL + LL +EL; Members	1.75
ACI	
Ductile Failure	3-3.5
Sudden Failures	3.5-4

The probability of failure P_f is equal to the ratio of the shaded area to the total area under the curve and is given by $\Phi(-\beta)$ where Φ is the standard normal cumulative probability function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = \frac{1}{2}\left[1 + erf\left(\frac{x}{\sqrt{2}}\right)\right] \tag{2}$$

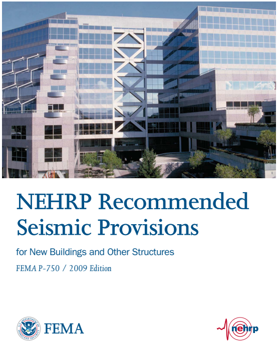
Target values for β


1/Probability of failure
10^6
10^4
10^2
10^0
0
1
2
3
4
β

1. Inconsistent: Linear analysis, but plastic design.
2. Ignores load redistribution near failure (though ACI implicitly accounts for some of it through reduction of negative moments).
3. Addresses only one level of hazard: failure of one structural component (and not the entire system), but how about quantification of damage due to more frequent events?

- PBE seeks first to identify discrete performance levels for the major structural components which significantly affect the building function and safety.
- ASCE 41 (ASCE 2007) (and other codes) generally provide guidance three performance levels
    - Immediate Occupancy where an essentially elastic behavior is sought by limiting structural damage (e.g., yielding of steel, significant cracking of concrete, and nonstructural damage.)
    - Life Safety Limit damage of structural and nonstructural components so as to minimize the risk of injury or casualties and to keep essential circulation routes accessible.
    - Collapse Prevention Ensure a small risk of partial or complete building collapse by limiting structural deformations and forces to the onset of significant strength and stiffness degradation.
- The engineer decides which performance levels
- Performance Based Engineering 1 Most recent code, FEMA 750-p developed by the Building Seismic Safety Council for FEMA. It builds on previous pre-Standards.

New Design	FEMA 310 (ASCE 1998)	ASCE/SEI 31 (2003)
Existing Buildings	FEMA 356 (ASCE 2000)	ASCE/SEI 41 (2006)



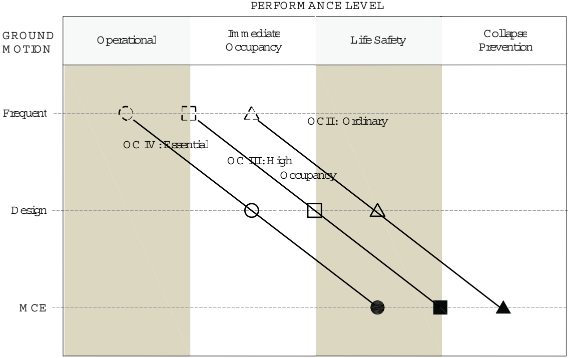
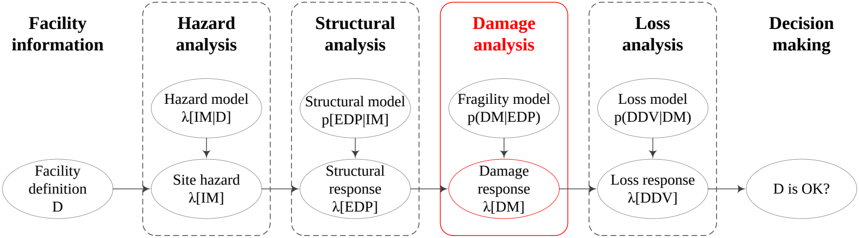


Figure C11.5-1 Expected performance as related to occupancy category (OC) and level of ground motion.

- First-generation procedures introduced the concept of performance in terms of discretely defined performance levels with names intended to connote the expected level of damage: Collapse, Collapse Prevention, Life Safety, Immediate Occupancy, and Operational Performance.
- They also introduced the concept of performance related to damage of both structural and nonstructural components. Performance Objectives were developed by linking one of these performance levels to a specific level of earthquake hazard.
- It is the state of the practice amongst high end companies. It is well established
- However:
    - Limit states are component-based not truly system-wide, (what if one component fails, does it trigger progressive collapse?)
    - treats only MCE event (2%/50years).
    - Limited treatment of uncertainty and probability.
    - Limited information for designing above code.

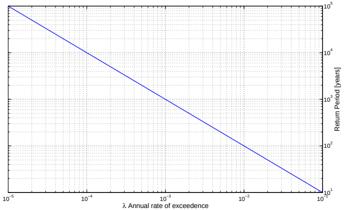
- Create new performance measures (e.g. repair costs, casualties, and time of occupancy interruption) that better relate to the decision-making needs of stakeholders.
- Create procedures for estimating probable repair costs, casualties, and time of occupancy interruption, for both new and existing buildings.

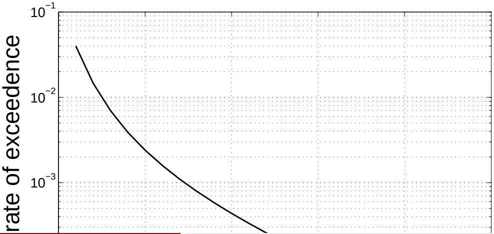


- We will need to identify specific engineering demand parameters (EDP) and appropriate acceptance criteria to quantitatively evaluate the performance levels.
- The demand parameters typically include peak (shear) forces and deformations, inter-story drifts, and floor accelerations in structural and nonstructural components.
- Performance is checked by comparing computed demands with acceptance criteria (capacity) for the desired performance level.
- Depending on the structural configuration, the results of nonlinear analyses can be sensitive to assumed input parameters and the types of models used.
- One must have clear expectations about those portions of the structure that are expected to undergo inelastic deformations and then use the analyses to
    1. Confirm the locations of inelastic deformations
    2. Characterize
        - Deformation demands of yielding elements
        - Force demands in non-yielding elements.
- Capacity design concepts can provide reliable performance.

- Capacity Design is indeed the approach where the engineer decides *a priori* which elements will yield (and thus need to be ductile) and those which will not yield (and will need to be stiff and with sufficient strength).
- Advantages
  - Safeguard against brittle failure of elements which can not be designed as ductile.
  - Limiting the location of the structure where expensive ductile detailing is required (they act as fuses).
  - Reliable energy dissipation by enforcing deformation modes where inelastic deformations are routed to ductile elements.
- Very similar to the structural design of a car.
- Example: strong column/weak beam.

- In the context of PBEE, one must first conduct a seismic hazard analysis (SHA) which includes location identification (with respect to a fault), geotechnical conditions (shear wave velocity), magnitude of previously recorded earthquakes, size of the rupture area, type of fault, crustal rock damping characteristics, rock properties.
- From the corresponding analysis one can determine annual rate of exceedance λ vs intensity measure (IM) a measure of the ground motion characteristic, typically the (peak or spectral) ground acceleration.



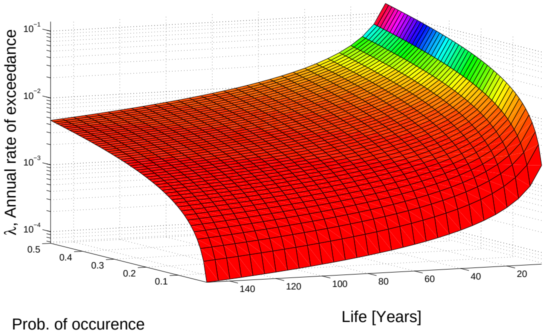


- The annual rate of exceedance of the ground motion amplitude, λ, (inverse of return period T_R) for Design Base Level (DBL) and Maximum Design Level (MDL) are determined from a Poisson probability model

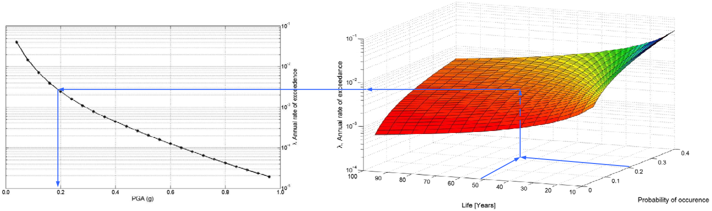
$$\lambda = -\frac{\text{Ln}\,(1 - P_E)}{t}$$

where P_E is the probability of occurrence of at least one event (i.e. an earthquake) during the life time t.

- t is usually taken as 50 years for buildings, and 100 years for dams.
- P_E for ground motion is usually assumed to be in the ranges [20%   64%] for DBL and [10%   20%] for MDL.
- Assuming a lifetime of 100 years, the corresponding $T_r = 1/\lambda$ is determined for 450 and 1,000 years for DBL and MDL, respectively from.


λ, Annual rate of exceedance
10^-1
10^-2
10^-3
10^-4
0.5
0.4
0.3
0.2
0.1
140
120
100
80
60
40
20
Prob. of occurence
Life [Years]

- Probability Seismic Hazard Analysis or PSHA=SHA+ESRA.
- Engineering Seismic Risk Analysis yielded annual rate of exceedance λ in terms of probability of occurrence of at least one event and life time t.
- Seismic hazard analysis yielded annual rate of exceedance λ vs intensity measure.
- Select λ from the first curve, and PGA from the second.



- with the PGA known, one selects (or generate) a set of n ground motion acceleration time histories to perform multiple analyses.
- From the corresponding analysis one plots

Intensity Measure (IM) a measure of the ground motion characteristic, typically the (peak or spectral) ground acceleration.

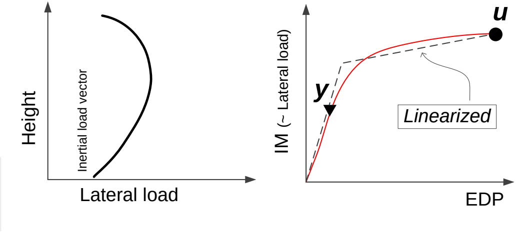
Engineering demand parameter (EDP) which corresponds to any outcome of the analysis of relevance to the safety assessment, such as base shear, drift.

- We repeat this process *m* times for different intensity levels.
- There are four types of analysis that can be performed.

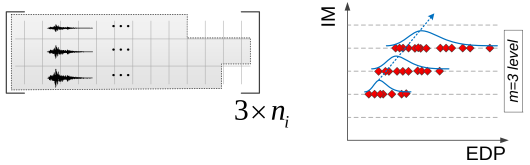
Method		S/D Analysis	*m*	*n*
Push Over Analysis	POA	Static	na	na
Multi Strip Analysis	MSA	Dynamic	3	*n*
Incremental Dynamic Analysis	IDA	Dynamic	Variable	*n*
Endurance Time Analysis	ETA	Dynamic	1	*n*

where *m* be the number of ground motion intensity levels (or strips), and *n* the number of ground motions for a given *m*.

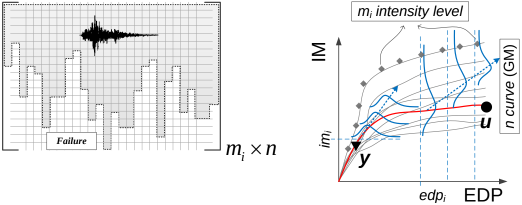
- In all cases we plot IM vs EDP (and not the other way around!)



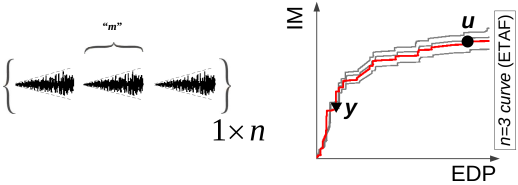
- Applies incrementally load or displacement
- Extensively used in building to capture failure mode in lieu of the more expensive transient nonlinear analysis.
- Assumed to be capable of mobilizing principal nonlinear modes of structural behavior up to collapse.



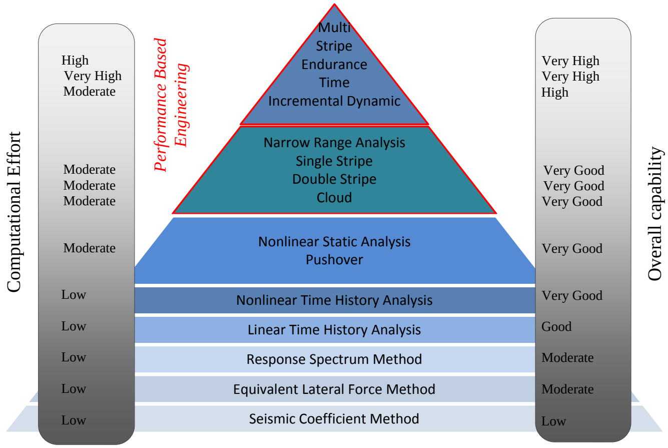
- Hinges on a deterministic number of ground motion intensity levels m (or strips)
- Typically $m = 3$ corresponding to the exceedance probabilities of 10% in 50-year, 5% in 50-year, and 2% in 50-year.
- To each strip correspond n ground motions.
- Two possibilities:
  - Selection of n different ground motions scaled at m different levels.
  - Selection of n_i ground motions for each of the intensity levels with no scaling.
- Following the analysis, and for each m the usual IM versus EDP results are first plotted.
- Then for each IM histograms are generated and the most suitable probability distribution function (normal or log-normal) is selected.



- Considers n ground motions which will all be incrementally scaled m times until failure.
- *a priori* m is unknown and each ground motion n will result in a corresponding failure at a different intensity level m_i .
- Following the analysis, the IDA curve connects the resulting m demand parameters for each of the n ground motions.
- Each one of those curve will be asymptotic to the corresponding failure.
- Capture of the overall response by a single measurable quantity at a given EDP (EDP = edp_i) can be determined through the corresponding probability distribution function.
- Similarly probability distribution function for a given IM (IM = im_i) can also be determined.
- Those curves can be used for the determination of the fragility plots, and probability of failure.



- The preceding two methods started with actual recorded ground motion and required up to $m \times n$ analysis, computationally expensive and may force the analysis to make greatly simplified assumption in their model. Such assumptions may lead to erroneous conclusions.
- ETA method starts with a synthetic ground motion and modify it to be characterized with an increasing amplitude.
- Substitute to the m intensity levels previously determined and n endurance time acceleration function (ETAF) are used.
- Outcome of the analysis, is the average of the n analyses in terms of IM versus EDP. The resulting curve is analogous to the one of the POA or 50% fractile of IDA.


Computational Effort
High
Very High
Moderate
Moderate
Moderate
Moderate
Moderate
Low
Low
Low
Low
Low
Performance Based Engineering
Multi
Stripe
Endurance
Time
Incremental Dynamic
Narrow Range Analysis
Single Stripe
Double Stripe
Cloud
Nonlinear Static Analysis
Pushover
Nonlinear Time History Analysis
Linear Time History Analysis
Response Spectrum Method
Equivalent Lateral Force Method
Seismic Coefficient Method
Very High
Very High
High
Very Good
Very Good
Very Good
Very Good
Very Good
Good
Moderate
Moderate
Low
Overall capability

Non Linear Structural Analysis

Introduction

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Fall 2020

Table of Contents I

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- It builds on previous pre-Standards: FEMA 310 (ASCE 1998) is the pre-code to ASCE/SEI 31 (2003) and they are both for existing buildings.
- ASCE 31 controls the evaluation of existing buildings, ASCE 41 covers procedures for retrofit of existing buildings. However, they have both been merged into one document now: ASCE 41-17

- Chapter 16 of ASCE 7-16 is the governance for PBEE of new design.

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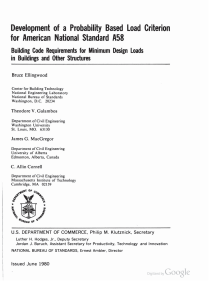
May need correction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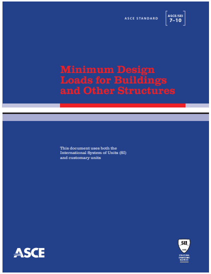
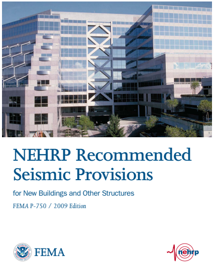

ASCE STANDARD
ASCE/SEI 7-10
Minimum Design Loads for Buildings and Other Structures
This document uses both the International System of Units (SI) and customary units
ASCE
SEI


Seismic Performance Assessment of Buildings
Volume 1 – Methodology
FEMA P-58-1 / September 2012
FEMA
nehrp


NEHRP Recommended Seismic Provisions
for New Buildings and Other Structures
FEMA P-750 / 2009 Edition
FEMA
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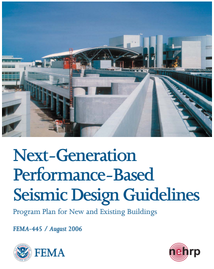

Next-Generation Performance-Based Seismic Design Guidelines
Program Plan for New and Existing Buildings
FEMA-445 / August 2006
FEMA
nehrp

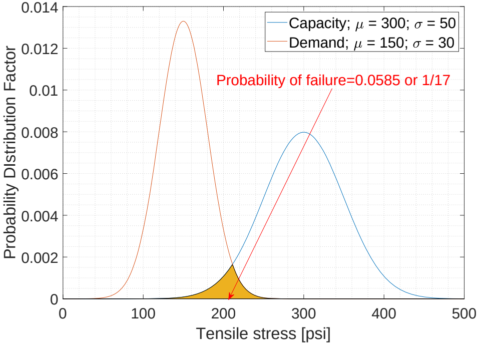
- Load and resistance are not deterministic quantities (as in the allowable stress design, ASD), but are random variables with their own probability distribution functions.
- There is a probability of failure.
- Load will be multiplied by a factor α, (ASCE-7-10) and we shall consider the ultimate resistance (reduced by Φ)
- We will assign α and Φ such that the probability of failure does not exceed a certain value.
- LRFD is generally expressed as

$$\Phi C_n \geq \Sigma \alpha_i D_i \tag{1}$$

 where C_n and D are the nominal capacity and demands (or nominal resistance and load).
- Limit state is generally determined from Plastic capacity without a nonlinear analysis.

- LRFD seeks to have a Reliability Index such that $\beta >\sim 3.5$. The Reliability Index is a "universal" indicator on the adequacy of a structure, and can be used as a metric to 1) assess the health of a structure, and 2) compare different structures targeted for possible remediation.

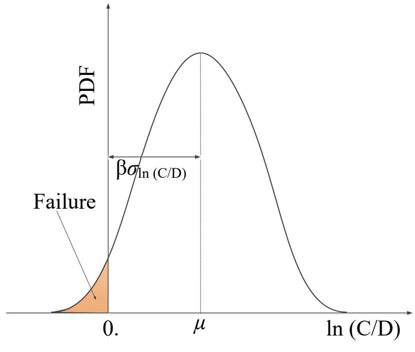
- Capacity C and demand D are both random variables (usually assumed to be normal, though a log-normal may be prefereable in some instances).



- Two approaches to determine β depending on how is the safety margin computed.

$$
\begin{aligned}
M &= C - D \\
\mu_M &= \mu_C - \mu_D \\
\sigma_M &= \sqrt{\sigma_C^2 + \sigma_D^2} \\
\beta &= \frac{\mu_M}{\sigma_M} \\
&= \frac{\mu_C - \mu_D}{\sqrt{\sigma_C^2 + \sigma_D^2}}
\end{aligned}
$$

$$
\begin{aligned}
M &= \ln C - \ln D \\
\mu_M &= \mu_C - \mu_D \quad \text{First order} \\
\sigma_M &= \sqrt{\frac{\sigma_C^2}{\mu_C^2} + \frac{\sigma_D^2}{\mu_D^2}} = \sqrt{V_C^2 + V_D^2} \\
\beta &= \frac{\mu_M}{\sigma_M} = \frac{\ln \mu_C - \ln \mu_D}{\sqrt{V_C^2 + V_D^2}} \\
&= \frac{\ln \mu_C / \mu_D}{\sqrt{V_C^2 + V_D^2}}
\end{aligned}
$$


PDF
βσ ln (C/D)
Failure
0.
μ
ln (C/D)

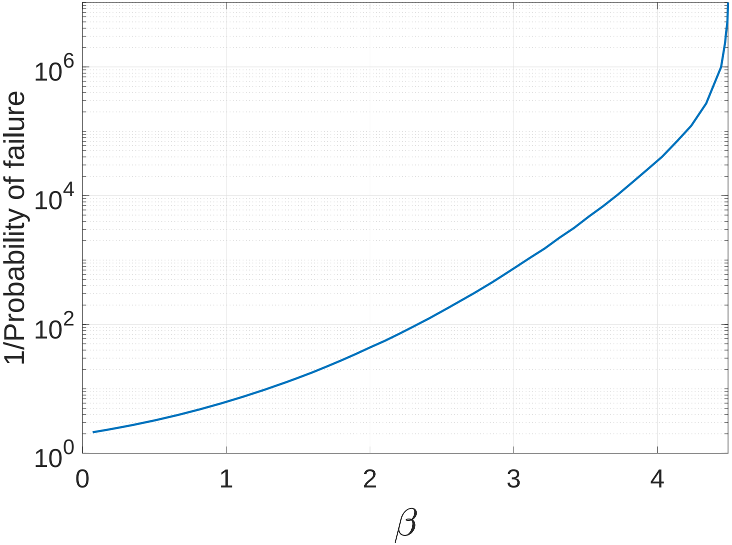
- β is selected to reflect failure consequences

Type of Load/Member	β
AISC	
DL + LL; Members	3.0
DL + LL; Connections	4.5
DL + LL + WL; Members	3.5
DL + LL +EL; Members	1.75
ACI	
Ductile Failure	3-3.5
Brittle Failures	3.5-4

The probability of failure P_f is equal to the ratio of the shaded area to the total area under the curve and is given by $\Phi(-\beta)$ where Φ is the standard normal cumulative probability function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = \frac{1}{2}\left[1 + erf\left(\frac{x}{\sqrt{2}}\right)\right] \quad (2)$$

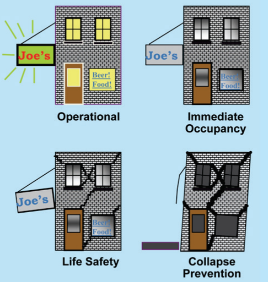
Target values for β


1/Probability of failure
10^0
10^2
10^4
10^6
0
1
2
3
4
β

1. Inconsistent: Linear analysis, but plastic design.
2. Ignores load redistribution near failure (though ACI implicitly accounts for some of it through reduction of negative moments).
3. Addresses only one level of hazard: failure of one structural component (and not the entire system), but how about quantification of damage due to more frequent events?

- Performance-based seismic design explicitly evaluates how a building is likely to perform, given the potential hazard it is likely to experience, considering uncertainties inherent in the quantification of potential hazard and uncertainties in assessment of the actual building response.
- Contrarily to LRFD, it does not limit itself to one level of hazard, but multiple.

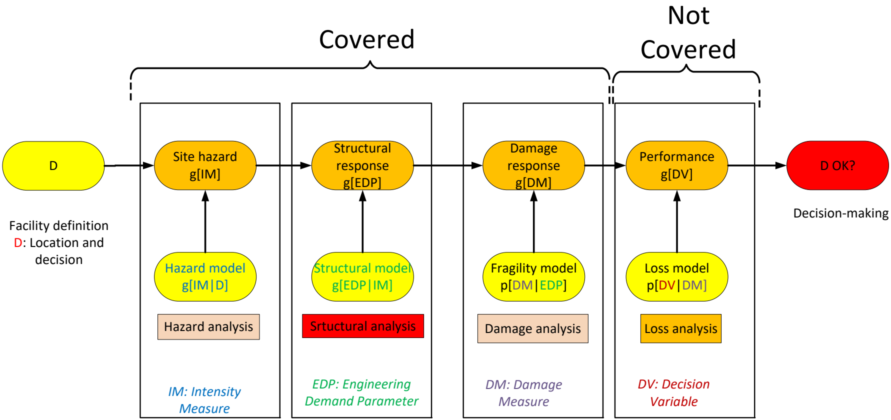
- Performance is measured in terms of the probability of incurring casualties, repair and replacement costs, repair time, and unsafe placarding.
- Performance can be assessed for a particular earthquake scenario or intensity, or considering all earthquakes that could occur, and the likelihood of each, over a specified period of time.
- Performance expressed in terms of a series of discrete performance levels identified as Operational, Immediate Occupancy, Life Safety, and Collapse Prevention.

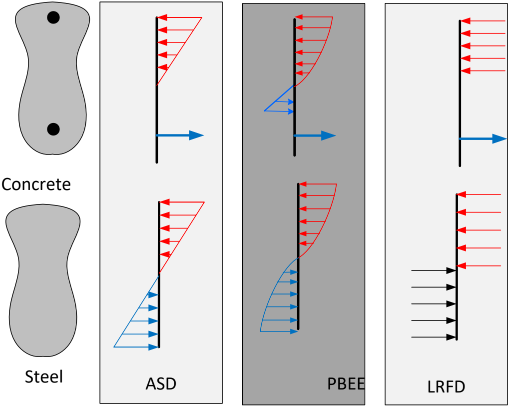


Present-generation discrete performance levels.

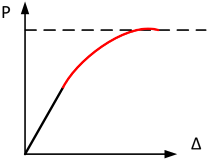
- Introduced the concept of performance related to damage of both structural and nonstructural components. Performance Objectives were developed by linking one of these performance levels to a specific level of earthquake hazard.
- It is the state of the practice among high end companies. It is well established
- However:
  - Limit states are component-based not truly system-wide, (what if one component fails, does it trigger progressive collapse?)
  - treats only MCE event (2%/50years).
  - Limited treatment of uncertainty and probability.
  - Limited information for designing above code.

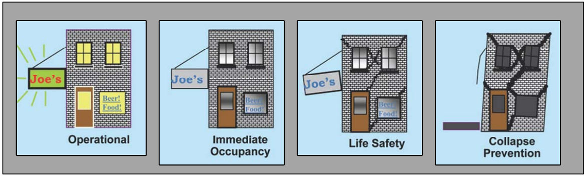
- New performance measures (e.g. repair costs, casualties, and time of occupancy interruption).
- Create procedures for estimating probable repair costs, casualties, and time of occupancy interruption, for both new and existing buildings.





	ASD/LRFD	PBEE
Focus on	Structural component	Structural system
Failure assessmsment	Pass/Fail	Full spectrum
Analysis	Linear	Non Linear





Eurocodes are a series of 10 European Standards,which will supersede national codes and should be enforced throughout Europe.

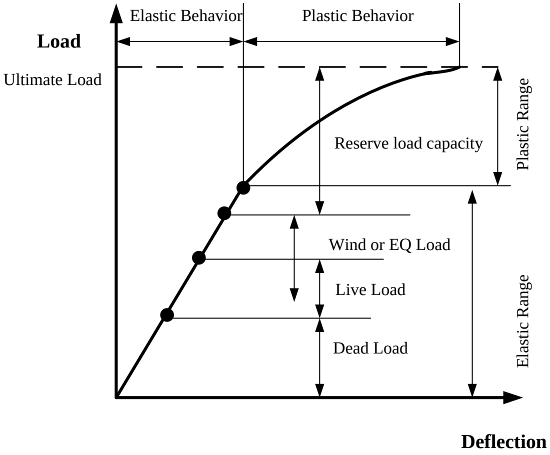
EN 1990	Eurocode:	Basis of structural design
EN 1991	Eurocode 1:	Actions on structures
EN 1992	Eurocode 2:	Design of concrete structures
EN 1993	Eurocode 3:	Design of steel structures
EN 1994	Eurocode 4:	Design of composite steel and concrete structures
EN 1995	Eurocode 5:	Design of timber structures
EN 1996	Eurocode 6:	Design of masonry structures
EN 1997	Eurocode 7:	Geotechnical design
EN 1998	Eurocode 8:	Design of structures for earthquake resistance
EN 1999	Eurocode 9:	Design of aluminium structures

Each one of them is divided in package, as for Eurocode 8

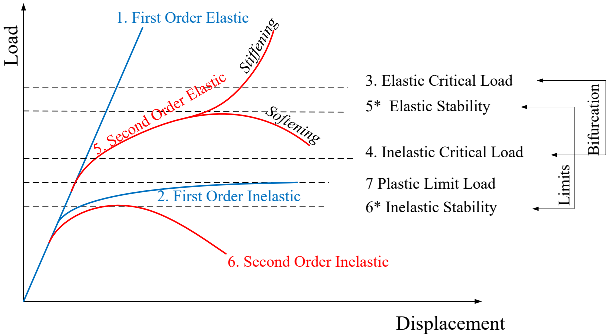
EN 1998-1:2004	Part 1: General rules, seismic actions and rules for buildings
EN 1998-2:2005	Part 2: Bridges
EN 1998-3:2005	Part 3: Assessment and retrofitting of buildings
EN 1998-4:2006	Part 4: Silos, tanks and pipelines
EN 1998-5:2004	Part 5: Foundations, retaining structures &geotechnical aspects
EN 1998-6:2005	Part 6: Towers, masts and chimneys

- EC8 and PBE require the completion of
  - Nonlinear Static Procedure or Nonlinear Pushover (NPO)
  - Nonlinear Dynamic Procedure or Nonlinear Time History (NTH)

- Emphasis will be on
  - Basic fundamental understanding of the analysis techniques as opposed to how to use them in the context of meeting code provisions.
  - How to perform NPO and NTH rather than going through the details of EC8 or PBEE code requirements (those can be easily studied individually).
  - 1D frame elements as opposed to continuum elements
  - Nonlinear static and dynamic analysis.
- Methodology presented constitutes the State of the Art as practiced only by a few “high end consulting firms”.
- Additional lectures
  - Performance Based Engineering
  - Examples of nonlinear analysis of structures (dams, nuclear reactors) using continuum elements.
  - Guest Lecture(s)
- Computer skills: Matlab
- Grading: 1-2 exams(?), homeworks, term project/report.


Elastic Behavior
Plastic Behavior
Load
Ultimate Load
Reserve load capacity
Plastic Range
Wind or EQ Load
Live Load
Dead Load
Elastic Range
Deflection

Levels of Structural Analysis



		Constitutive Equations			
		Undeformed Shape		Deformed Shape	
		Elastic (Linear)	Inelastic (Non Linear)	Elastic (Linear)	Inelastic (Non Linear)
Kinematic Eq.	1^{st} Order (Linear)	1 (C:L-K:L)	2 (C:NL-K:L)	Critical Load	
				3 Elastic	4 Inelastic
	2^{nd} Order (non Linear)	Deformed Shape			
		5 (C:L-K:NL)	6 (C:NL-K:NL)	-	-

	1	2	3	4	5	6	7
CM	L	NL	L	NL	L	NL	NL
Analysis	Lin.	Inc	Bif.	Bif.	Inc	Inc	Bif
FBD	US	DS	DS	DS	DS	DS	DS

L: Linear elastic; NL: Nonlinear;
Lin: Linear; Incr: Nonlinear incremental analysis; Bif: Bifurcation (Eigenvalue) analysis;
US: Undeformed state (Lagranfian); DS: Deformed state (Eulerian);

First Order Elastic excludes any nonlinearities. If the equilibrium equation is written in terms of

- **1 (C:L-K:L); Undeformed Shape** This is the most common case, linear elastic. It is usually acceptable for service loads. For time dependent cases, we must consider visco-elastic models.
- **3 Bifurcation; Deformed shape** (or 'zero order") an eigenvalue analysis which would lead to the Elastic Critical Load. Note that we do not have a corresponding load-displacement curve, but rather "buckling modes".

First Order Inelastic Accounts for material non-linearity. In such an analysis, the inelastic region (plastic zone) develops gradually, and it will provide a good estimate of the elasto-plastic response (note that instability is not addressed). We consider

- Non-linear Elasticity: reversible non-linear stress-strain (upon unloading, the strain goes back to zero).
- Plasticity, non reversible non-linear stress-strain.
- Damage

If the equilibrium equation is written in terms of

2 (C:NL-K:L); Undeformed Shape Second most common form of analysis, typically conducted for ultimate/unusual loads.

4 Bifurcation; Deformed shape: an eigenvalue analysis which would lead to the Inelastic Critical Load. Note that we do not have a corresponding load-displacement curve, but rather "buckling modes". This inelastic critical load will be smaller than the elastic one.
For time dependent cases, we consider visco-plasticity, or fatigue, or continuous damage models.

Second Order: Need to draw FBD in the deformed shape:

5 (C:L-K:NL); Elastic: accounts for the effects of finite deformation and displacements, equilibrium equations are written in terms of the geometry of the deformed shape (Eulerian), does not account for material non-linearities, may be able to detect bifurcation and or increased stiffness (when a member is subjected to a tensile axial load). Analysis of cables, nets, catenary structures.

6 (C:NL-K:NL); Inelastic equations of equilibrium written in terms of the geometry of the deformed shape, can account for both geometric and material nonlinearities. Most suitable to determine failure or ultimate loads. By far the most complex form of analysis, used in Metal Forming simulation, fragmentation of structures (missile impact).

How does it relate to the stiffness matrix

$$\mathrm{K} = \int_{\Omega} \mathrm{B}^{T}\mathrm{D}\mathrm{B}\,d\Omega$$

- Geometric nonlinearity impacts B
- Material nonlinearity impacts D

Non Linear Structural Analysis

Matlab; Advanced Features for NSA

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University of Colorado, Boulder

Fall 2020

Table of Contents I

Motivation I

- Ultimately we rely on computer programs to perform structural analysis.
- Commercial codes are widely available for linear analysis.
- "Modern Non Linear" analysis codes are still in infancy, these include
  - OpenSEES developed by PEER/NEES, open source, c++, tcl. Very powerful, yet modification is not simple for most students.
  - FEADAS, Prof. Filippou/Berkeley, Matlab, closed source code.
  - Mercury, Prof. Saouma's group, primarily for hybrid simulation, however two identical versions are available: Matlab and c++
  - Matlab is far "friendlier" than c++ for programming.
  - You will be asked to modify the Matlab version as part of homeworks.
  - Though you are expected to have had some exposure to Matlab, there are certain features, widely used in modern codes, that you may not have been exposed to.
- Matlab: Interpreter, slow (even the compiled version), expensive.

Motivation II

- Octave is a Matlab-alike free program,
- For new programming language: use Python
- There are many textbooks, and hundred of on-line tutorials.

Functions I

- A script is the simplest kind of program file because there is no input or output arguments. It is an external file that contains a sequence of MATLAB statements. There are no local variables in a script.
- A function which accepts input from and returns output to its caller. Functions operate on variables within their own workspace. This workspace is separate from the base workspace; it allows for local variable which do not interfere with the ones of the calling entity. Ideally, each function is stored in an `.m` file with the same name.

```
function [out1, out2, ...] = myfun(in1, in2, ...)
```

Listing 1: Function Defintion

Functions II

```
function [str, ele] = FEAnalysis(str, ele, sec, mat, fos)
```

Listing 2: Example of Function Definition

```
[str, ele] = FEAnalysis(str, ele, sec, mat, fos)
```

Listing 3: Example Invocation of function

Switch Case

http://blogs.mathworks.com/pick/2008/01/02/matlab-basics-switch-case-vs-if-elseif/

```
switch eletype
        case 'Simple2DTruss'
            [tmpeleinfo] = Simple2DTrussInfo(eleinfo);
        case 'Simple3DTruss'
            [tmpeleinfo] = Simple3DTrussInfo(eleinfo);
        case 'StiffnessBasedBeam'
            [tmpeleinfo] = StiffnessBasedBeamInfo(eleinfo);
        case 'StiffnessBased2DBeamColumn'
            [tmpeleinfo] = StiffnessBased2DBeamColumnInfo(eleinfo);
        case 'StiffnessBased3DBeamColumn'
            [tmpeleinfo] = StiffnessBased3DBeamColumnInfo(eleinfo);
        case 'Grid'
            [tmpeleinfo] = GridInfo(eleinfo);
    end
```

Data Types

- Numeric Types: Integer and floating-point data
- Characters and Strings: Characters and arrays of characters
- Cell Arrays Data of varying types and sizes stored in cells of array.
- Structures: Data of varying types and sizes stored in fields of a structure

Cell Arrays

- String arrays must have all entries with the same length. `c=['steel';'concrete']` is not acceptable (`CAT arguments dimensions are not consistent.`)
- Cell arrays may contain
  - Strings of various length
    `C = {'Steel';'Concrete';'Structu` `Analysis' }`
    creates a 3-by-1 cell array that requires no padding because each row of the array can have a different length:
    `C =`
    `'Steel'`
    `'Concrete'`

```
elements = { {1, 'Simple2DTruss', 1, 2, 1};
             {2, 'Simple2DTruss', 2, 3, 1};
             {3, 'Simple2DTruss', 1, 4, 1};
             {4, 'Simple2DTruss', 2, 4, 1};
             {5, 'Simple2DTruss', 3, 4, 1};
             {6, 'Simple2DTruss', 4, 5, 1} };
>> elements{1}
ans =
    [1]    'Simple2DTruss'    [1]    [2]    [1]
>> elements{1}{2}
ans =
Simple2DTruss
>> elements{1}{3}
ans =
     1
size(elements)
ans =
     6     1
>> elements
elements =
    {1x5 cell}
    {1x5 cell}
    {1x5 cell}
    {1x5 cell}
    {1x5 cell}
    {1x5 cell}
```

Cell Arrays; Example

- variable length cell array.
- Example of input data for material properties, length of cell array depends on the constitutive model selected.

```
materials = {{1, 'ModKP', 3.57, 0.0026, 1.19, 0.0078,0.3,0.448121077,549.2307692,0};
            {2, 'ModKP', 3.9, 0.00284, 3.51, 0.00852,0.3,0.46837485,549.4505495,0};
            {3, 'ModGMP',27300,80,0.01,15,0.925,0.15,0,0,55,0,55};};
>> materials
materials =
    {1x10 cell}
    {1x10 cell}
    {1x13 cell}
>> materials{2}{2}
ans =
ModKP
>> {materials{2}{9}
ans =
549.4505
```

$$\texttt{materials} \quad = \quad \texttt{mattag}_i,\ \texttt{mattype}_i,\ \texttt{modulus}_i,\ \texttt{density}_i,\ \{\,\texttt{MatProp}_i\,\} \quad ; \quad \text{where:}$$

- $\texttt{mattag}_i$: Consecutive integer number identifying material at i^{th} material
- $\texttt{mattype}_i$: Material type at i^{th} material

Structures I

- Structures are like cell arrays, in that they allow one to group collections of dissimilar data into a single variable. However, instead of addressing elements by number, structure elements are addressed by names called fields.
- Cell arrays use curly braces to access data, structures use dot notation.
- Structures are multidimensional arrays with elements accessed by textual field designators. For example, `S.name = 'Barack Obama'; S.score = 83; S.grade = 'B+'` as opposed to a cell array which would look like `S = {'Barack Obama', '83', 'B+'}` and `S(2)='83'`.
- In this example we have created a scalar structure with three fields:

```
S =
     name: 'Barack Obama'
    score: 83
    grade: 'B+'
```

Structures II

- Like everything else in the MATLAB environment, structures are arrays, so you can insert additional elements. In this case, each element of the array is a structure with several fields. The fields can be added one at a time,
  `S(2).name = 'Ronald Reagan'; S(2).score = 91; S(2).grade = 'A-';`
  or an entire element can be added with a single statement:
  `S(3) = struct('name','George Washington','score',70,'grade','C')`
- Now the structure is large enough that only a summary is printed:

```
S =
1x3 struct array with fields:
    name
    score
    grade
```

Structures III

- There are several ways to reassemble the various fields into other MATLAB arrays. They are mostly based on the notation of a comma-separated list. If you type `S.score` it is the same as typing `S(1).score, S(2).score, S(3).score` which is a comma-separated list.

Mercury Example I

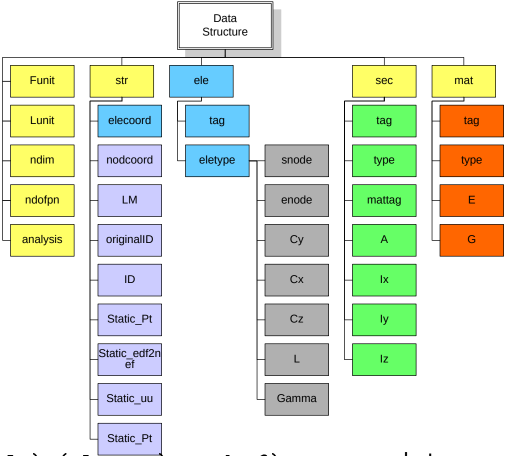
```
function [str, ele] = ElementInfo(str, elements)
%
str.nele = size(elements,1);
for iele = 1:str.nele
    eletag  = elements{iele}{1};
    eletype = elements{iele}{2};
    eleinfo = elements{iele};
    ele(eletag).type = eletype;
    switch eletype
        case 'Simple2DTruss'
            [tmpeleinfo] = Simple2DTrussInfo(eleinfo);
        case 'StiffnessBased2DBeamColumn'
            [tmpeleinfo] = StiffnessBased2DBeamColumnInfo(eleinfo);
        ...
    end
    ele(eletag).(eletype) = tmpeleinfo;
end
```

```
% Make LM matrix
for iele = 1:str.nele
    eletype = ele(iele).type;
    str.LM(iele,1) = str.ID(ele(iele).(eletype).snode,1);
    ...
    str.elecoord(iele,1) = str.nodcoord(ele(iele).(eletype).snode,1);
    ...
end
```

Mercury Example II

```
for iele = 1:str.nele
    eletype = ele(iele).type;
    switch eletype
        case 'ZeroLength2D'
            % Do nothing
        case 'ZeroLength2DSection'
            % Do nothing
        otherwise
            xs      = str.elecoord(iele,1);
            ys      = str.elecoord(iele,2);
            xe      = str.elecoord(iele,3);
            ye      = str.elecoord(iele,4);
            dx      = xe    xs;
            dy      = ye    ys;
            ele(iele).(eletype).L  = sqrt( dx*dx + dy*dy );
            ele(iele).(eletype).Cx = dx/ele(iele).(eletype).L;
            ele(iele).(eletype).Cy = dy/ele(iele).(eletype).L;
    end
end
```

Structure Data



- `str.ID(ele(iele).(eletype).enode,6)` corresponds to xxx;
- `str.elecoord(iele,4) = str.nodcoord(ele(iele).(eletype).enode,2)` corresponds to xxx;;

Input File Example I

```
AnalysisType = 2;
%
% Preface
Unit     = {'kN', 'mm'};
StrMode = {2, 2};
%
% Control block
Iteration = {'static',    { {'Linear'} } ;
             'transient', { {'Linear'} };
            };
if (AnalysisType == 2)
    Integration = {'Newmark', 0, 1/4, 1/2, 0, 0};
    eigens      = {0.02, 0.02};
end
%
% Geometry block
nodcoord    = {1,    0, 0;
               2, 1500, 0;
               3, 3000, 0;
               4, 1500, 2000;
               5, 3000, 2000};
constraint = {3, 1, 1;
              5, 1, 1};
%
% Element block
elements = { {1, 'Simple2DTruss', 1, 2, 1};
             {2, 'Simple2DTruss', 2, 3, 1};
```

Input File Example II

```
            {3, 'Simple2DTruss', 1, 4, 1};
            {4, 'Simple2DTruss', 2, 4, 1};
            {5, 'Simple2DTruss', 3, 4, 1};
            {6, 'Simple2DTruss', 4, 5, 1} };
%
% Section block
sections = { 1, 'General', {1, 400, 0, 0, 0} };
%
% Material block
materials = { {1, 'Elastic', 200, 0, 7850*10^9} };
%
% Force block
if (AnalysisType == 1)
    forces = { 1, 'Static', {'NodalForces', {1, 2, 30;
                                            2, 2, 20} } };
elseif (AnalysisType == 2)
    ga = load('ElCentro_g_0_01_Matlab.txt');
    nga = size(ga, 1);
    for i = 1:nga
        groundacceleration{i,1} = ga(i,1);
        groundacceleration{i,2} = ga(i,2);
        groundacceleration{i,3} = ga(i,3);
    end
    forces = { 1, 'Static', {'NodalForces', {1, 2, 0} };
              2, 'Acceleration', {9810, groundacceleration} };
end
%
```

Input File Example III

Save Data

- Often times, there is a need to store some or all the data in a binary file.
- Examples: Results of an experiment, or data from an analysis for subsequent post-processing.
- This can be easily accomplished by the `save` and `load` commands.
- Try to load the `Recorder_6.mat` file which was generated by an experiment.

```
% clear all data
clear
% define an array and perform a dummy operation
x=[1:1:20];
y=2*x;
% save only the x array in a binary file my_data
%(which will be assigned the extension .mat)
save('my_data','x')
% clear all the data
clear
% convince ourself that x is gone
x
%load the data stored in my_data.mat
load('my_data.mat')
% verify that we recover x
x
```

Non Linear Structural Analysis

Strong to Weak Formulations

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Motivation I

- Structural engineering (and mechanics) can be approached from two different angles:
  1. Newtonian approach, equations of equilibrium.
  2. Lagrangian approach: thermodynamics (balance of energy).
- So far we have pursued the former, from this point onward, we shall focus on the second which will provide the formalism needed to develop the finite element method.
- Some of the concepts will look familiar (first law of thermodynamic, principle of virtual force, minimum potential energy) at first.
- This lecturewill
  1. Bring together the various "energy methods" and show that they are all (essentially) the same.
  2. Develop the principle of virtual displacement as a prelude to the finite element method.

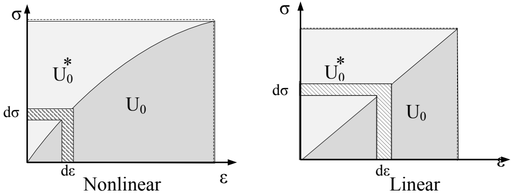
Motivation II

3. Show the duality between the so-called strong form (differential equation) and the weak form (satisfy a principle in an average sense).
4. Formalize the definition of Natural and Essential boundary conditions.

First law of Thermodynamics

- First Law of Thermodynamics: The time-rate of change of the total energy (i.e., sum of the kinetic energy K and the internal energy U) is equal to the sum of the rate of work done by the external forces W_e and the change of heat content per unit time H: $\frac{\mathrm{d}}{\mathrm{d}t}(K+U) = W_e + H$
- For an adiabatic system (no heat exchange) and if loads are applied in a quasi static manner (no kinetic energy), the above relation simplifies to: $W_e = U$

InternalEnergy I



Strain energy density :

$$U_0 \stackrel{\text{def}}{=} \int_0^{\varepsilon} \sigma d\varepsilon \tag{1}$$

Complementary strain energy density :

$$U_0^* \stackrel{\text{def}}{=} \int_0^{\sigma} \varepsilon d\sigma \tag{2}$$

InternalEnergy II

strain and complementary strain energy :

$$U \stackrel{\text{def}}{=} \int_{\Omega} U_0 d\Omega \quad (3)$$

$$U^* \stackrel{\text{def}}{=} \int_{\Omega} U_0^* d\Omega \quad (4)$$

Stress Strain Relation :

$$\boldsymbol{\sigma} = \mathrm{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) + \boldsymbol{\sigma}_0 \quad (5)$$

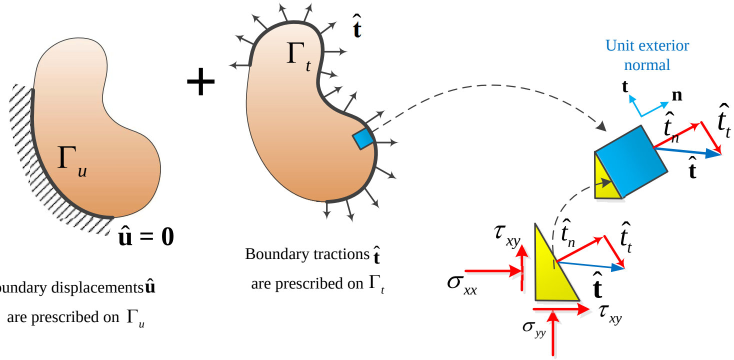
Strain Energy for Linear Systems :

$$U = \frac{1}{2}\int_{\Omega} \boldsymbol{\epsilon}^T \mathrm{D}\boldsymbol{\epsilon} d\Omega - \int_{\Omega} \boldsymbol{\epsilon}^T \mathrm{D}\boldsymbol{\epsilon}_0 d\Omega + \int_{\Omega} \boldsymbol{\epsilon}^T \boldsymbol{\sigma}_0 d\Omega \quad (6)$$

External Work and Virtual Work I

Forces Only two types of forces:

- Surface traction $\hat{\mathbf{t}}$



Boundary displacements $\hat{\mathbf{u}}$ are prescribed on Γ_u

Boundary tractions $\hat{\mathbf{t}}$ are prescribed on Γ_t

External Work and Virtual Work II

- Body force b

External work $W_e \stackrel{\text{def}}{=} \int_\Omega \mathbf{u}^T \mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{u}^T \hat{\mathbf{t}} d\Gamma$

Point Force/Moment $W_e = \int_0^{\Delta_f} P d\Delta + \int_0^{\theta_f} M d\theta$

Internal Strain Energy/Virtual Work $\delta\overline{U} = -\delta\overline{W}_i \stackrel{\text{def}}{=} \int_\Omega \sigma \delta\overline{\varepsilon} d\Omega$

External Virtual Work $\delta\overline{W}_e \stackrel{\text{def}}{=} \int_{\Gamma_t} \delta\overline{\mathbf{u}}^t \hat{\mathbf{t}} d\Gamma + \int_\Omega \delta\overline{\mathbf{u}}^t \mathbf{b} d\Omega$

Complementary Internal Strain Energy-Internal Virtual Work

$$\delta\overline{U}^* = -\delta\overline{W}_i^* \stackrel{\text{def}}{=} \int_\Omega \varepsilon \delta\overline{\sigma} d\Omega$$

Complementary External Virtual Work $\delta\overline{W}_e^* \stackrel{\text{def}}{=} \int_{\Gamma_u} \hat{\mathbf{u}}^t \delta\overline{\mathbf{t}} d\Gamma$

Variables

- The complementary internal virtual strain energy is expressed in terms of strain or internal displacements ($u(x)$, $v(x)$).
- It will lead to the formulation at the root of the finite element method.

Axial Members

Strains and displacements constitute the virtual quantities identified by δ.

Elastic System

$$\left.\begin{array}{rcl} \delta\overline{U} & = & \int_\Omega \sigma\delta\overline{\varepsilon}d\Omega \\ d\Omega & = & Adx \end{array}\right\} \delta\overline{U} = A\int_0^L \sigma\delta\overline{\varepsilon}dx \tag{7}$$

Linear Elastic

$$\left.\begin{array}{rcl} \delta\overline{U} & = & \int \sigma\delta\overline{\varepsilon}d\Omega \\ \sigma_x & = & E\varepsilon_x = E\frac{du}{dx} \\ \delta\overline{\varepsilon} & = & \frac{d(\delta\overline{u})}{dx} \\ d\Omega & = & Adx \end{array}\right\} \delta\overline{U} = \int_0^L \underbrace{E\frac{du}{dx}}_{"\sigma"} \underbrace{\frac{d(\delta\overline{u})}{dx}}_{"\delta\overline{\varepsilon}"} \underbrace{Adx}_{d\Omega} \tag{8}$$

Flexural Members I

Elastic System

$$\left.\begin{array}{rcl} \delta\overline{U} &=& \int \sigma_x \delta\overline{\varepsilon}_x d\Omega \\ M &=& \int_A \sigma_{xx} y dA \Rightarrow \frac{M}{y} = \int_A \sigma_{xx} dA \\ \delta\overline{\phi} &=& \frac{\delta\overline{\varepsilon}}{y} \Rightarrow \delta\overline{\phi} y = \delta\overline{\varepsilon} \\ d\Omega &=& \int_0^L \int_A dA dx \end{array}\right\} \delta\overline{U} = \int_0^L M(x)\delta\overline{\phi} dx \tag{9}$$

Linear Elastic

$$\left.\begin{array}{rcl} & & \delta\overline{U} = \int_\Omega \sigma_x \delta\overline{\varepsilon}_x d\Omega \\ \left.\begin{array}{rcl} \sigma_x &=& \frac{My}{I_z} \\ M &=& \frac{d^2v}{dx^2} EI_z \end{array}\right\} & & \sigma_x = \underbrace{\frac{d^2v}{dx^2}}_{\kappa} Ey \\ & & \delta\overline{\varepsilon}_x = \frac{\delta\overline{\sigma}_x}{E} = \frac{d^2(\delta\overline{v})}{dx^2} y \\ & & d\Omega = dAdx \end{array}\right\} \delta\overline{U} = \int_0^L \int_A \frac{d^2v}{dx^2} Ey \frac{d^2(\delta\overline{v})}{dx^2} y dA dx$$

Flexural Members II

Since $\int_A y^2 \mathrm{d}A = I_z \Rightarrow$

$$\delta\overline{U} = \int_0^L \underbrace{EI_z \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}}_{"\sigma"} \underbrace{\frac{\mathrm{d}^2(\delta\overline{v})}{\mathrm{d}x^2}}_{"\delta\overline{\varepsilon}"} \mathrm{d}x \tag{10}$$

Potential Energy I

Potential of external work W

$$\mathcal{W}_e \stackrel{\text{def}}{=} \int_\Omega \mathrm{u}^T \mathrm{b} d\Omega + \int_{\Gamma_t} \mathrm{u}^T \hat{\mathrm{t}} d\Gamma + \mathrm{uP} \tag{11}$$

Potential energy

$$\Pi \stackrel{\text{def}}{=} U - \mathcal{W}_e = \int_\Omega U_0 d\Omega - \left(\int_\Omega \mathrm{ubd}\Omega + \int_{\Gamma_t} \mathrm{u}\hat{\mathrm{t}} d\Gamma + \mathrm{uP} \right) \tag{12}$$

Complementary potential energy

$$\Pi \stackrel{\text{def}}{=} U^* - \mathcal{W}_e^* = \int_\Omega U_0^* d\Omega - \left(\int_\Omega \mathrm{ubd}\Omega + \int_{\Gamma_t} \mathrm{u}\hat{\mathrm{t}} d\Gamma + \mathrm{uP} \right) \tag{13}$$

Summary I

	U	Virtual Displacement $\delta\overline{U}$ $-\int_\Omega \delta\overline{\mathbf{u}}^T\left(\mathbf{L}^T\sigma+\mathbf{b}\right)d\Omega$ $+\int_{\Gamma_t}\delta\overline{\mathbf{u}}^T(\mathbf{t}-\hat{\mathbf{t}})d\Gamma=0$		Virtual Force $\delta\overline{U}^*$ $\int_\Omega\left(\varepsilon_{ij}-u_{i,j}\right)\delta\overline{\sigma}_{ij}d\Omega$ $-\int_{\Gamma_u}(u_i-\hat{u})\,\delta\overline{t}_i d\Gamma=0$	
		Essential BC		Natural BC	
		Elastic	El. Linear	Elastic	El. Linear
Axial	$\frac{1}{2}\int_0^L \frac{P^2}{AE}dx$	$A\int_0^L \sigma\delta\overline{\varepsilon}dx$	$\int_0^L \underbrace{E\frac{du}{dx}}_{\sigma}\underbrace{\frac{d(\delta\overline{u})}{dx}}_{\delta\overline{\varepsilon}}\underbrace{Adx}_{d\Omega}$	$A\int_0^L \delta\overline{\sigma}\varepsilon dx$	$\int_0^L \underbrace{\delta\overline{P}}_{\delta\overline{\sigma}}\underbrace{\frac{P}{AE}}_{\varepsilon}dx$
Flexure	$\frac{1}{2}\int_0^L \frac{M^2}{EI_z}dx$	$\int_0^L M\delta\overline{\phi}dx$	$\int_0^L \underbrace{EI_z\frac{d^2v}{dx^2}}_{\sigma}\underbrace{\frac{d^2(\delta\overline{v})}{dx^2}}_{\delta\overline{\varepsilon}}dx$	$\int_0^L \delta\overline{M}\phi dx$	$\int_0^L \underbrace{\delta\overline{M}}_{\delta\overline{\sigma}}\underbrace{\frac{M}{EI_z}}_{\varepsilon}dx$
P	$\Sigma_i\frac{1}{2}P_i\Delta_i$	$\Sigma_i P_i\delta\overline{\Delta}_i$		$\Sigma_i\delta\overline{P}_i\Delta_i$	
M	$\Sigma_i\frac{1}{2}M_i\theta_i$	$\Sigma_i M_i\delta\overline{\theta}_i$		$\Sigma_i\delta\overline{M}_i\theta_i$	
w	$\int_0^L w(x)v(x)dx$	$\int_0^L w(x)\delta\overline{v}(x)dx$		$\int_0^L \delta\overline{w}(x)v(x)dx$	

Formulation	Potential Energy Displacement	Complementary Force
Axial	$\frac{1}{2}\int_0^L E\left(\frac{du}{dx}\right)^2dx$	$\frac{1}{2}\int_0^L \frac{P^2}{AE}dx$
Flexural	$\frac{1}{2}\int_0^L EI_z(v'')^2dx$	$\frac{1}{2}\int_0^L \frac{M^2}{EI_z}dx$
P	$\Sigma_i P_i\Delta_i$	
M	$\Sigma_i M_i\theta_i$	
w	$\int_0^L w(x)v(x)dx$	

Summary II

Need to derive potential energy in terms of displacements in beamer and book

Strong/Weak; Natural Essential

Strong/Weak We will refer to a strong form a derivation stemming from a differential equation, and one which is exactly satisfied.
The weak form will be only satisfied in an average sense over a volume Ω.

Boundary Conditions A more detailed coverage of B.C. entails calculus of variation, and derivation of the Euler equation associated with a potential.

Γ	Traction	Displ.	Math.	Structural Mechanics			DOF
Γ_t	$\mathtt{t}\surd$	$\mathtt{u}^?$	Dirichlet	Essential	Primary	Kinematic	Free
Γ_u	$\mathtt{t}^?$	$\mathtt{u}\surd$	Neuman	Natural	Secondary	Static	Fixed/Constrained

Principle of Virtual Work and Complementary Virtual Work

- The principles of Virtual Work and Complementary Virtual Work relate

  force systems which satisfy the requirements of equilibrium
  deformation systems which satisfy the requirement of compatibility.

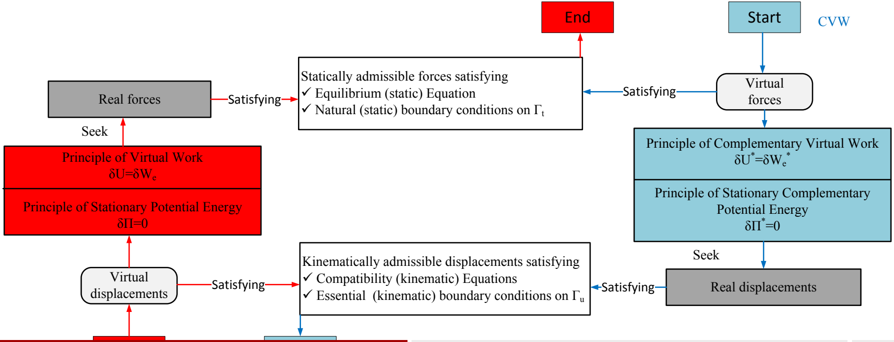
	Force		Deformation		IVW	Formulation
	External	Internal	External	Internal		
1	$\delta\overline{\mathrm{p}}$	$\delta\overline{\sigma}$	du	$d\varepsilon$	$\delta\overline{U}^*$	CVW/Flexibility
2	dp	$d\sigma$	$\delta\overline{\mathrm{u}}$	$\delta\overline{\varepsilon}$	$\delta\overline{U}$	VW/Stiffness

- The principle of Complementary Virtual Work (of Principle of Virtual Force) is what we have already seen previously (unit force method).
- The Principle of Virtual work is new, and is at the basis of the finite element method.

Approaches

Principle	Real	BC	Virtual	BC	Proves
VW	Equilibrium Ω	Natural Γ_t	Kinematic Ω	Essential Γ_u	$\delta\overline{U} = \delta\overline{W}_e$
CVW	Kinematic Ω	Essential Γ_u	EquilibriumΩ	Natural Γ_t	$\delta\overline{U}^* = \delta\overline{W}_e^*$

Principle	Primary Variable Real & Virtual	Satisfying (Strong)	on BC	Apply	Weak Form of
VW	Displacements	Kinematic	Essential	$\delta\overline{U} = \delta\overline{W}_e$	Equilibrium
CVW	Forces	Equilibrium	Natural	$\delta\overline{U}^* = \delta\overline{W}_e^*$	Kinematic



Principle of Virtual Work; Derivation I

1. Derivation of the principle of virtual work starts with the assumption that forces are in equilibrium and satisfaction of the natural (tractions) boundary conditions.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0 \quad (14)$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + b_y = 0 \quad (15)$$

where $\mathbf{b}$ representing the body force.

2. In matrix form, this can be rewritten as

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = 0 \quad (16)$$

or Strong Form

$$\underbrace{\mathbf{L}^T \boldsymbol{\sigma}}_{\nabla.\boldsymbol{\sigma}} + \mathbf{b} = 0 \quad (17)$$

Principle of Virtual Work; Derivation II

3. The surface Γ of the solid can be decomposed into two parts Γ_t and Γ_u

$$\Gamma = \Gamma_t \cup \Gamma_u$$

where tractions and displacements are respectively specified.

4. Essential B.C.

$$\mathbf{t} - \hat{\mathbf{t}} = 0 \quad \text{on } \Gamma_t \text{ Essential B.C.} \tag{18}$$

where $\hat{t}$ are known traction along Γ_t.

5. Equations 17 and 18 constitute a statically admissible stress field.

6. We are going to enforce satisfaction of the local condition of equilibrium Eq. 17 and the static boundary condition Eq. 18 in global (or integral/weak) form. This is accomplished by multiplying both equations by a virtual displacement $\delta\overline{\mathbf{u}}$

Principle of Virtual Work; Derivation III

7. The Weak Form is thus given by

$$-\int_{\Omega} \delta\overline{\mathrm{u}}^{T} \underbrace{\left(\mathrm{L}^{T}\sigma + \mathrm{b}\right)}_{\text{Equil.}} d\Omega + \int_{\Gamma_t} \delta\overline{\mathrm{u}}^{T} \underbrace{(\mathrm{t} - \hat{\mathrm{t}})}_{\text{Essential B.C.}} d\Gamma = 0 \tag{19}$$

8. An important requirement on the (virtual) displacements, is that they must satisfy the requirement of compatibility (by contrast, in the principle of complementary virtual work, stresses had to be statically admissible). The virtual displacements must satisfy the essential boundary condition:

$$\mathrm{L}\delta\overline{\mathrm{u}} = \mathrm{div}\delta\overline{\mathrm{u}} = \delta\overline{\varepsilon} \tag{20}$$

9. Focus on $\int_{\Gamma_t} \delta\overline{\mathrm{u}}^{T}\mathrm{t}d\Gamma$

$$\int_{\Gamma_t} \delta\overline{\mathrm{u}}^{T}\mathrm{t}d\Gamma = \int_{\Gamma} \delta\overline{\mathrm{u}}^{T}\mathrm{t}d\Gamma - \int_{\Gamma_u} \delta\overline{\mathrm{u}}^{T}\mathrm{t}d\Gamma \tag{21}$$

and we seek to convert into a volume integral through Gauss Theorem.

Principle of Virtual Work; Derivation IV

10. Recall the definition of the traction vector

$$\mathbf{t} = \boldsymbol{\sigma}.\mathbf{n} \quad \text{or} \quad t_i = \sigma_{ij} n_j \tag{22}$$

applying Gauss theorem we obtain

$$\int_\Gamma \delta\overline{\mathbf{u}}^T \mathbf{t} d\Gamma = \int_\Gamma (\delta\overline{\mathbf{u}}^T \boldsymbol{\sigma}) \mathbf{n} d\Gamma = \int_\Omega \text{div}(\delta\overline{\mathbf{u}}^T \boldsymbol{\sigma}) d\Omega \tag{23}$$

$$= \int_\Omega \text{div}\delta\overline{\mathbf{u}}^T \boldsymbol{\sigma} d\Omega + \int_\Omega \delta\overline{\mathbf{u}}^T \text{div}\boldsymbol{\sigma} d\Omega \tag{24}$$

However, $\text{div}\boldsymbol{\sigma} = \mathbf{L}^T \boldsymbol{\sigma}$ thus

$$\int_\Gamma \delta\overline{\mathbf{u}}^T \mathbf{t} d\Gamma = \int_\Omega \text{div}\delta\overline{\mathbf{u}}^T \boldsymbol{\sigma} d\Omega + \int_\Omega \delta\overline{\mathbf{u}}^T \mathbf{L}^T \boldsymbol{\sigma} d\Omega \tag{25}$$

Principle of Virtual Work; Derivation V

11. Following some skipped exciting derivation, the above equation yields the Principle of Virtual Work (Displacements)

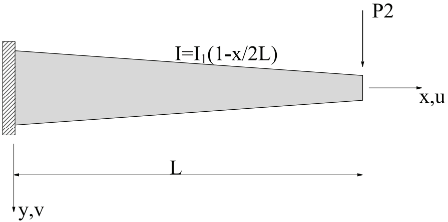
$$\underbrace{\int_\Omega \delta\overline{\varepsilon}^T \sigma d\Omega}_{-\delta\overline{W}_i=\delta\overline{U}_i} \underbrace{- \int_\Omega \delta\overline{u}^T b d\Omega - \int_{\Gamma_t} \delta\overline{u}^T \hat{t} d\Gamma}_{-\delta\overline{W}_e} = 0 \Rightarrow \delta\overline{U}_i = \delta\overline{W}_e \tag{26}$$

A deformable system is in equilibrium (Eq. 17) if the sum of the external virtual work and the internal virtual work is zero for virtual displacements $\delta\overline{u}$ that satisfy the kinematic equation and kinematic boundary conditions (Eq. 20).

12. For one dimensional elements, this reduces to

$$\underbrace{\int \sigma\delta\overline{\varepsilon} d\Omega}_{\delta\overline{U}} = \underbrace{P\delta\overline{v}}_{\delta\overline{W}} \tag{27}$$

Example; PVW I



- In applying the PVW, we need to have an approximation of the actual displacement v and the virtual one $\delta\overline{v}$. Those expressions must satisfy the essential boundary conditions (displacement and slope for beams).
- The approximate solutions proposed to this problem are

$$v = \left(1 - \cos\frac{\pi x}{2L}\right) v_2 \quad (28)$$

$$v = \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] v_2 \quad (29)$$

- They satisfy the essential B.C: $v = v' = 0$ at $x = 0$.
- We consider 3 cases:

Solution	Real	Virtual
1	Eqn. 50	Eqn. 51
2	Eqn. 50	Eqn. 50
3	Eqn. 51	Eqn. 51

Example; PVW II

- Application of the PVW requires evaluation of the functions second derivatives.

	Trigonometric (Eqn. 50)	Polynomial (Eqn. 51)
v	$\left(1-\cos\frac{\pi x}{2L}\right)v_2$	$\left[3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3\right]v_2$
$\delta\overline{v}$	$\left(1-\cos\frac{\pi x}{2L}\right)\delta\overline{v}_2$	$\left[3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3\right]\delta\overline{v}_2$
v''	$\frac{\pi^2}{4L^2}\cos\frac{\pi x}{2L}v_2$	$\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)v_2$
$\delta\overline{v}''$	$\frac{\pi^2}{4L^2}\cos\frac{\pi x}{2L}\delta\overline{v}_2$	$\left[\frac{6}{L^2}-\frac{12x}{L^3}\right]\delta\overline{v}_2$

$$\delta\overline{U}=\int_0^L EI_z\frac{d^2v}{dx^2}\frac{d^2(\delta\overline{v})}{dx^2}dx; \qquad \delta\overline{W}=P_2\delta\overline{v}_2$$

Example; PVW III

Solution 1:

$$\begin{aligned}
\delta\overline{U} &= \int_0^L \underbrace{\frac{\pi^2}{4L^2}\cos\left(\frac{\pi x}{2L}\right) v_2}_{v''} \underbrace{\left(\frac{6}{L^2} - \frac{12x}{L^3}\right)\delta\overline{v}_2}_{\delta\overline{v}''} \underbrace{EI_1\left(1 - \frac{x}{2L}\right)}_{EI} \mathrm{d}x \\
&= \frac{3\pi EI_1}{2L^3}\left[1 - \frac{10}{\pi} + \frac{16}{\pi^2}\right] v_2\delta\overline{v}_2 \\
\delta\overline{W} &= P_2\delta\overline{v}_2
\end{aligned}$$

which yields:

$$\boxed{v_2 = \frac{P_2L^3}{2.648EI_1}}$$

Example; PVW IV

Solution 2:

$$\begin{aligned}
\delta\overline{U} &= \int_0^L \underbrace{\frac{\pi^4}{16L^4}\cos^2\left(\frac{\pi x}{2L}\right)v_2}\underbrace{\delta\overline{v}_2 EI_1\left(1-\frac{x}{2L}\right)}dx \\
&= \frac{\pi^4 EI_1}{32L^3}\left(\frac{3}{4}+\frac{1}{\pi^2}\right)v_2\delta\overline{v}_2 \\
\delta\overline{W} &= P_2\delta\overline{v}_2
\end{aligned}$$

which yields:

$$\boxed{v_2 = \frac{P_2L^3}{2.57EI_1}}$$

Example; PVW V

Solution 3:

$$
\begin{aligned}
\delta\overline{U} &= \int_0^L \left(\frac{6}{L^2} - \frac{12x}{L^3}\right)^2 \left(1 - \frac{x}{2L}\right) EI_1 \delta\overline{v}_2 v_2 dx \\
&= \frac{9EI}{L^3} v_2 \delta\overline{v}_2 \\
\delta\overline{W} &= P_2 \delta\overline{v}_2
\end{aligned}
$$

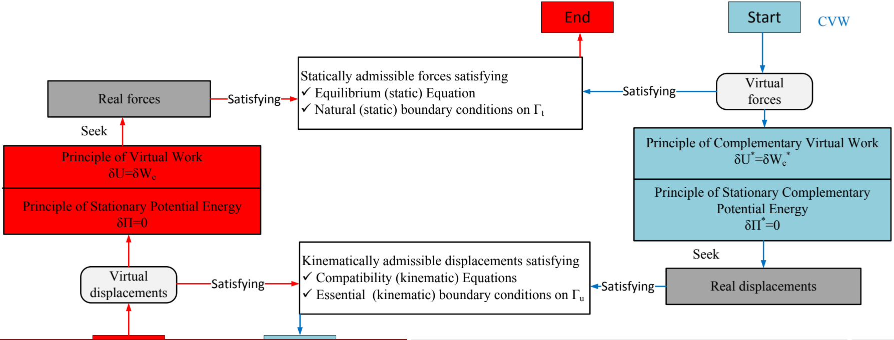
which yields:

$$\boxed{v_2 = \frac{P_2 L^3}{9EI_1}}$$

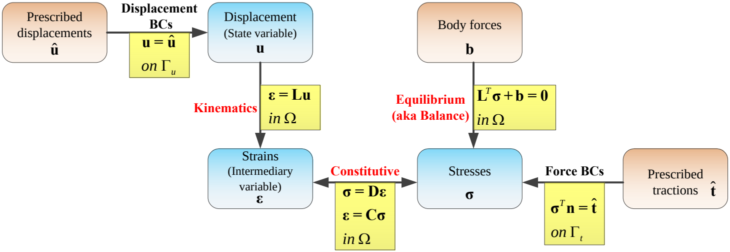
Summary

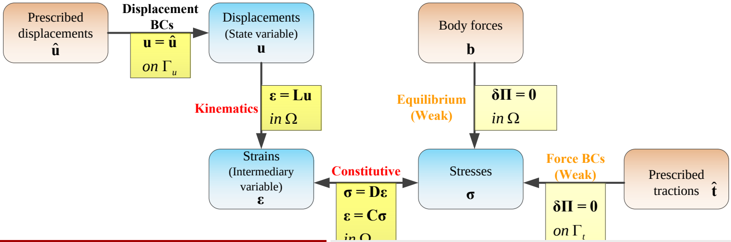
Principle	Real	BC	Virtual	BC	Proves
VW	Equilibrium Ω	Natural Γ_t	Kinematic Ω	Essential Γ_u	$\delta\overline{U} = \delta\overline{W}_e$
CVW	Kinematic Ω	Essential Γ_u	EquilibriumΩ	Natural Γ_t	$\delta\overline{U}^* = \delta\overline{W}_e^*$

Principle	Primary Variable Real & Virtual	Satisfying (Strong)	on BC	Apply	Weak Form of
VW	Displacements	Kinematic	Essential	$\delta\overline{U} = \delta\overline{W}_e$	Equilibrium
CVW	Forces	Equilibrium	Natural	$\delta\overline{U}^* = \delta\overline{W}_e^*$	Kinematic



Tonti Diagrams





Principle of Total and Complementary Potential Energy

- A completely different but related approach will now be presented.
- Rather than convoluting real and virtual quantities, we will simply seek to minimize the total (or complementary) potential energy.
- Those two principles will be derived from those of the virtual strain energy (and the inverse operation is naturally possible).
- The δ operator will assume its full mathematical meaning: differential, whereas before it implied a virtual quantity.

Total Potential Energy I

- If U_0 is a potential function, we take its differential $\delta U_0 = \frac{\partial U_0}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij}$ and the fundamental theorem of calculus states that $\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x f(u)\mathrm{d}u = f(x)$. Thus

$$\left.\begin{array}{rcl} \frac{\delta U_0}{\delta \varepsilon_{ij}} & = & \frac{\partial U_0}{\partial \varepsilon_{ij}} \\ U_0 & = & \int_0^{\varepsilon_{ij}} \sigma_{ij} \mathrm{d}\varepsilon_{ij} \\ \frac{\delta}{\delta \varepsilon_{ij}} \int_0^x \sigma_{ij} \mathrm{d}\varepsilon_{ij} & = & \sigma_{ij} \end{array}\right\} \frac{\delta U_0}{\delta \varepsilon_{ij}} = \sigma_{ij} \tag{30}$$

- We now define the variation of the strain energy density at a point (Note that the variation of strain energy density is, $\delta U_0 = \sigma_{ij} \delta \varepsilon_{ij}$, and the variation of the strain energy itself is $\delta U = \int_\Omega \delta U_0 \mathrm{d}\Omega$.) Thus

$$\delta U_0 = \sigma_{ij} \delta \varepsilon_{ij} \tag{31}$$

Total Potential Energy II

- The principle of virtual work $\int_{\Omega}\delta\overline{\varepsilon}_{ij}\sigma_{ij}\mathrm{d}\Omega-\int_{\Omega}\delta\overline{u}_i b_i\mathrm{d}\Omega-\int_{\Gamma_t}\delta\overline{u}_i\hat{t}_i\mathrm{d}\Gamma=0$ can now be rewritten as

$$\int_{\Omega}\delta U_0\mathrm{d}\Omega-\int_{\Omega}\delta u_i b_i\mathrm{d}\Omega-\int_{\Gamma_t}\delta u_i\hat{t}_i\mathrm{d}\Gamma=0 \tag{32}$$

- If nor the surface tractions, nor the body forces alter their magnitudes or directions during deformation, the previous equation can be rewritten as

$$\delta\left[\underbrace{\overbrace{\int_{\Omega}U_0\mathrm{d}\Omega}^{U}\overbrace{-\int_{\Omega}u_i b_i\mathrm{d}\Omega-\int_{\Gamma_t}u_i\hat{t}_i\mathrm{d}\Gamma}^{-\mathcal{W}_{\urcorner}}}_{\Pi}\right]=0 \tag{33}$$

Total Potential Energy III

- Comparing this last equation, with $\Pi \stackrel{\text{def}}{=} U - \mathcal{W}_e = \int_\Omega U_0 \mathrm{d}\Omega - \left(\int_\Omega \mathbf{u}\mathbf{b}\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{u}\hat{\mathbf{t}}\mathrm{d}\Gamma + \mathbf{u}\mathbf{P} \right)$ we show that the variation of the potential energy is zero.

$$\delta\Pi = 0 \tag{34}$$

- The principle of stationary value of the potential energy can now be stated as follows:

  Of all kinematically admissible deformations (displacements satisfying the essential boundary conditions), the actual deformations (those which correspond to stresses which satisfy equilibrium) are the ones for which the total potential energy assumes a stationary value.

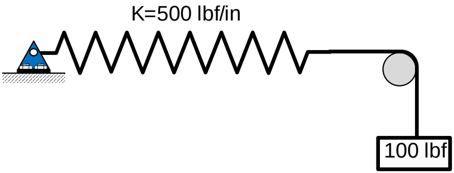
Total Potential Energy IV

- For problems involving multiple degrees of freedom,

$$\delta\Pi = \frac{\partial\Pi}{\partial\Delta_1}\delta\Delta_1 + \frac{\partial\Pi}{\partial\Delta_2}\delta\Delta_2 + \ldots + \frac{\partial\Pi}{\partial\Delta_n}\delta\Delta_n = 0 \tag{35}$$

or n equations with n unknowns.

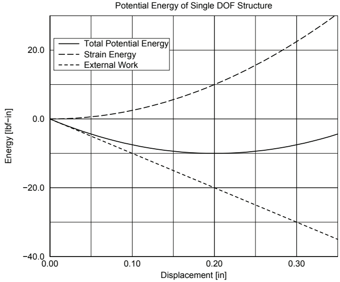
Example 1



The strain energy U and potential of the external work $\mathcal{W}$ are given by

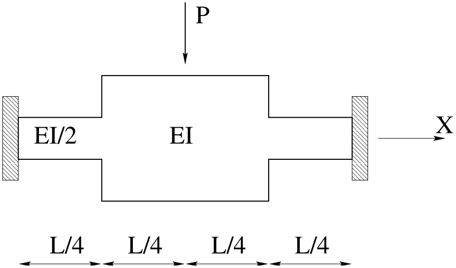
$$U = \frac{1}{2}u(Ku) = 250u^2 \qquad \mathcal{W}_e = mgu = 100u$$

Note that there is no 1/2 in $\mathcal{W}_e$ because the force is constant during the work. Thus the total potential energy is given by $\Pi = 250u^2 - 100u$ and will be stationary for $\delta\Pi = \frac{d\Pi}{du} = 0 \Rightarrow 500u - 100 = 0 \Rightarrow$ $\boxed{u = 0.2 \text{ in}}$



Obviously, similar result could have been obtained from statics.

Example 2 I



- Let us assume that
$v = a_1 x^3 + a_2 x^2 + a_3 x + a_4$
- This solution must satisfy the essential B.C.: $v = v' = 0$ at $x = 0$; Secondly, $v = v_{max}$ and $v' = 0$ at $x = \frac{L}{2}$.
- This will be enforced by determining the four parameters in terms of a single unknown quantity (4 equations and 4 B.C.'s):

$$
\begin{array}{lll}
@x = 0 & v = 0 & \Rightarrow a_4 = 0 \\
@x = 0 & \frac{dv}{dx} = 0 & \Rightarrow a_3 = 0 \\
@x = \frac{L}{2} & v = v_{max} & \Rightarrow v_{max} = a_1 \frac{L^3}{8} + \\
@x = \frac{L}{2} & \frac{dv}{dx} = 0 & \Rightarrow \frac{3}{4} a_1 L^2 + a_2 L =
\end{array}
$$

- upon substitution, we obtain:

$$v = \left(-\frac{16x^3}{L^3} + \frac{12x^2}{L^2} \right) v_{max} \quad (36)$$

Example 2 II

- Hence, in this problem the solution is in terms of only one unknown variable v_{max}.
- In order to apply the principle of Minimum Potential Energy we should evaluate:

Internal Strain Energy U: for flexural members, expressed in terms of displacements (a must in this method) is given by

$$U = 2\left[\frac{1}{2}\int_0^{L/2} E\left(\frac{d^2v}{dx^2}\right)^2 I_z dx\right] \text{ thus we must evaluate } \frac{d^2v}{dx^2}:$$

$$\frac{dv}{dx} = \left(-\frac{48x^2}{L^3} + \frac{24x}{L^2}\right) v_{max}; \qquad \frac{d^2v}{dx^2} = -\frac{24}{L^2}\left(1 - \frac{4x}{L}\right) v_{max}$$

Substituting

$$\frac{U}{2} = \frac{E}{2}\int_0^{\frac{L}{4}} \frac{24^2}{L^4}\left(1 - \frac{4x}{L}\right)^2 v_{max}^2 \frac{I_z}{2} dx + \frac{E}{2}\int_{\frac{L}{4}}^{\frac{L}{2}} \frac{24^2}{L^4}\left(1 - \frac{4x}{L}\right)^2 v_{max}^2 I_z$$

Potential of the External Work $\mathcal{W}_e$: For a point load, $\mathcal{W}_e = Pv_{max}$

Example 2 III

- Finally,

$$\frac{\partial \Pi}{\partial v_{max}} = 0; \qquad \frac{\partial U}{\partial v_{max}} - \frac{\partial \mathcal{W}_\rceil}{\partial v_{max}} = 0 \qquad \frac{144 E I_z}{L^3} v_{max} = P \qquad \Rightarrow v_{max} = \boxed{\frac{PL^3}{144 E I_z}}$$

Total Complementary Potential Energy

Mildly relevant, not covered.

Rayleigh Ritz; Derivation I

- In the minimization of the total potential energy, we expressed the potential energy in terms of physical quantities (displacements/rotations) at certain nodes, and then stated that the potential is stationary, i.e.

$$\delta\Pi = \frac{\partial\Pi}{\partial\Delta_1}\delta\Delta_1 + \frac{\partial\Pi}{\partial\Delta_2}\delta\Delta_2 + \ldots + \frac{\partial\Pi}{\partial\Delta_n}\delta\Delta_n \tag{37}$$

- A more general (and still approximate) approach is to express the displacements, and thus the potential in terms of unknown coefficients in a function such as:

$$u_1 \approx \sum_{i=1}^{n} c_i^1 \phi_i^1 + \phi_0^1 \quad u_2 \approx \sum_{i=1}^{n} c_i^2 \phi_i^2 + \phi_0^2 \quad u_3 \approx \sum_{i=1}^{n} c_i^3 \phi_i^3 + \phi_0^3 \tag{38}$$

where c_i^j denote undetermined parameters, and ϕ are appropriate functions of positions.

Rayleigh Ritz; Derivation II

- In the PTPE, we had a single displacement field in terms of *n* variables (unknown displacements), now we have multiple expressions of the displacement field in terms of *n* coefficients *c*.
- ϕ should satisfy three conditions
    1. Be continuous.
    2. Must be admissible, i.e. satisfy the essential boundary conditions (the natural boundary conditions are included already in the variational statement. However, if ϕ also satisfy them, then better results are achieved).
    3. Must be independent and complete (which means that the exact displacement and their derivatives that appear in Π can be arbitrary matched if enough terms are used. Furthermore, lowest order terms must also be included).
- In general ϕ is a polynomial or trigonometric function.

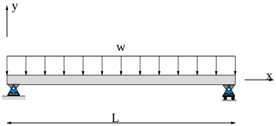
Rayleigh Ritz; Derivation III

- We determine the parameters c_i^j satisfying the stationarity of Π for arbitrary variations δc_i^j. or $\delta\Pi(u_1, u_2, u_3) = \sum_{i=1}^{n} \left(\frac{\partial \Pi}{\partial c_i^1} \delta c_i^1 + \frac{\partial \Pi}{\partial c_i^2} \delta c_i^2 + \frac{\partial \Pi}{\partial c_i^3} \delta c_i^3 \right) = 0$ for arbitrary and independent variations of δc_i^1, δc_i^2, and δc_i^3, thus it follows that

$$\frac{\partial \Pi}{\partial c_i^j} = 0 \qquad i = 1, 2, \cdots, n; j = 1, 2, 3 \tag{39}$$

- Thus we obtain a total of $3n$ linearly independent simultaneous equations. From these displacements, we can then determine strains and stresses (or internal forces). Hence we have replaced a problem with an infinite number of d.o.f by one with a finite number.

Rayleigh Ritz; Example I



let us assume a solution given by the following infinite series:

$$v = a_1 x(L - x) + a_2 x^2 (L - x)^2 + \dots \tag{40}$$

for this particular solution, let us retain only the first term:

$$v = a_1 x(L - x) \tag{41}$$

We observe that:

1. Contrarily to the previous example problem the essential (or geometric) B.C. are immediately satisfied at both $x = 0$ and $x = L$.

Rayleigh Ritz; Example II

2. We can keep v in terms of a_1 and take $\frac{\partial \Pi}{\partial a_1} = 0$ (If we had left v in terms of a_1 and a_2 we should then have to take both $\frac{\partial \Pi}{\partial a_1} = 0$, and $\frac{\partial \Pi}{\partial a_2} = 0$).
3. Or we can solve for a_1 in terms of v_{max} at $x = \frac{L}{2}$ and take $\frac{\partial \Pi}{\partial v_{max}} = 0$.

$$\Pi = U - \mathcal{W}_\rceil \tag{42}$$

$$= \int_0^L \frac{EI_z}{2} \left(\frac{d^2 v}{dx^2} \right)^2 dx - \int_0^L w v(x) dx \tag{43}$$

$$= \int_0^L \left[\frac{EI_z}{2} (-2a_1)^2 - a_1 w x (L - x) \right] dx \tag{44}$$

$$= \frac{EI_z}{2} 4 a_1^2 L - a_1 w \frac{L^3}{2} + a_1 w \frac{L^3}{3} \tag{45}$$

$$= 2 a_1^2 EI_z L - \frac{a_1 w L^3}{6} \tag{46}$$

Rayleigh Ritz; Example III

If we now take $\frac{\partial \Pi}{\partial a_1} = 0$, we would obtain:

$$4a_1 EI_z L - \frac{wL^3}{6} = 0 \quad (47)$$

$$a_1 = \frac{wL^2}{24EI_z} \quad (48)$$

Having solved the displacement field in terms of a_1, we now determine v_{max} at $\frac{L}{2}$:

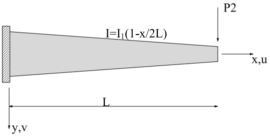
$$v = \underbrace{\frac{wL^4}{24EI_z}}_{a_1}\left(\frac{x}{L} - \frac{x^2}{L^2}\right) = \boxed{\frac{wL^4}{96EI_z}} \quad (49)$$

This is to be compared with the exact value of $v_{\text{max}}^{exact} = \frac{5}{384}\frac{wL^4}{EI_z} = \frac{wL^4}{76.8EI_z}$ which constitutes $\approx 17\%$ error.

Note: If two terms were retained, then we would have obtained: $a_1 = \frac{wL^2}{24EI_z}$ and $a_2 = \frac{w}{24EI_z}$ and v_{max} would be equal to v_{max}^{exact}. (Why?)

Example; PVW I

In applying the PVW, we need to have an approximation of the actual displacement v and the virtual one δv. Those expressions must satisfy the essential boundary conditions (displacement and slope for beams).



The approximate solutions proposed to this problem are

$$v = \left(1 - \cos\frac{\pi x}{2L}\right) v_2 \tag{50}$$

$$v = \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] v_2 \tag{51}$$

They satisfy the essential B.C: $v = v' = 0$ at $x = 0$.
We consider 3 cases:

Solution	Real	Virtual
1	Eqn. 50	Eqn. 51
2	Eqn. 50	Eqn. 50
3	Eqn. 51	Eqn. 51

Example; PVW II

Application of the PVW requires evaluation of the functions second derivatives.

	Trigonometric (Eqn. 50)	Polynomial (Eqn. 51)
v	$\left(1 - \cos\frac{\pi x}{2L}\right) v_2$	$\left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] v_2$
δv	$\left(1 - \cos\frac{\pi x}{2L}\right) \delta v_2$	$\left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] \delta v_2$
v''	$\frac{\pi^2}{4L^2}\cos\frac{\pi x}{2L} v_2$	$\left(\frac{6}{L^2} - \frac{12x}{L^3}\right) v_2$
$\delta v''$	$\frac{\pi^2}{4L^2}\cos\frac{\pi x}{2L} \delta v_2$	$\left[\frac{6}{L^2} - \frac{12x}{L^3}\right] \delta v_2$

$$\delta U = \int_0^L EI_z \frac{d^2 v}{dx^2}\frac{d^2(\delta v)}{dx^2}dx; \qquad \delta W = P_2 \delta v_2$$

Solution 1:

$$\begin{aligned}
\delta U &= \int_0^L \underbrace{\frac{\pi^2}{4L^2}\cos\left(\frac{\pi x}{2L}\right) v_2}_{v''} \underbrace{\left(\frac{6}{L^2} - \frac{12x}{L^3}\right)\delta v_2}_{\delta v''} \underbrace{EI_1\left(1 - \frac{x}{2L}\right)}_{EI} dx = \frac{3\pi EI_1}{2L^3}\left[1 - \frac{10}{\pi} + \frac{16}{\pi^2}\right] v_2 \delta v_2 \\
\delta W &= P_2 \delta v_2
\end{aligned}$$

Example; PVW III

which yields:

$$\boxed{v_2 = \frac{P_2 L^3}{2.648 EI_1}}$$

Solution 2:

$$\begin{aligned}
\delta U &= \int_0^L \underbrace{\frac{\pi^4}{16L^4}\cos^2\left(\frac{\pi x}{2L}\right) v_2}\,\underbrace{\delta v_2 EI_1 \left(1 - \frac{x}{2L}\right)} \mathrm{d}x = \frac{\pi^4 EI_1}{32L^3}\left(\frac{3}{4} + \frac{1}{\pi^2}\right) v_2 \delta v_2 \\
\delta W &= P_2 \delta v_2
\end{aligned}$$

which yields:

$$\boxed{v_2 = \frac{P_2 L^3}{2.57 EI_1}}$$

Solution 3:

$$\begin{aligned}
\delta U &= \int_0^L \left(\frac{6}{L^2} - \frac{12x}{L^3}\right)^2 \left(1 - \frac{x}{2L}\right) EI_1 \delta v_2 v_2 \mathrm{d}x = \frac{9EI}{L^3} v_2 \delta v_2 \\
\delta W &= P_2 \delta v_2
\end{aligned}$$

which yields:

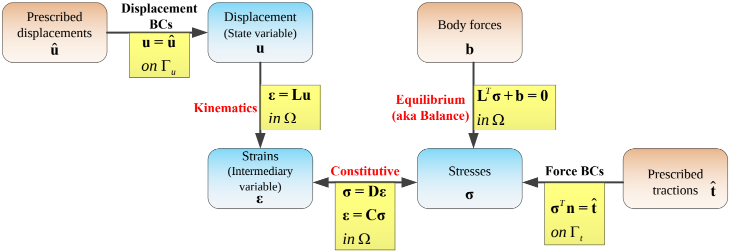
$$\boxed{v_2 = \frac{P_2 L^3}{9 EI_1}}$$

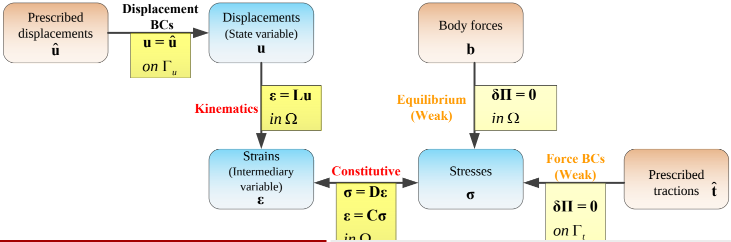
Summary

Princ.	Real/Weak			Virtual/Strong			Proves
	Var.	Satisfies	BC	Var.	Satisfies	BC	
VW	$\boldsymbol{\sigma}$	Equil.	Γ_t	$\mathbf{u}$	Kinem.	Γ_u	$\delta U = \delta W_e$
CVW	$\mathbf{u}$	Kinem.	Γ_u	$\boldsymbol{\sigma}$	Equil.	Γ_t	$\delta U^* = \delta W_e^*$

Γ_t: Natural B.C.; Γ_u: Essential B.C.
Note: in VW displacements do not satisfy equilibrium, ($M \neq EI\frac{d^2v}{dx^2}$); more about this later.

Tonti Diagram





Shape Functions; Definitions I

Expression for the generalized displacement (translation or rotation), Δ at any point in terms of all its known nodal ones, $\overline{\Delta}$.

$$\Delta = \sum_{i=1}^{n} N_i(x)\overline{\Delta}_i = \lfloor \mathrm{N}(x) \rfloor \{\overline{\Delta}\}$$

$\overline{\Delta}_i$ is the (generalized) nodal displacement corresponding to d.o.f i

1. N_i is an interpolation function, or shape function which has the following characteristics: $N_i = 1$ at node i and $N_i = 0$ at node j where $i \neq j$.
2. Summation of N at any point is equal to unity $\Sigma N = 1$.

Shape functions should

1. Be continuous, of the type required by the variational principle.
2. Exhibit rigid body motion (i.e. $v = a_1 + \ldots$)
3. Exhibit constant strain.

Shape functions should be complete, and meet the same requirements as the coefficients of the Rayleigh Ritz method. Shape functions can often be written in non-dimensional coordinates (i.e. $\xi = \frac{x}{l}$). This will be exploited later by the so-called isoparametric elements.

Generalization

$$u = a_1 x + a_2 = \underbrace{\lfloor\ x \quad 1\ \rfloor}_{[\mathrm{p}]} \underbrace{\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}}_{\{\mathrm{a}\}}$$

where $[\mathrm{p}]$ corresponds to the polynomial approximation, and $\{\mathrm{a}\}$ is the coefficient vector. We next apply the boundary conditions:

$$\underbrace{\begin{Bmatrix} \overline{u}_1 \\ \overline{u}_2 \end{Bmatrix}}_{\{\overline{\Delta}\}} = \underbrace{\begin{bmatrix} 0 & 1 \\ L & 1 \end{bmatrix}}_{[\mathcal{L}]} \underbrace{\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}}_{\{\mathrm{a}\}}$$

Following inversion of $[\mathcal{L}]$, this leads to

$$\underbrace{\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}}_{\{\mathrm{a}\}} = \underbrace{\frac{1}{L}\begin{bmatrix} -1 & 1 \\ L & 0 \end{bmatrix}}_{[\mathcal{L}]^{-1}} \underbrace{\begin{Bmatrix} \overline{u}_1 \\ \overline{u}_2 \end{Bmatrix}}_{\{\overline{\Delta}\}}$$

Substituting this last equation, we obtain:

$$u = \underbrace{\underbrace{\lfloor\ (1 - \tfrac{x}{L}) \quad \tfrac{x}{L}\ \rfloor}_{[\mathrm{p}][\mathcal{L}]^{-1}}}_{[\mathrm{N}]} \underbrace{\begin{Bmatrix} \overline{u}_1 \\ \overline{u}_2 \end{Bmatrix}}_{\{\overline{\Delta}\}}$$

Hence, the shape functions $[\mathrm{N}]$ can be directly obtained from

$$[\mathrm{N}] = [\mathrm{p}][\mathcal{L}]^{-1}$$

C^1, Flexural Shape Functions I

We have 4 d.o.f.'s, $\{\Delta\}_{4\times 1}$: and hence will need 4 shape functions, N_1 to N_4, and those will be obtained through 4 boundary conditions. Therefore we need to assume a polynomial approximation for displacements of degree 3.

$$
\begin{aligned}
v &= a_1x^3 + a_2x^2 + a_3x + a_4 \\
\theta &= \frac{dv}{dx} = 3a_1x^2 + 2a_2x + a_3
\end{aligned}
$$

Note that v can be rewritten as:

$$
\left\{ \begin{array}{c} v \\ \frac{dv}{dx} \end{array} \right\} = \left[\begin{array}{cccc} x^3 & x^2 & x & 1 \\ 3x^2 & 2x & 1 & 0 \end{array} \right] \underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \right\}}_{\{\mathrm{a}\}}
$$

C^1, Flexural Shape Functions II

We now apply the boundary conditions:

$$
\begin{array}{lll}
v = \overline{v}_1 & \text{at} & x = 0 \\
v = \overline{v}_2 & \text{at} & x = L \\
\theta = \overline{\theta}_1 = \frac{dv}{dx} & \text{at} & x = 0 \\
\theta = \overline{\theta}_2 = \frac{dv}{dx} & \text{at} & x = L
\end{array}
$$

or:

$$
\underbrace{\left\{ \begin{array}{c} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{array} \right\}}_{\{\overline{\Delta}\}} = \underbrace{\left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ L^3 & L^2 & L & 1 \\ 3L^2 & 2L & 1 & 0 \end{array} \right]}_{[\mathcal{L}]} \underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \right\}}_{\{\mathtt{a}\}}
$$

which when inverted yields:

$$
\underbrace{\left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \right\}}_{\{\mathtt{a}\}} = \underbrace{\frac{1}{L^3} \left[\begin{array}{cccc} 2 & L & -2 & L \\ -3L & -2L^2 & 3L & -L^2 \\ 0 & L^3 & 0 & 0 \\ L^3 & 0 & 0 & 0 \end{array} \right]}_{[\mathcal{L}]^{-1}} \underbrace{\left\{ \begin{array}{c} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{array} \right\}}_{\{\overline{\Delta}\}}
$$

C^1, Flexural Shape Functions III

Combining, we obtain:

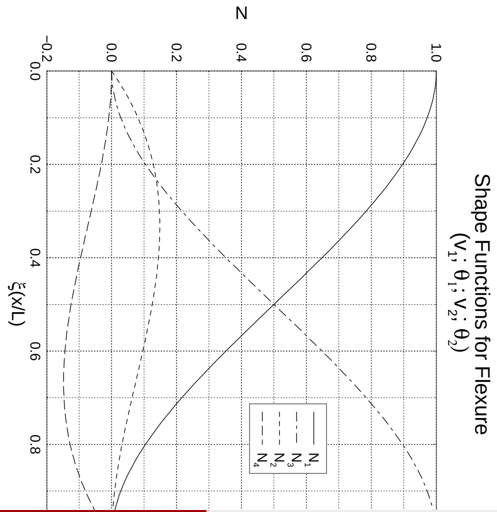
$$\Delta = \underbrace{\lfloor\ x^3 \quad x^2 \quad x \quad 1\ \rfloor}_{[\mathrm{p}]} \underbrace{\frac{1}{L^3}\begin{bmatrix} 2 & L & -2 & L \\ -3L & -2L^2 & 3L & -L^2 \\ 0 & L^3 & 0 & 0 \\ L^3 & 0 & 0 & 0 \end{bmatrix}}_{[\mathcal{L}]^{-1}} \underbrace{\begin{Bmatrix} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{Bmatrix}}_{\{\overline{\Delta}\}}$$

$$= \underbrace{\lfloor\ \underbrace{(1+2\xi^3-3\xi^2)}_{N_1} \quad \underbrace{x(1-\xi)^2}_{N_2} \quad \underbrace{(3\xi^2-2\xi^3)}_{N_3} \quad \underbrace{x(\xi^2-\xi)}_{N_4}\ \rfloor}_{\underbrace{[\mathrm{p}][\mathcal{L}]^{-1}}_{[\mathrm{N}]}} \underbrace{\begin{Bmatrix} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{Bmatrix}}_{\{\overline{\Delta}\}}$$

where $\xi = \frac{x}{L}$. Hence, the shape functions for the flexural element are given by:

$$\begin{aligned} N_1 &= (1+2\xi^3-3\xi^2) \\ N_2 &= x(1-\xi)^2 \\ N_3 &= (3\xi^2-2\xi^3) \\ N_4 &= x(\xi^2-\xi) \end{aligned}$$

C^1, Flexural Shape Functions IV



C^1, Flexural Shape Functions V

Function	$\xi = 0$		$\xi = 1$	
	N_i	$N_{i,x}$	N_i	$N_{i,x}$
$N_1 = (1 + 2\xi^3 - 3\xi^2)$	1	0	0	0
$N_2 = \xi(1-\xi)^2$	0	1	0	0
$N_3 = (3\xi^2 - 2\xi^3)$	0	0	1	0
$N_4 = \xi(\xi^2 - \xi)$	0	0	0	1

The displacements can now be expressed as

$$\left\{ \begin{array}{c} u \\ \theta \end{array} \right\} = \left[\begin{array}{cccc} N_1 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_4 \end{array} \right] \left\{ \begin{array}{c} \overline{u}_1 \\ \overline{\theta}_1 \\ \overline{u}_2 \\ \overline{\theta}_2 \end{array} \right\}$$

Strain Displacement Relations

- The displacement Δ at any point inside an element can be written in terms of the shape functions $\lfloor \mathrm{N} \rfloor$ and the nodal displacements $\{\overline{\Delta}\}$ as $\Delta(x) \stackrel{\text{def}}{=} \lfloor \mathrm{N}(x) \rfloor \{\overline{\Delta}\}$
- The strain is then defined as $\varepsilon(x) \stackrel{\text{def}}{=} [\mathrm{B}(x)]\{\overline{\Delta}\}$ where $[\mathrm{B}]$ is the matrix which relates nodal displacements to strain field and is clearly expressed in terms of derivatives of N.

Strain Displacement Relations; Axial

$$
\begin{aligned}
u(x) &= \underbrace{\lfloor \underbrace{(1-\frac{x}{L})}_{N_1} \quad \underbrace{\frac{x}{L}}_{N_2} \rfloor}_{\lfloor \mathrm{N} \rfloor} \underbrace{\left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \end{array} \right\}}_{\{\overline{\Delta}\}} \\
\varepsilon(x) &= \varepsilon_{xx} = \frac{\mathrm{d}u}{\mathrm{d}x} = \underbrace{\left\lfloor \underbrace{-\frac{1}{L}}_{\frac{\partial N_1}{\partial x}} \quad \underbrace{\frac{1}{L}}_{\frac{\partial N_2}{\partial x}} \right\rfloor}_{[\mathrm{B}]} \underbrace{\left\{ \begin{array}{c} \overline{u}_1 \\ \overline{u}_2 \end{array} \right\}}_{\{\overline{\Delta}\}}
\end{aligned}
$$

Strain Displacement Relations; Flexural Members

Using the shape functions for flexural elements previously derived in

$$
\begin{aligned}
\varepsilon &= \frac{y}{\rho} = y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \\
&= y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \\
&= \underbrace{y\left\lfloor \underbrace{\frac{6}{L^2}(2\xi-1)}_{\frac{\partial^2 N_1}{\partial x^2}} \quad \underbrace{-\frac{2}{L}(3\xi-2)}_{\frac{\partial^2 N_2}{\partial x^2}} \quad \underbrace{\frac{6}{L^2}(-2\xi+1)}_{\frac{\partial^2 N_3}{\partial x^2}} \quad \underbrace{-\frac{2}{L}(3\xi-1)}_{\frac{\partial^2 N_4}{\partial x^2}} \right\rfloor}_{[\mathrm{B}]} \underbrace{\begin{Bmatrix} \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{Bmatrix}}_{\{\overline{\Delta}\}}
\end{aligned}
$$

Virtual Displacement and Strain

In anticipation of the application of the principle of virtual displacement, we define the vectors of virtual displacements and strain in terms of nodal displacements and shape functions:

$$\delta\Delta(x) = [N(x)]\{\delta\overline{\Delta}\} \tag{52}$$

$$\delta\varepsilon(x) = [B(x)]\{\delta\overline{\Delta}\} \tag{53}$$

Element Stiffness Matrix I

$$\{\sigma\} = [D]\{\varepsilon\} - [D]\{\varepsilon^0\} \tag{54}$$

where $[D]$ is the constitutive matrix which relates stress and strain vectors. and $q(x)$ is the load acting on its surface.
Let us now apply the principle of virtual displacement and restate some known relations:

$$\begin{aligned}
\delta U &= \delta W && (55)\\
\delta U &= \int_\Omega \lfloor \delta\varepsilon \rfloor \{\sigma\} d\Omega && (56)\\
\{\sigma\} &= [D]\{\varepsilon\} - [D]\{\varepsilon^0\} && (57)\\
\{\varepsilon\} &= [B]\{\overline{\Delta}\} && (58)\\
\{\delta\varepsilon\} &= [B]\{\delta\overline{\Delta}\} && (59)\\
\lfloor \delta\varepsilon \rfloor &= \lfloor \delta\overline{\Delta} \rfloor [B]^T && (60)
\end{aligned}$$

Combining Eqns. 55, 56, 57, 60, and 58, the internal virtual strain energy is given by:

$$\begin{aligned}
\delta U &= \int_\Omega \underbrace{\lfloor \delta\overline{\Delta} \rfloor [B]^T}_{\lfloor \delta\varepsilon \rfloor} \underbrace{[D][B]\{\overline{\Delta}\}}_{\{\sigma\}} d\Omega - \int_\Omega \underbrace{\lfloor \delta\overline{\Delta} \rfloor [B]^T}_{\lfloor \delta\varepsilon \rfloor} \underbrace{[D]\{\varepsilon^0\}}_{\{\sigma^0\}} d\Omega \\
&= \lfloor \delta\overline{\Delta} \rfloor \int_\Omega [B]^T [D][B]\, d\Omega \{\overline{\Delta}\} - \lfloor \delta\overline{\Delta} \rfloor \int_\Omega [B]^T [D]\{\varepsilon^0\} d\Omega
\end{aligned} \tag{61}$$

Element Stiffness Matrix II

The virtual external work in turn is given by:

$$\delta W = \underbrace{\lfloor \delta \overline{\Delta} \rfloor}_{\text{Virt. Nodal Displ.}} \underbrace{\{\overline{\mathrm{F}}\}}_{\text{Nodal Force}} + \int_l \lfloor \delta \overline{\Delta} \rfloor q(x) \mathrm{d}x \tag{62}$$

combining this equation with $\{\delta \Delta\} = [\mathrm{N}]\{\delta \overline{\Delta}\}$ yields:

$$\delta W = \lfloor \delta \overline{\Delta} \rfloor \{\overline{\mathrm{F}}\} + \lfloor \delta \overline{\Delta} \rfloor \int_0^l [\mathrm{N}]^T q(x)\, \mathrm{d}x \tag{63}$$

Equating the internal strain energy Eqn. 61 with the external work Eqn. 63, we obtain:

$$\underbrace{\lfloor \delta \overline{\Delta} \rfloor \underbrace{\int_\Omega [\mathrm{B}]^T [\mathrm{D}][\mathrm{B}]\, \mathrm{d}\Omega}_{[\mathrm{k}]} \{\overline{\Delta}\} - \lfloor \delta \overline{\Delta} \rfloor \underbrace{\int_\Omega [\mathrm{B}]^T [\mathrm{D}]\{\varepsilon^0\} \mathrm{d}\Omega}_{\{\overline{\mathrm{F}}^0\}}}_{\delta U} = \underbrace{\lfloor \delta \overline{\Delta} \rfloor \{\overline{\mathrm{F}}\} + \lfloor \delta \overline{\Delta} \rfloor \underbrace{\int_0^l [\mathrm{N}]^T q(x)\, \mathrm{d}x}_{\{\overline{\mathrm{F}}^e\}}}_{\delta W} \tag{64}$$

Element Stiffness Matrix III

or

$$\boxed{[\mathrm{k}]\{\overline{\Delta}\} - \{\overline{\mathrm{F}}^{o}\} = \{\overline{\mathrm{F}}\} + \{\overline{\mathrm{F}}^{e}\}} \tag{65}$$

which is the counterpart of Eq. 54.
Canceling out the $\lfloor \delta\overline{\Delta} \rfloor$ term, this is the same equation of equilibrium as the one written earlier on. It relates the (unknown) nodal displacement $\left\{\overline{\Delta}\right\}$, the structure stiffness matrix $[\mathrm{k}]$, the external nodal force vector $\left\{\overline{\mathrm{F}}\right\}$, the distributed element force $\left\{\overline{\mathrm{F}}^{e}\right\}$, and the vector of initial displacement.
From this relation we define:

The element stiffness matrix:

$$\boxed{[\mathrm{k}] = \int_{\Omega} [\mathrm{B}]^{T}[\mathrm{D}][\mathrm{B}]\mathrm{d}\Omega} \tag{66}$$

Element initial force vector:

$$\boxed{\{\overline{\mathrm{F}}^{0}\} = \int_{\Omega} [\mathrm{B}]^{T}[\mathrm{D}]\{\varepsilon^{0}\}\mathrm{d}\Omega} \tag{67}$$

Element equivalent load vector:

$$\boxed{\{\overline{\mathrm{F}}^{e}\} = \int_{0}^{L} [\mathrm{N}]\, q(x)\, \mathrm{d}x} \tag{68}$$

Element Stiffness Matrix IV

and the general equation of equilibrium can be written as:

$$\boxed{[\mathtt{k}]\{\overline{\Delta}\} - \{\overline{\mathtt{F}}^0\} = \{\overline{\mathtt{F}}\} + \{\overline{\mathtt{F}}^e\}} \tag{69}$$

or Internal forces equal external forces

Stress Recovery I

$$\begin{aligned} \{\sigma\} &= [D]\{\varepsilon\} & (70) \\ \{\varepsilon\} &= [B]\{\overline{\Delta}\} & (71) \end{aligned}$$

With the vector of nodal displacement $\{\Delta\}$ known, those two equations would yield:

$$\boxed{\{\sigma\} = [D] \cdot [B]\{\overline{\Delta}\}} \quad (72)$$

We note that the secondary variables (strain and stresses) are derivatives of the primary variables (displacement), and as such may not always be determined with the same accuracy.

Stiffness Matrix of the Truss Element

The shape functions of the truss element were derived earlier:

$$N_1 = 1 - \frac{x}{L}$$
$$N_2 = \frac{x}{L}$$

The corresponding strain displacement relation $[\mathrm{B}]$ is given by:

$$\begin{aligned}\varepsilon_{xx} &= \frac{\mathrm{du}}{\mathrm{d}x} \\ &= \left[\begin{array}{cc} \frac{dN_1}{\mathrm{d}x} & \frac{dN_2}{\mathrm{d}x}\end{array}\right] \\ &= \underbrace{\left[\begin{array}{cc} -\frac{1}{L} & \frac{1}{L}\end{array}\right]}_{[\mathrm{B}]}\end{aligned}$$

For the truss element, the constitutive matrix $[\mathrm{D}]$ reduces to the scalar E; Hence, substituting into Eq. 66, with $\mathrm{d}\Omega = \mathrm{d}A\mathrm{d}x$: $[\mathrm{k}] = \int_\Omega [\mathrm{B}]^T[\mathrm{D}][\mathrm{B}]\mathrm{d}\Omega$ and with $\mathrm{d}\Omega = A\mathrm{d}x$ for element with constant cross sectional area we obtain:

$$[\mathrm{k}] = A\int_0^L \left\{\begin{array}{c} -\frac{1}{L} \\ \frac{1}{L}\end{array}\right\} \cdot E \cdot \lfloor \begin{array}{cc} -\frac{1}{L} & \frac{1}{L}\end{array} \rfloor \mathrm{d}x$$

$$\boxed{\begin{aligned}[\mathrm{k}] &= \frac{AE}{L^2}\int_0^L \left[\begin{array}{cc} 1 & -1 \\ -1 & 1\end{array}\right]\mathrm{d}x \\ &= \frac{AE}{L}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1\end{array}\right]\end{aligned}}$$

Stiffness Matrix of Beam Element

For a beam element, for which we have previously derived the shape functions and the $[\mathrm{B}]$ matrix. Substituting in Eq. 66:

$$[\mathrm{k}] = \int_0^L \int_A [\mathrm{B}]^T [\mathrm{D}][\mathrm{B}]\, y^2\, dA\, dx$$

and noting that $\displaystyle\int_A y^2\, dA = I_z$ Eq. 66

reduces to

$$[\mathrm{k}] = \int_0^L [\mathrm{B}]^T [\mathrm{D}][\mathrm{B}] I_z\, dx$$

For this simple case, we have: $[\mathrm{D}] = E$, thus:

$$[\mathrm{k}] = EI_z \int_0^l [\mathrm{B}]^T [\mathrm{B}]\, dx$$

Using the shape function for the beam element, and noting the change of integration variable from dx to $d\xi$, we obtain

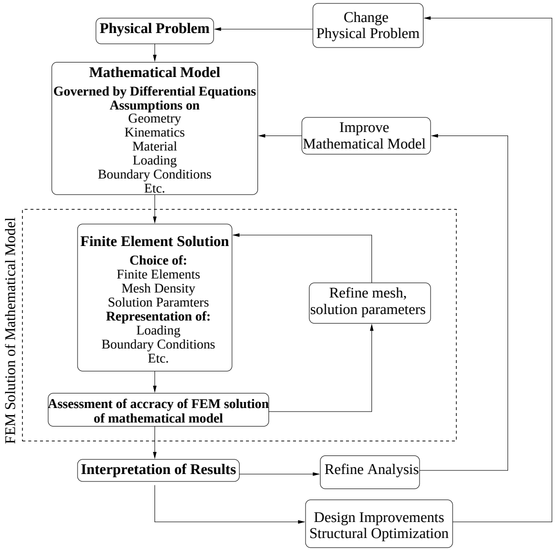
$$[\mathrm{k}] = EI_z \int_0^1 \left\{ \begin{array}{c} \frac{6}{L^2}(2\xi - 1) \\ -\frac{2}{L}(3\xi - 2) \\ \frac{6}{L^2}(-2\xi + 1) \\ -\frac{2}{L}(3\xi - 1) \end{array} \right\} \lfloor \; \frac{6}{L^2}(2\xi - 1) \quad -\frac{2}{L}(3\xi$$

or

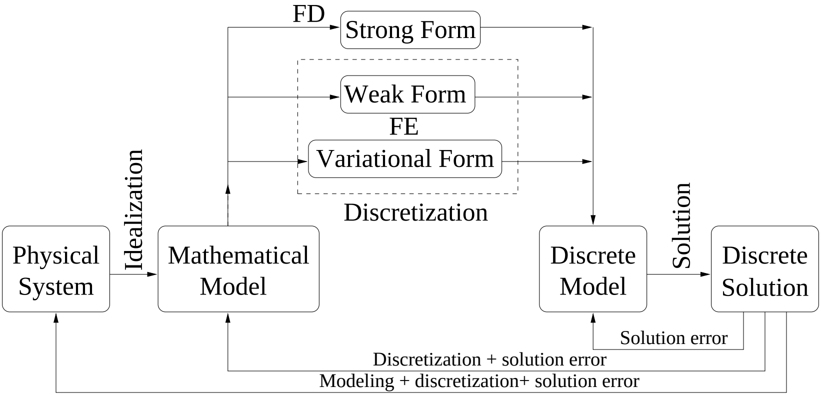
$$[\mathrm{k}] =$$

$$\begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \begin{array}{c} \begin{array}{cccc} \overline{v}_1 & \overline{\theta}_1 & \overline{v}_2 & \overline{\theta}_2 \end{array} \\ \left[\begin{array}{cccc} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{array} \right] \end{array}$$

FEA Process


Physical Problem
Change
Physical Problem
Mathematical Model
Governed by Differential Equations
Assumptions on
Geometry
Kinematics
Material
Loading
Boundary Conditions
Etc.
Improve
Mathematical Model
FEM Solution of Mathematical Model
Finite Element Solution
Choice of:
Finite Elements
Mesh Density
Solution Paramters
Representation of:
Loading
Boundary Conditions
Etc.
Refine mesh,
solution parameters
Assessment of accracy of FEM solution
of mathematical model
Interpretation of Results
Refine Analysis
Design Improvements
Structural Optimization

Computer Simulation



Non Linear Structural Analysis

Euler Equations; Boundary Conditions

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Table of Contents I

- Solutions to many physical problems require maximizing or minimizing some parameter F.
  - Distance
  - Time
  - Surface Area
- Parameter F dependent on function u (field variable), and variable x

$$\Pi(u) = \int_{\Omega} F(x, u(x), u'(x))dx \tag{1}$$

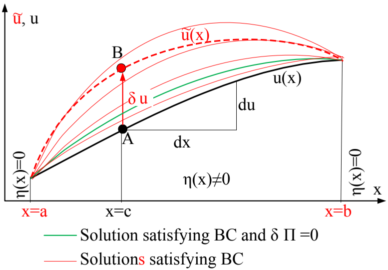
- A problem may be formulated as a partial differential equation, or as a variational one (maximize/minimize function).
- For example equation of equilibrium and minimization of total potential energy are analogous.
- Question: Can we go back and forth from one formulation to the other? and in so doing what are the boundary conditions?
- Reason: it may be easier to solve a problem one way or another.
- We will resort to calculus of variation that deals with minima or maxima of functionals.
- The origin of CV can be traced to the brachistochrone problem (*find the path that will carry a point-like body from one place to another in the least amount of time*).

- Differential calculus involves a function of one or more variable, variational calculus involves a function of a function, or a functional
- We seek a function $u(x)$ such that

$$\Pi(u) = \int_a^b F(x, u(x), u'(x))\mathrm{d}x \tag{2}$$

is stationary. Or, $\delta\Pi = 0$ where δ indicates the variation operator.

- $u(x)$ is a function of x in the interval (a, b), and F to be a known real function (such as the energy density).
- The domain of a functional is the collection of admissible functions belonging to a class of functions in function space rather than a region in coordinate space (as is the case for a function).
- We seek the function $u(x)$ which extremizes Π.
- Letting $\tilde{u}(x)$ to be a family of neighboring paths of the extremizing function $u(x)$ and we assume that at the end points $x = a, b$ they coincide.
- We define $\tilde{u}(x)$ as the sum of the extremizing path and some arbitrary variation.



$$\tilde{u}(x,\varepsilon) = u(x) + \varepsilon\eta(x) = u(x) + \delta u(x) \tag{3}$$

where ε is a small parameter, and $\delta u(x)$ is the variation of $u(x)$

$$\delta u(x) = \tilde{u}(x,\varepsilon) - u(x) \tag{4}$$
$$= \varepsilon\eta(x) \tag{5}$$

and $\eta(x)$ is twice differentiable, has undefined amplitude but is such that $\eta(a) = \eta(b) = 0$. We note that $\tilde{u}$ coincides with u if $\varepsilon = 0$.

- Again, to reinforce the distinction between differential calculus (DC) and variational calculus (VC) it should be noted that:
  - The necessary condition to extremize a value in DC is that the first derivative be equal to zero, and that the first variation be zero in VC.
  - The result of the extremization is a single variable x in DC, and $u(x)$ in VC.
- Variation and derivation operators are commutative

$$\left.\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}x}(\delta u) & = & \tilde{u}'(x,\varepsilon) - u'(x) \\ \delta u' & = & \tilde{u}'(x,\varepsilon) - u'(x) \end{array}\right\} \boxed{\frac{\mathrm{d}}{\mathrm{d}x}(\delta u) = \delta\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)} \tag{6}$$

- Variational operator δ and the differential calculus operator d can be similarly used, i.e.

$$\delta(u')^2 = 2u'\delta u' \tag{7}$$

$$\delta(u+v) = \delta u + \delta v \tag{8}$$

$$\delta\left(\int u\mathrm{d}x\right) = \int(\delta u)\mathrm{d}x \tag{9}$$

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y \tag{10}$$

however, they have clearly different meanings. du is associated with a neighboring point at a distance dx, however δu is a small *arbitrary* change in the function u for a given x (there is no associated δx).

- For boundaries where u is specified, its variation must be zero, and it is arbitrary elsewhere. The variation δu of u is said to undergo a virtual change.

- Define $\Phi(\varepsilon)$

$$\Phi(\varepsilon) \stackrel{\text{def}}{=} \Pi(u+\varepsilon\eta) = \int_a^b F(x, u+\varepsilon\eta, u'+\varepsilon\eta')dx \tag{11}$$

- Using this "trick" we now Cast the variational formulation ($\delta\Pi = 0$) into a differential one $\frac{d\Phi(\varepsilon)}{d\varepsilon} = 0$
- Since $\tilde{u} \to u$ as $\varepsilon \to 0$, the necessary condition for Π to be an extremum is

$$\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0 \tag{12}$$

- From Eq. 3 $\tilde{u} = u + \varepsilon\eta$, and $u(\tilde{x})' = u'(x) + \varepsilon\eta'(x)$, and applying the chain rule

$$\frac{d\Phi(\varepsilon)}{d\varepsilon} = \int_a^b \left(\frac{\partial F}{\partial \tilde{u}}\frac{d\tilde{u}}{d\varepsilon} + \frac{\partial F}{\partial \tilde{u}'}\frac{d\tilde{u}'}{d\varepsilon}\right)dx = \int_a^b \left(\eta\frac{\partial F}{\partial \tilde{u}} + \eta'\frac{\partial F}{\partial \tilde{u}'}\right)dx \tag{13}$$

for $\varepsilon = 0$, $\tilde{u} = u$, thus

$$\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \int_a^b \left(\eta\frac{\partial F}{\partial u} + \eta'\frac{\partial F}{\partial u'}\right)dx = 0 \tag{14}$$

- Integration by part of the second term leads to

$$\int_a^b \left(\eta' \frac{\partial F}{\partial u'}\right) \mathrm{d}x = \eta \frac{\partial F}{\partial u'}\bigg|_a^b - \int_a^b \eta(x) \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'}\right) \mathrm{d}x \qquad (15)$$

- Substituting,

$$\left.\frac{\mathrm{d}\Phi(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \underbrace{\int_a^b \eta(x)\left[\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'}\right]}_{\mathrm{I}\ (x\in[a,b])} + \underbrace{\eta(x)\frac{\partial F}{\partial u'}\bigg|_a^b}_{\mathrm{II}\ (x=a,b)} = 0 \qquad (16)$$

We will force each one of the two terms to be equal to zero.

First term : will give rise to the governing partial differential equation (or Euler equation).

Secon term : will enable us to define the boundary conditions

- The fundamental Lemma of the calculus of variation states that for continuous $\Psi(x)$ in $a \leq x \leq b$, and with arbitrary continuous function $\eta(x)$ which vanishes at a and b, then

$$\int_a^b \eta(x)\Psi(x)\mathrm{d}x = 0 \Leftrightarrow \Psi(x) = 0 \tag{17}$$

Thus, part I in Eq. 16 yields

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} = 0 \quad \text{in } a < x < b \tag{18}$$

- This differential equation is called the Euler-Lagrange equation associated with Π and is a necessary condition for $u(x)$ to extremize Π.

- Generalizing for a functional Π which depends on two field variables, $u = u(x, y)$ and $v = v(x, y)$

$$\Pi = \int\int F(x, y, u, v, u_{,x}, u_{,y}, u_{,xx}, v_{,yy})\mathrm{d}x\mathrm{d}y \tag{19}$$

There would be as many Euler equations as dependent field variables

$$\left\{ \begin{array}{lcl} \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_{,x}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial u_{,y}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial u_{,xx}} + \frac{\partial^2}{\partial x \partial y}\frac{\partial F}{\partial u_{,xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial u_{,yy}} & = & 0 \\ \frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\frac{\partial F}{\partial v_{,x}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial v_{,y}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial v_{,xx}} + \frac{\partial^2}{\partial x \partial y}\frac{\partial F}{\partial v_{,xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial v_{,yy}} & = & 0 \end{array} \right. \tag{20}$$

- We note that the Functional and the corresponding Euler Equations, Eq. 2 and 18, or Eq. 19 and 20 describe the same problem.
- The Euler equations usually correspond to the governing differential equation and are referred to as the **strong form** (or classical form).
- The functional is referred to as the weak form (or generalized solution). This classification stems from the fact that equilibrium is enforced in an average sense over the body.

- The field variable is differentiated m times in the weak form , and $2m$ times in the strong form.
- From above, $m = 1$ ($u_{,xx}$ in Eq. 19 and $u_{,xxxx}$ in Eq. 20.
- It can be shown that in the principle of virtual displacements, the Euler equations are the equilibrium equations, whereas in the principle of virtual forces, they are the compatibility equations.
- Euler equations are differential equations which can not always be solved by exact methods.
- An alternative method consists in bypassing the Euler equations and go directly to the variational statement of the problem to the solution of the Euler equations.
- Finite Element formulation are based on the weak form, whereas the formulation of Finite Differences are based on the strong form.

- In the preceding section we have just shown that $d\Phi(\varepsilon)/d\varepsilon$ leads to the Euler-Lagrange equation. We still have to define $\delta\Pi$. The first variation of a functional expression is

$$\left.\begin{array}{rcl} \delta F & = & \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \\ \delta\Pi & = & \int_a^b \delta F \mathrm{d}x \end{array}\right\} \delta\Pi = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right)\mathrm{d}x \tag{21}$$

Integration by parts of the second term (as in Eq. 14) yields

$$\delta\Pi = \int_a^b \delta u \left(\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'}\right)\mathrm{d}x \tag{22}$$

- We have just shown that finding the stationary value of Π by setting $\delta\Pi = 0$ is equivalent to finding the extremal value of Π by setting $\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}$ equal to zero.
- We could have also applied the fundamental Lemma of the calculus of variation to obtain Euler's Equation from Eq. 22 since δu is arbitrary.
- Similarly, it can be shown that as with second derivatives in calculus, the second variation $\delta^2\Pi$ can be used to characterize the extremum as either a minimum or maximum.
- An important observation is that the variational formulation is a scalar one, whereas the Eulerian one is vectorial.

- Revisiting the second part of Eq. 16, we had

$$\underbrace{\underbrace{\eta(x)}_{\text{Ess.}}\underbrace{\left.\frac{\partial F}{\partial u'}\right|_a^b}_{\text{Nat.}}}_{\text{Boundary Cond.}} = 0 \tag{23}$$

This can be achieved through the following combinations

$$\begin{array}{rclll} \eta(a) = 0 & \text{and} & \eta(b) & = 0 & \text{Essential} & \Gamma_u \\ \eta(a) = 0 & \text{and} & \frac{\partial F}{\partial u'}(b) & = 0 & \text{Mixed} & \Gamma_u \bigcup \Gamma_t \\ \frac{\partial F}{\partial u'}(a) = 0 & \text{and} & \eta(b) & = 0 & \text{Mixed} & \Gamma_u \bigcup \Gamma_t \\ \frac{\partial F}{\partial u'}(a) = 0 & \text{and} & \frac{\partial F}{\partial u'}(b) & = 0 & \text{Natural} & \Gamma_t \end{array} \tag{24}$$

- For example in the previously investigated column with one end fixed and the other hinged we had:

$$\textbf{Essential:}\quad v|_{x=0} = 0; \quad v|_{x=L} = 0; \quad \underbrace{v_{,x}|_{x=L}}_{\theta|_{x=L}} = 0;$$

$$\textbf{Natural:}\quad \underbrace{v_{,xx}|_{x=0}}_{M|_{x=0}} = 0$$

- Generalizing, for a problem with, one field variable, in which the highest derivative in the governing differential equation is of order $2m$ (or simply m in the corresponding functional), then we have

Essential (or forced, or geometric) boundary conditions, (because it was essential for the derivation of the Euler equation) if $\eta(a)$ or $\eta(b)$ =0. Essential boundary conditions, involve derivatives of order zero (the field variable itself) through m-1. Trial displacement functions are explicitly required to satisfy this B.C. Mathematically, this corresponds to *Dirichlet boundary-value problems*.

Natural (or natural or static) if we left η to be arbitrary, then it would be necessary to use $\frac{\partial F}{\partial u'} = 0$ at $x = a$ or b. Natural boundary conditions, involve derivatives of order m and up. This B.C. is implied by the satisfaction of the variational statement but not explicitly stated in the functional itself. Mathematically, this corresponds to *Neuman boundary-value problems*.

Mixed Boundary-Value/Robin problems, are those in which both essential and natural boundary conditions are specified on complementary portions of the boundary (such as Γ_u and Γ_t).

Problem	Axial Member Distributed load	Flexural Member Distributed load
Differential Equation	$AE\frac{d^2u}{dx^2} + q = 0$	$EI\frac{d^4w}{dx^4} - q = 0$
m	1	2
Essential B.C. $[0, m-1]$	u	$w, \frac{dw}{dx}$
Natural B.C. $[m, 2m-1]$	$\frac{du}{dx}$ or $\sigma_{xx} = Eu_{,x}$	$\frac{d^2w}{dx^2}$ and $\frac{d^3w}{dx^3}$ or $M = EIw_{,xx}$ and $V = EIw_{,xxx}$

- The total potential energy Π of an axial member of length L, modulus of elasticity E, cross sectional area A, fixed at left end and subjected to an axial force P at the right one is given by

$$\Pi = \int_0^L \frac{EA}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x - Pu(L) \tag{25}$$

  where the first term represents the strain energy sotred in the bar, and the second term denotes the work done on the bar by the load P in displacing the end $x = L$ through displacement $u(L)$.
- Determine the Euler Equation by requiring that Π be a minimum.

Solution I The first variation of Π is given by

$$\delta\Pi = \int_0^L \frac{EA}{2} 2 \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) \delta \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) \mathrm{d}x - P\delta u(L) \tag{26}$$

Integrating by parts we obtain

$$\begin{aligned} \delta\Pi &= \int_0^L -\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right)\delta u \mathrm{d}x + EA\frac{\mathrm{d}u}{\mathrm{d}x}\delta u\Big|_0^L - P\delta u(L) = 0 \qquad (27) \\ &= -\int_0^L \delta u \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right)}_{\text{Euler Eq.}} \mathrm{d}x + \underbrace{\left[\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right)\Big|_{x=L} - P\right]}_{\text{B.C.}} \delta u(L) \\ &\quad \underbrace{- \left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right)\Big|_{x=0} \delta u(0)}_{0} \qquad (28) \end{aligned}$$

The last term is zero because of the specified essential boundary condition which implies that $\delta u(0) = 0$. Recalling that δ in an arbitrary operator which can be assigned any value, we set the coefficients of

δu between $(0, L)$ and those for δu at $x = L$ equal to zero separately, and obtain

Euler Equation:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right) = 0 \quad 0 < x < L \tag{29}$$

Natural Boundary Condition:

$$EA\frac{\mathrm{d}u}{\mathrm{d}x} - P = 0 \quad \text{at} \quad x = L \tag{30}$$

Solution II We have

$$F(x, u, u') = \frac{EA}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \tag{31}$$

(note that since P is an applied load at the end of the member, it does not appear as part of $F(x, u, u')$. To evaluate the Euler Equation from Eq. 18, we evaluate

$$\frac{\partial F}{\partial u} = 0 \;\; \& \;\; \frac{\partial F}{\partial u'} = EAu' \tag{32}$$

Thus, substituting into Eq. 18, we obtain

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} = 0 \Rightarrow -\frac{\mathrm{d}}{\mathrm{d}x}(EAu') = 0 \text{ Euler Equation} \quad (33)$$

$$EA\frac{\mathrm{d}u}{\mathrm{d}x} = 0 \text{ B.C.} \quad (34)$$

- The total potential energy of a beam supporting a uniform load p is given by

$$\Pi = \int_0^L \left(\frac{1}{2}M\kappa - pw\right) \mathrm{d}x = \int_0^L \underbrace{\left(\frac{1}{2}(EIw'')w'' - pw\right)}_{F} \mathrm{d}x \tag{35}$$

Derive the first variational of Π.

- Extending Eq. 21, and integrating by part twice

$$\begin{aligned}
\delta\Pi &= \int_0^L \delta F \mathrm{d}x = \int_0^L \left(\frac{\partial F}{\partial w''}\delta w'' + \frac{\partial F}{\partial w}\delta w\right) \mathrm{d}x && (36)\\
&= \int_0^L (EIw''\delta w'' - p\delta w)\mathrm{d}x && (37)\\
&= (EIw''\delta w')\big|_0^L - \int_0^L \left[(EIw'')'\delta w' + p\delta w\right] \mathrm{d}x && (38)\\
&= \underbrace{\underbrace{(EIw''}_{\text{Nat.}}\ \underbrace{\delta w')|_0^L}_{\text{Ess.}} - \underbrace{[(EIw'')'}_{\text{Nat.}}\ \underbrace{\delta w]|_0^L}_{\text{Ess.}}}_{\text{BC}} + \int_0^L \underbrace{\left[(EIw'')'' - p\right]}_{\text{Euler Eq.}} \delta w \mathrm{d}x = 0 && (39)
\end{aligned}$$

Or

$$(EIw'')'' = p \quad \text{for all x}$$

which is the governing differential equation of beams and

Essential		Natural
$\delta w' = 0$	or	$EIw'' = -M = 0$
$\delta w = 0$	or	$(EIw'')' = -V = 0$

at $x = 0$ and $x = L$

Non Linear Structural Analysis

Geometric Non-Linearities

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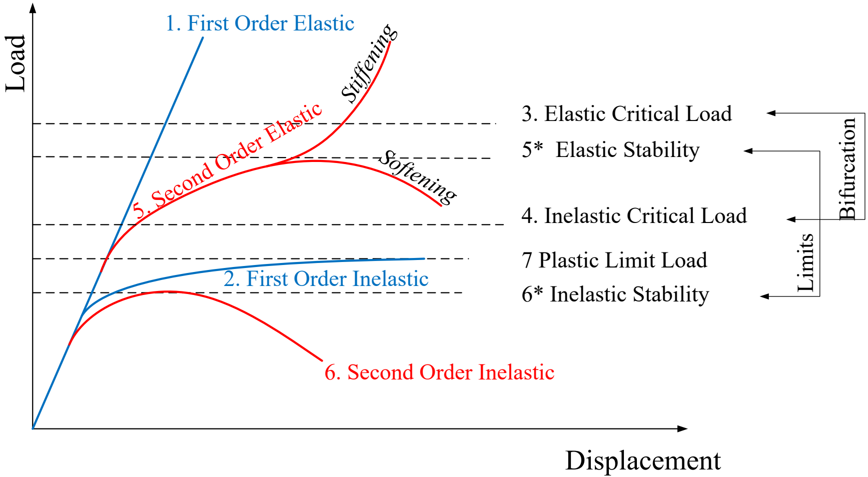
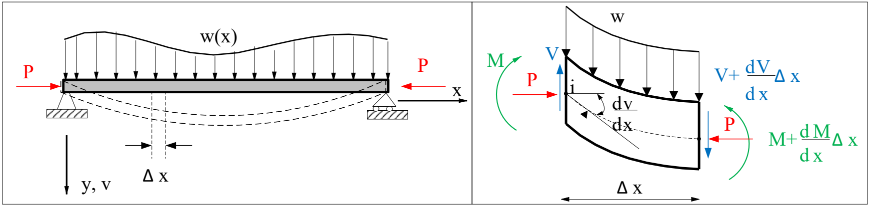
Table of Contents I

- There are two sources of nonlinearities: Material and Geometric.
- Geometric nonlinearity, in the context of analysis of skeletal structures, refers to
  - Effect if initial member imperfection which could result in instability or buckling.
  - $P - \Delta$ effects, secondary moments equal to vertical loads time the corresponding lateral displacements. It is a structural effect.
  - $P - \delta$ effects is the "stress stiffening" of an element on account of the axial load. It is an member effect.
- We will focus on the former, and in so doing will also address the interaction between axial and flexural stiffnesses (through the geometric stiffness matrix)


Load
1. First Order Elastic
Stiffening
5. Second Order Elastic
Softening
2. First Order Inelastic
6. Second Order Inelastic
Displacement
3. Elastic Critical Load
5* Elastic Stability
4. Inelastic Critical Load
7 Plastic Limit Load
6* Inelastic Stability
Bifurcation
Limits

		Constitutive Equations			
		Undeformed Shape		Deformed Shape	
		Elastic (Linear)	Inelastic (Non Linear)	Elastic (Linear)	Inelastic (Non Linear)
Kinematic Eq.	1^{st} Order (Linear)	1 (C:L-K:L)	2 (C:NL-K:L)	Critical Load	
				3 Elastic	4 Inelastic
	2^{nd} Order (Non Linear)	5 (C:L-K:NL)	6 (C:NL-K:NL)	-	-



- Summing moments wrt i using the deformed shape:

$$\underbrace{M - \left(M + \frac{dM}{dx}\Delta x\right) + w\cancelto{0}{\frac{(\Delta x)^2}{2}} + \left(V + \cancelto{0}{\frac{dV}{dx}\Delta x}\right)\Delta x}_{} + \underbrace{P\left(\frac{dv}{dx}\right)\Delta x}_{} = 0$$

- Neglecting the terms in Δx^2, and then differentiating each term with respect to x

$$-\frac{d^2M}{dx^2} + \frac{dV}{dx} + P\frac{d^2v}{dx^2} = 0$$

- Equilibrium in the y direction gives $\frac{dV}{dx} = -w$, and beam theory $M = -EI\frac{d^2v}{dx^2}$.
- Combining

$$EI\frac{d^4v}{dx^4} + P\frac{d^2v}{dx^2} = w$$

- Let $k^2 = \frac{P}{EI} \Rightarrow v = C_1 \sin kx + C_2 \cos kx + C_3 x + C_4$

- For a hinge-hinge column, BC:

$$\begin{array}{lrclrcl} \textbf{Essential:} & v|_{x=0} & = & 0; & v|_{x=L} & = & 0; \\ \textbf{Natural:} & \underbrace{v_{,xx}|_{x=0}}_{M|_{x=0}} & = & 0; & \underbrace{v_{,xx}|_{x=L}}_{M|_{x=L}} & = & 0 \end{array}$$

- substitution of the two conditions at $x = 0$ leads to $C_2 = C_4 = 0$.
- From the remaining conditions, we obtain $C_1 \sin kL + C_3 L = 0$ and $-C_1 k^2 \sin kl = 0 \Rightarrow kL = n\pi$ for $n = 1, 2, 3 \cdots$.
- For $n = 1$,

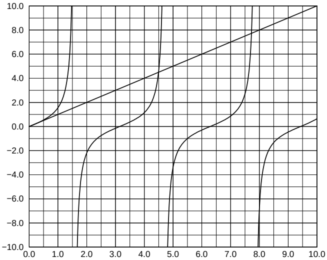
$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

- Consider a column with one end fixed (at $x = L$), and one end hinged (at $x = 0$)

$$\textbf{Essential:} \quad v|_{x=0} = 0; \quad v|_{x=L} = 0; \quad \underbrace{v_{,x}|_{x=L}}_{\theta|_{x=L}} = 0;$$

$$\textbf{Natural:} \quad \underbrace{v_{,xx}|_{x=0}}_{M|_{x=0}} = 0$$

- $\Rightarrow C_2 = C_4 = 0$ and $\sin kL - kL \cos kL = 0$ or $\tan kL = kL$ which is a transcendental algebraic equation and can only be solved numerically.



- Smallest positive root is $kL = 4.4934$, since $k^2 = \frac{P}{EI}$, the smallest critical load is $P_{cr} = \frac{(4.4934)^2}{L^2} EI = \frac{\pi^2}{(0.699L)^2} EI$
- Note that if we were to solve for x such that $v_{,xx} = 0$ (i.e. an inflection point), then $x = 0.699L$.

- In order to discretize the problem (through finite element), we need to first obtain the weak form of the governing differential equation.
- We will (as before) apply the principle of virtual strain energy.
- Contrarily to before, the expression of the strain will be enriched by additional higher order terms (initially neglected).

- Axial:

$$\varepsilon_{xx} = \underbrace{u_{,x}}_{\text{First Order}} + \underbrace{\frac{1}{2}(u_{,x}^2 + v_{,x}^2 + w_{,x}^2)}_{\text{Second Order}}$$

  u and v are the axial and transversal displacements respectively. Second order term is the “counterpart” of writing the equilibrium equation in the deformed shape in the analytical solution

- Flexural

$$\left.\begin{array}{lcl} \frac{d^2v}{dx^2} & = & \frac{M}{EI} \\ \sigma_{xx} & = & -\frac{My}{I} \end{array}\right\} \varepsilon_{xx} = -y\frac{d^2v}{dx^2}$$

- Total strain would be

$$\varepsilon_{xx}(x,y) = \underbrace{\underbrace{\frac{du}{dx}}_{\text{Axial}} \underbrace{-y\left(\frac{d^2v}{dx^2}\right)}_{\text{Flexure}}}_{\text{Small Deformation}} + \underbrace{\frac{1}{2}\left(\frac{dv}{dx}\right)^2}_{\text{Large Deformation}} \tag{1}$$

  Note that second term is negative since for positive y (top) we have compressive stresses, and the first and second terms are the familiar components of axial and flexural strains respectively, and the third one (which is nonlinear) is obtained from large-deflection strain-displacement.

- The (elastic) virtual Strain energy of the element is given by

$$\delta U_i^{(e)} = \frac{1}{2}\int_{\Omega} \sigma_{xx}\delta\varepsilon_{xx} d\Omega = \frac{1}{2}\int_{\Omega} E\varepsilon_{xx}\delta\varepsilon_{xx} d\Omega \tag{2}$$

- Substituting Eq. **??** into $U_i^{(e)}$ we obtain

$$\begin{aligned}\delta U_i^{(e)} = & \tfrac{1}{2}\int_L\int_A \left[\frac{du}{dx}\frac{d\delta u}{dx} + y^2\frac{d^2v}{dx^2}\frac{d^2\delta v}{dx^2} + \frac{1}{4}\left(\frac{dv}{dx}\right)^2\left(\frac{d\delta v}{dx}\right)^2\right. \\ & \left. -2y\left(\tfrac{du}{dx}\right)\tfrac{dv}{dx}\tfrac{d\delta v}{dx} - y\left(\tfrac{dv}{dx}\tfrac{d\delta v}{dx}\right)\left(\tfrac{dv}{dx}\right)^2 + \left(\tfrac{du}{dx}\right)\left(\tfrac{dv}{dx}\tfrac{d\delta v}{dx}\right)\right] EdAdx\end{aligned} \tag{3}$$

- Recalling that $\int_A dA = A$, and $\int_A ydA \stackrel{\text{def}}{=} 0$ and $\int_A y^2 dA \stackrel{\text{def}}{=} I$,
- Discarding highest order term and under the assumption of an independent prebuckling analysis for axial loading where $P^{(e)} = EA\frac{du}{dx} \Rightarrow A\frac{du}{dx} = \frac{P^{(e)}}{E}$

- We obtain

$$\delta U_i^{(e)} = \delta U_i^{(e,a)} + \delta U_i^{(e,f)} \tag{4}$$

$$\delta U_i^{(e,a)} = \frac{1}{2}\int_L EA \frac{d\delta u}{dx}\frac{du}{dx}dx \tag{5}$$

$$\delta U_i^{(e,f)} = \frac{1}{2}\int_L \left[EI \frac{d^2 v}{dx^2}\frac{d^2 \delta v}{dx^2} + P^{(e)}\frac{dv}{dx}\frac{d\delta v}{dx}\right] dx \tag{6}$$

- Assuming a functional representation of the transverse displacements in terms of the four joint displacements $v = \mathrm{N}\overline{\mathrm{u}}$, $\frac{dv}{dx} = \mathrm{N}_{,x}\overline{\mathrm{u}}$, $\frac{d^2v}{dx^2} = \mathrm{N}_{,xx}\overline{\mathrm{u}}$,
- The internal virtual strain energy must be equal to the external virtual work

$$
\begin{aligned}
\delta U_i^{(e,f)} &= \frac{1}{2}\int_L \left[EI\frac{d^2v}{dx^2}\frac{d^2\delta v}{dx^2} + P^{(e)}\frac{dv}{dx}\frac{d\delta v}{dx}\right] dx \\
\delta W_e &= P^{(e)}\int_0^L \left[\left(\frac{d\delta u}{dx}\frac{du}{dx}\right) + \left(\frac{d\delta v}{dx}\frac{dv}{dx}\right)\right] dx \\
[\mathrm{K}_e + \mathrm{K}_g]\,\mathrm{u}_e &= \mathrm{P} \\
\left[\mathrm{k}_e^{(e)}\right] &= \left[\int_L EI \underbrace{\{\mathrm{N}_{,xx}\}}_{\mathrm{B}^T} \underbrace{\lfloor \mathrm{N}_{,xx} \rfloor}_{\mathrm{B}} dx\right] \\
\left[\mathrm{k}_g^{(e)}\right] &= \left[P^{(e)}\int_L \{\mathrm{N}_{,x}\}\lfloor \mathrm{N}_{,x}\rfloor dx\right] \quad \text{Geometric Stiffness Matrix}
\end{aligned}
$$

Note that the geometric stiffness matrix terms solely depend on geometric parameters (length).

- Substituting the shape functions:

$$\mathrm{k}_g^{(e)} = \frac{P}{L} \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2}{15}L^2 & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{L}{10} & 0 & \frac{6}{5} & -\frac{L}{10} \\ 0 & \frac{L}{10} & -\frac{L^2}{30} & 0 & -\frac{L}{10} & \frac{2}{15}L^2 \end{bmatrix} \end{array}$$

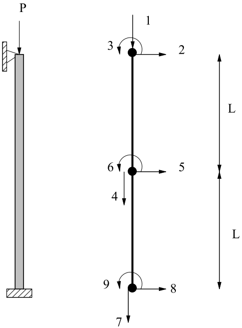
- The equilibrium relation is $\mathrm{k}\overline{\mathrm{u}} = \overline{\mathrm{P}}$ and $\mathrm{k}^{(e)} = \mathrm{k}_e^{(e)} + \mathrm{k}_g^{(e)}$ and in a global formulation,we would have $\mathrm{K} = \mathrm{K}_e + \mathrm{K}_g$
- We note that the structure becomes stiffer for tensile load P applied through K_g, and weaker in compression.

- We seek to determine the multiplier λ of an initial load vector $\overline{\mathrm{P}}^*$ obtained from a first order linear elastic analysis which will cause buckling, $\overline{\mathrm{P}}_{cr} = \lambda\overline{\mathrm{P}}^*$
- Since the geometric stiffness matrix is proportional to the internal forces, $\mathrm{K}_g = \lambda\mathrm{K}_g^*$ where K_g^* corresponds to the geometric stiffness matrix for the reference load $\overline{\mathrm{P}}^*$ (usually set to unity).
- The elastic stiffness matrix K_e remains a constant, hence we can write $(\mathrm{K}_e + \lambda\mathrm{K}_g^*)\overline{\mathrm{u}} - \underbrace{\lambda\overline{\mathrm{P}}^*}_{\overline{\mathrm{P}}_{cr}} = 0$
- The displacements are in turn given by $\overline{\mathrm{u}} = (\mathrm{K}_e + \lambda\mathrm{K}_g^*)^{-1}\lambda\overline{\mathrm{P}}^*$ and for the displacements to tend toward infinity (i.e buckling/bifurcation/instability), then $|\mathrm{K}_e + \lambda\mathrm{K}_g^*| = 0$ which can also be expressed as $|\mathrm{K}_g^{*-1}\mathrm{K}_e + \lambda\mathrm{I}| = 0$ which is an eigenvalue problem from which we can solve the eigenvalues $\boldsymbol{\lambda}$.
- Since K_g^* has some zero terms along the diagonal, we use an alternate formulation

$$|\mathrm{K}_e^{-1}\mathrm{K}_g^* + \frac{1}{\lambda}\mathrm{I}| = 0$$

however, $\mathrm{K}_e^{-1}\mathrm{K}_g^*$ may not be symmetric.

- The lowest value of λ, λ_{crit} will give the buckling load for the structure and the buckling loads will be given by $\overline{\mathrm{P}}_{crit} = \lambda_{crit}\overline{\mathrm{P}}^*$



$$
\mathbf{k}_e^1 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array}
$$

$$
\mathbf{k}_g^1 = \frac{-P}{L} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2}{15}L^2 & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{L}{10} & 0 & \frac{6}{5} & -\frac{L}{10} \\ 0 & \frac{L}{10} & -\frac{L^2}{30} & 0 & -\frac{L}{10} & \frac{2}{15}L^2 \end{bmatrix} \end{array}
$$

- The structure's stiffness matrices $\mathbf{K}_e$ and $\mathbf{K}_g$ can now be assembled from the element stiffnesses.

- Eliminating rows and columns 2, 7, 8, 9 corresponding to zero displacements in the column, we obtain

$$
\mathrm{K}_e = \frac{EI}{L^3}
\begin{array}{c}
\begin{array}{ccccc} 1 & 4 & 3 & 5 & 6 \end{array} \\
\begin{bmatrix}
\frac{AL^2}{I} & -\frac{AL^2}{I} & 0 & 0 & 0 \\
-\frac{AL^2}{I} & 2\frac{AL^2}{I} & 0 & 0 & 0 \\
0 & 0 & 4L^2 & -6L & 2L^2 \\
0 & 0 & -6L & 24 & 0 \\
0 & 0 & 2L^2 & 0 & 8L^2
\end{bmatrix}
\end{array}
$$

$$
\mathrm{K}_g = \frac{-P}{L}
\begin{array}{c}
\begin{array}{ccccc} 1 & 4 & 3 & 5 & 6 \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{15}L^2 & \frac{-L}{10} & \frac{-L^2}{30} \\
0 & 0 & \frac{-L}{10} & \frac{12}{5} & 0 \\
0 & 0 & \frac{-L^2}{30} & 0 & \frac{4}{15}L^2
\end{bmatrix}
\end{array}
$$

- Noting that in this case $\mathrm{K}_g^* = \mathrm{K}_g$ for $P = 1$, the determinant $|\mathrm{K}_e + \lambda \mathrm{K}_g^*| = 0$ leads to

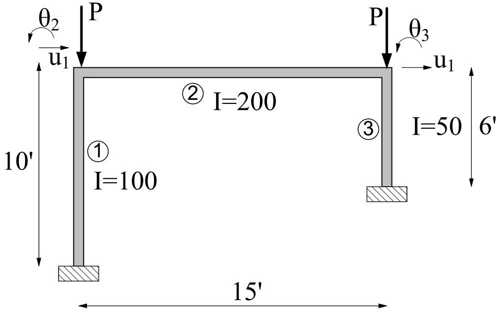
$$\begin{array}{c|ccccc|l} & 1 & 4 & 3 & 5 & 6 & \\ 1 & \frac{AL^2}{I} & -\frac{AL^2}{I} & 0 & 0 & 0 & \\ 4 & -\frac{AL^2}{I} & 2\frac{AL^2}{I} & 0 & 0 & 0 & \\ 3 & 0 & 0 & 4L^2 - \frac{2}{15}\frac{\lambda L^4}{EI} & -6L + \frac{1}{10}\frac{\lambda L^3}{EI} & 2L^2 + \frac{1}{30}\frac{\lambda L^4}{EI} & = 0 \\ 5 & 0 & 0 & -6L + \frac{1}{10}\frac{\lambda L^3}{EI} & 24 - \frac{12}{5}\frac{\lambda L^2}{EI} & 0 & \\ 6 & 0 & 0 & 2L^2 + \frac{1}{30}\frac{\lambda L^4}{EI} & 0 & 8L^2 - \frac{4}{15}\frac{\lambda L^4}{EI} & \end{array}$$

- At his point we can either solve numerically or algebraically. Let us be brave enoughand go with the second:

- Introducing $\phi = \frac{AL^2}{I}$ and $\mu = \frac{\lambda L^2}{EI}$, the determinant becomes

$$\begin{array}{c|ccccc|l} & 1 & 4 & 3 & 5 & 6 & \\ 1 & \phi & -\phi & 0 & 0 & 0 & \\ 4 & -\phi & 2\phi & 0 & 0 & 0 & \\ 3 & 0 & 0 & 2\left(2 - \frac{\mu}{15}\right) & -6L + \frac{\mu}{10} & 2 + \frac{\mu}{30} & = 0 \\ 5 & 0 & 0 & -6L + \frac{\mu}{10} & 12\left(2 - \frac{\mu}{5}\right) & 0 & \\ 6 & 0 & 0 & 2 + \frac{\mu}{30} & 0 & 4\left(2 - \frac{\mu}{15}\right) & \end{array}$$

- Expanding the determinant, we obtain the cubic equation in μ $3\mu^3 - 220\mu^2 + 3,840\mu - 14,400 = 0$ and the lowest root of this equation is $\mu = 5.1772$. Hence the buckling load of the column of length $2L$ is $P_{cr} = \lambda = \frac{5.1772EI}{L^2}$
- The exact solution for a column of length L is $P_{cr} = \frac{(4.4934)^2}{L^2}EI = \frac{(4.4934)^2}{(2L)^2}EI = \boxed{5.0477\frac{EI}{L^2}}$
- Thus, the numerical value is about 2.6 percent higher than the exact one.



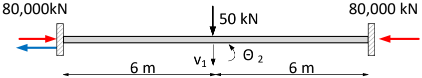
$$\left(\mathrm{K}_e - P\mathrm{K}_g\right)\mathrm{u} = 0$$

$$\begin{array}{ccc} u_1 & \theta_2 & \theta_3 \end{array}$$

$$\begin{vmatrix} (66.75) - P(0.026666) & (1,208.33) - P(0.1) & (1,678.24) - P(0.1) \\ (1,208.33) - P(0.1) & (225,556.) - P(16.) & (64,444.4) - P(0) \\ (1,678.24) - P(0.1) & (64,444.) - P(0) & (209,444.) - P(9.6) \end{vmatrix} = 0$$

The smallest buckling load amplification factor λ is equal to 2,017 kips.

```
(* Initialize constants *)
a1=0 a2=0 a3=0 i1=100 i2=200 i3=50 l1=10 12 l2=15 12 l3=6 12 e1=29000 e2=e1 e3=e1
(* Define elastic stiffness matrices *)
ke[e_,a_,l_,i_]:={
{e a/l  , 0          , 0         , -e a/l , 0          , 0          },
{0      , 12 e i/l^3 , 6 e i/l^2 , 0      , -12 e i/l^3 , 6 e i/l^2 },
{0      , 6 e i/l^2  , 4 e i/l   , 0      , -6 e i/l^2 , 2 e i/l    },
{-e a/l , 0          , 0         , e a/l  , 0          , 0          },
{ 0     , -12 e i/l^3 , -6 e i/l^2 , 0    , 12 e i/l^3 , -6 e i/l^2 },
{ 0     , 6e i/l^2   , 2 e i/l   , 0      , -6 e i/l^2 , 4 e i/l    } }
ke1=ke[e1,a1,l1,i1]; ke2=ke[e2,a2,l2,i2]; ke3=ke[e3,a3,l3,i3]
(* Define geometric stiffness matrices *)
kg[l_,p_]:=p/l{
{0 , 0    , 0        , 0 , 0      , 0        }, {0 , 6/5  , l/10     , 0 , - 6/5 , l/10     },
{0 , l/10 , 2 l^2/15 , 0 , - l/10 , - l^2/30 }, {0 , 0    , 0        , 0 , 0     , 0        },
{0 , -6/5 , - l/10   , 0 , 6/5    , - l/10   }, {0 , l/10 , - l^2/30 , 0 , - l/10 , 2 l^2/15 }
      }
kg1=kg[l1,1]
kg3=kg[l3,1]
(* Assemble structure elastic and geometric stiffness matrices *)
ke={
{ ke1[[2,2]]+ke3[[2,2]] , ke1[[2,3]]            , ke3[[2,3]]            },
{ ke1[[3,2]]            , ke1[[3,3]]+ke2[[3,3]] , ke2[[3,6]]            },
{ ke3[[3,2]]            , ke2[[6,3]]            , ke2[[6,6]]+ke3[[3,3]] } }
kg={
{ kg1[[2,2]]+kg3[[2,2]] , kg1[[2,3]] , kg3[[2,3]] },
{ kg1[[3,2]]            , kg1[[3,3]] , 0          },
{ kg3[[3,2]]            , 0          , kg3[[3,3]] } }
(* Determine critical loads in terms of p (note p=1) *)
p=1
keigen=Inverse[kg] . ke; pcrit=N[Eigenvalues[keigen]]; modshap=N[Eigensystems[keigen]]
```



- Using two elements for the beam column, the only degrees of freedom are the deflection and rotation at midspan (we neglect the axial deformation).
- Assembling the stiffness and geometric matrices:

$$\left\{ \begin{array}{c} v_1 \\ \theta_2 \end{array} \right\} = \underbrace{\left\{ \begin{array}{c} -0.00012123 \\ 0 \end{array} \right\}}_{P=-80,000} = \underbrace{\left\{ \begin{array}{c} -0.0001125 \\ 0 \end{array} \right\}}_{P=0} = \underbrace{\left\{ \begin{array}{c} -0.000104944 \\ 0 \end{array} \right\}}_{P=80,000}$$

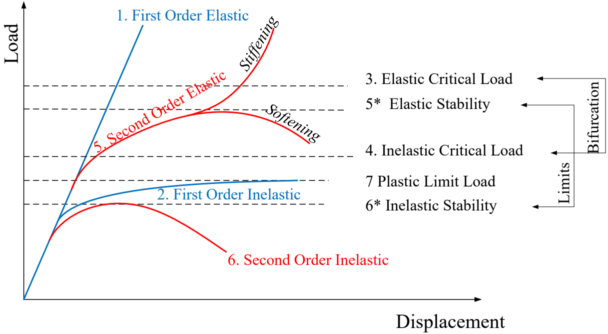
- The member end forces for element 1 are given by

$$\left\{ \begin{array}{c} P_{lft} \\ V_{lft} \\ M_{lft} \\ P_{rgt} \\ V_{rgt} \\ M_{rgt} \end{array} \right\} = \left[[\mathbf{k}_e^1] + [\mathbf{k}_g^1] \right] \left\{ \begin{array}{c} u_{lft} \\ v_{lft} \\ \theta_{lft} \\ u_{rgt} \\ v_{rgt} \\ \theta_{rgt} \end{array} \right\} = \underbrace{\left\{ \begin{array}{c} 0 \\ 25. \\ 79.8491 \\ 0. \\ -25. \\ 79.8491 \end{array} \right\}}_{P=-80,000} = \underbrace{\left\{ \begin{array}{c} 0 \\ 25. \\ 75. \\ 0. \\ -25. \\ 75. \end{array} \right\}}_{P=0} = \underbrace{\left\{ \begin{array}{c} 0 \\ 25. \\ 70.8022 \\ 0. \\ -25. \\ 70.8022 \end{array} \right\}}_{P=80,000}$$

- Compressive force increased the displacements and the end moments, whereas a tensile one stiffens the structure by reducing them.

```
(* Initialize constants *)
OpenWrite["mat.out"]; a1=1; a2=1; i1=1 l^3/12;i2=i1; l1=12;l2=12; e1=200000;e2=e1;e3=e1;
theta1=N[Pi/8];theta2=Pi-theta1; load i1=i i2=i1 l=6 l1=l l2=6 p=-80000 load={-50,0}
(* Define elastic stiffness matrices *)
ke[e_,a_,l_,i_]:={
{e a/l   , 0            , 0          , -e a/l , 0            , 0          },
{0       , 12 e i/l^3   , 6 e i/l^2  , 0      , -12 e i/l^3  , 6 e i/l^2  },
{0       , 6 e i/l^2    , 4 e i/l    , 0      , -6 e i/l^2   , 2 e i/l    },
{-e a/l  , 0            , 0          , e a/l  , 0            , 0          },
{ 0      , -12 e i/l^3  , -6 e i/l^2 , 0      , 12 e i/l^3   , -6 e i/l^2 },
{ 0      , 6e i/l^2     , 2 e i/l    , 0      , -6 e i/l^2   , 4 e i/l    } }
ke1=N[ke[e,a1,l1,i1]]; ke2=N[ke[e,a2,l2,i2]]
(* Assemble structure elastic stiffness matrices *)
ke=N[{
{ ke1[[5,5]]+ke2[[2,2]], ke1[[5,6]]+ke2[[2,3]]},
{ ke1[[6,5]]+ke2[[3,2]], ke1[[6,6]]+ke2[[3,3]]} }]
WriteString["mat.out",MatrixForm[ke1]]; WriteString["mat.out",MatrixForm[ke2]];
WriteString["mat.out",MatrixForm[ke]]
(* Define geometric stiffness matrices *)
kg[p_,l_]:=p/l {
{0 , 0    , 0        , 0 , 0     , 0        }, {0 , 6/5  , l/10     , 0 , - 6/5 , l/10     },
{0 , l/10 , 2 l^2/15 , 0 , - l/10 , - l^2/30 }, {0 , 0    , 0        , 0 , 0     , 0        },
{0 , -6/5 , - l/10   , 0 , 6/5   , - l/10   }, {0 , l/10 , - l^2/30 , 0 , - l/10 , 2 l^2/15 }
     }
kg1=N[kg[p,l1]]; kg2=N[kg[p,l2]]
(* Assemble structure geometric stiffness matrices *)
kg=N[{
{ kg1[[5,5]]+kg2[[2,2]], kg1[[5,6]]+kg2[[2,3]]},
{ kg1[[6,5]]+kg2[[3,2]], kg1[[6,6]]+kg2[[3,3]]} }]
```

```
(* Determine critical loads and normalize wrt p  *)
keigen=Inverse[kg] . ke; pcrit=N[Eigenvalues[keigen] p]
(* Note that this gives lowest pcrit=1.11  10^6, exact value is 1.095 10^6 *)
(* Add elastic to geometric structure stiffness matrices *)
k=ke+kg
(* Invert stiffness matrix and solve for displacements *)
km1=Inverse[k] dis=N[km1 . load]
(* Displacements of element 1*)
dis1={0, 0, 0, 0, dis[[1]], dis[[2]]}
k1=ke1+kg1
(* Member end forces for element 1 with axial forces  *)
endfrc1=N[k1 . dis1]
(* Member end forces for element 1 without axial forces  *)
knopm1=Inverse[ke]; disnop=N[knopm1 . load]; disnop1={0, 0, 0, 0, disnop[[1]], disnop[[2]]}
(* Displacements of element 1*)
endfrcnop1=N[ke1 . disnop1]
```

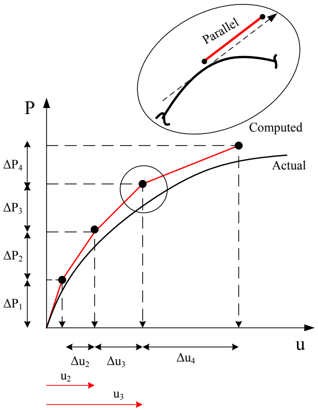


- So far, we have focused on bifurcation analysis (type 3), we now seek to perform a second order elastic analysis (type 5).

$$[\mathrm{K}_e + \mathrm{K}_g]\,\overline{\mathrm{u}} = \overline{\mathrm{P}}$$

- Since k_g depends on the magnitude of $P^{(e)}$, which itself may be an unknown in a framework, then we do have a geometrically non-linear problem.
- A simple way to solve this nonlinear equation is to use a step-by-step or incremental procedure. The linearized incremental formulation can be obtained by applying an incremental operator Δ

$$\{\Delta\overline{\mathrm{u}}\}_i = [\mathrm{K}_e + \mathrm{K}_g]_i^{-1}\{\Delta\overline{\mathrm{P}}\}_i$$



In an incremental analysis, we should:

Apply an incremental load ΔP_i at each increment i.

At the end of each increment:

Update the total displacement $u_i = u_{i-1} + \Delta u_i$.

Update the geometry $\mathtt{x}_i = \mathtt{x}_{i-1} + \Delta\mathtt{u}_i$, get new lengths.

Update the transformation matrix.

Update the elastic and geometric stiffness matrix

Note that we are not checking for equilibrium at the end of each increment (more about this later), and if we take sufficiently small steps, solution should not diverge too much.

Update of geometry should also take care of $P - \Delta$ effects.

- Keep in mind that this is an incremental analysis, each analysis is one associated with an increment of the load. At the each of each increment:

$$\begin{array}{rll} \text{Displacements} & \Delta \mathrm{u}^i; & \mathrm{u}^i = \mathrm{u}^{i-1} + \Delta \mathrm{u}^i \\ \text{Nodal coordinates} & & \mathrm{x}^i = \mathrm{x}^{i-1} + \Delta \mathrm{u}^i \\ \text{Internal forces} & \Delta \mathrm{F}^i; & \mathrm{F}^i = \mathrm{F}^{i-1} + \Delta \mathrm{F}^i \\ \text{Total Reactions} & \Delta \mathrm{R}^i; & \mathrm{R}^i = \mathrm{R}^{i-1} + \Delta \mathrm{R}^i \end{array}$$

- Recompute at the beginning of each increment:
  - Lengths, direction cosines, transformation matrix.
  - Element stiffness matrix based on updated nodal coordinates and transformation matrix.
  - Geometric stiffness matrix based on updated total axial force (from element internal forces) for each element.
- Use k and k_g to compute the internal forces.

$P - \Delta$ a structural effect achieved by updating the geometry Augmented Lagrangian formulation: $\mathrm{x}_i = \mathrm{x}_{i-1} + \Delta \mathrm{u}_i$. It does not account for the deformed shape of the member. It accounts for the effect of axial load on equilibrium.

$P - \delta$ a member effect accounted for through the addition of the geometric stiffness matrix. It accounts for the effect of axial load on internal forces in the deformed configuration.

- If many elements model a column, in the limit one can ignore K_g, and eventually, there will be a large displacement at the corresponding “buckling load". This can be easily verified.

Non Linear Structural Analysis

Plasticity I; Material & Mechanics

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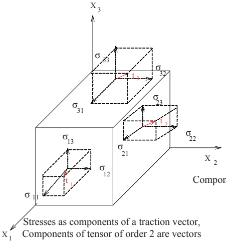
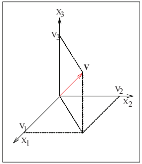
Table of Contents I

Table of Contents II

- Material nonlinearity is the dominant source of nonlinearity in the structural response of most civil engineering structures.
- In the context of this course, plasticity would refer to both steel and concrete.
- Coverage will follow a three tier approach:
  - Material (stress-strain) level, uniaxial and multiaxial (though not as relevant in this course).
  - Section (Moment-Curvature)
  - Structural.


Stresses as components of a traction vector,
Components of tensor of order 2 are vectors



Components of a vector are scalars

$$\begin{aligned}
t_{n_1} &= \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 \\
t_{n_2} &= \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 \\
t_{n_3} &= \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 \\
t_{n_i} &= \sigma_{ji} n_j \\
\mathbf{t}_n &= \mathbf{n} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \cdot \mathbf{n}
\end{aligned}$$

Cauchy stress tensor:

$$\sigma_{ij} = \begin{Bmatrix} \mathbf{t}^{(1)} \\ \mathbf{t}^{(2)} \\ \mathbf{t}^{(3)} \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Note that this stress tensor is really defined in the undeformed space (Eulerian), it could be defined in terms of the deformed space (Lagrangian).

$$\boldsymbol{\sigma} = \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{Bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{Bmatrix}$$

We seek to determine the traction (or stress vector) $\mathbf{t}$ passing through P and parallel to the plane ABC where $A(4, 0, 0)$, $B(0, 2, 0)$ and $C(0, 0, 6)$.
The vector normal to the plane can be found by taking the cross products of vectors AB and AC:

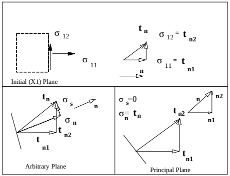
$$\mathbf{N} = AB \times AC = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -4 & 2 & 0 \\ -4 & 0 & 6 \end{vmatrix} = 12\mathbf{e}_1 + 24\mathbf{e}_2 + 8\mathbf{e}_3$$ The unit normal of N is given by $\mathbf{n} = \frac{3}{7}\mathbf{e}_1 + \frac{6}{7}\mathbf{e}_2 + \frac{2}{7}\mathbf{e}_3$ Hence the stress vector (traction) will be

$$\begin{bmatrix} \frac{3}{7} & \frac{6}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{9}{7} & \frac{5}{7} & \frac{10}{7} \end{bmatrix}$$

and thus $\mathbf{t} = -\frac{9}{7}\mathbf{e}_1 + \frac{5}{7}\mathbf{e}_2 + \frac{10}{7}\mathbf{e}_3$

- Stress transformation for the second order stress tensor is given by $[\sigma] = [A][\overline{\sigma}][A]^T$ where $[A]$ is the transformation matrix composed of the direction cosines.
- Note analogy with the transformation of the stiffness matrix from local to global coordinate system $\mathbf{K} = \mathbf{\Gamma}^T \mathbf{k} \mathbf{\Gamma}$
- For the 2D plane stress this simplifies to

$$\left\{ \begin{array}{c} \overline{\sigma}_{xx} \\ \overline{\sigma}_{yy} \\ \overline{\sigma}_{xy} \end{array} \right\} = \left[\begin{array}{ccc} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \cos\alpha\sin\alpha & \cos^2\alpha - \sin^2\alpha \end{array} \right] \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\}$$



- Choose a special set of axis through the point so that the shear stress components vanish when the stress components are referred to this system of axis $\Rightarrow$principal axes of the principal stresses.
- $\mathbf{n}$ unit vector in one of the unknown directions. λ: the principal-stress component on the plane whose normal is $\mathbf{n}$ (note both $\mathbf{n}$ and λ are yet unknown). Since there is no shear stress component on the plane perpendicular to $\mathbf{n}$, the stress vector on this plane must be parallel to $\mathbf{n}$ and $\mathbf{t}_n = \lambda\mathbf{n} \Rightarrow \mathbf{n}\cdot\boldsymbol{\sigma} = \lambda\mathbf{n}$ or $\mathbf{n}\;(\boldsymbol{\sigma} - \lambda\mathbf{I}) = 0$

$$\begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda \end{vmatrix} = 0$$

The three lambdas correspond to the three principal stresses $\sigma_{(1)} > \sigma_{(2)} > \sigma_{(3)}$.

Stress tensor $\boldsymbol{\sigma} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, determine the principal stress values and the corresponding directions. $\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 0-\lambda & 2 \\ 1 & 2 & 0-\lambda \end{vmatrix} = 0$ Or upon expansion (and simplification) $(\lambda+2)(\lambda-4)(\lambda-1) = 0$, thus the roots are $\sigma_{(1)} = -2$, $\sigma_{(2)} = 1$ and $\sigma_{(3)} = 4$. We also note that those are the three eigenvalues of the stress tensor. If we let $\overline{x}_1$ axis be the one corresponding to the direction of $\sigma_{(1)}$ and $n_i^{(1)}$ be the direction cosines of this axis, then

$$\left\{ \begin{array}{rcl} (3+2)n_1^{(1)} + n_2^{(1)} + n_3^{(1)} & = & 0 \\ n_1^{(1)} - 2n_2^{(1)} + 2n_3^{(1)} & = & 0 \\ n_1^{(1)} + 2n_2^{(1)} - 2n_3^{(1)} & = & 0 \end{array} \right.$$

thus

$$\begin{array}{llllll} n_1^{(1)} = 0 & & n_2^{(1)} = \frac{1}{\sqrt{2}} & & n_3^{(1)} = -\frac{1}{\sqrt{2}} \\ n_1^{(2)} = \frac{1}{\sqrt{3}} & & n_2^{(2)} = -\frac{1}{\sqrt{3}} & & n_3^{(2)} = -\frac{1}{\sqrt{3}} \\ n_1^{(3)} = -\frac{2}{\sqrt{6}} & & n_2^{(3)} = -\frac{1}{\sqrt{6}} & & n_3^{(3)} = -\frac{1}{\sqrt{6}} \end{array}$$

Finally, we can convince ourselves that the two stress tensors have the same invariants I_1, I_2 and I_3.

- Principal stresses are physical quantities, whose values do not depend on the coordinate system in which the components of the stress were initially given. They are therefore invariants of the stress state.
- When the determinant in the characteristic equation is expanded, the cubic equation is: $\lambda^3 - I_1\lambda^2 - I_2\lambda - I_3 = 0$ where I_1, I_2 and I_3 (in terms of principal stresses are given by

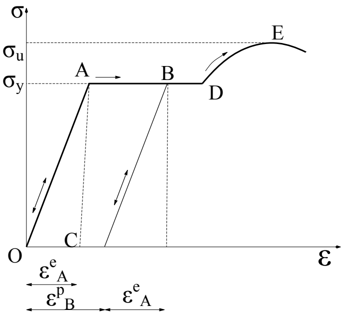
$$
\begin{aligned}
I_1 &= \sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)} \\
I_2 &= -\left(\sigma_{(1)}\sigma_{(2)} + \sigma_{(2)}\sigma_{(3)} + \sigma_{(3)}\sigma_{(1)}\right) \\
I_3 &= \sigma_{(1)}\sigma_{(2)}\sigma_{(3)}
\end{aligned}
$$

where $\sigma_{(1)} < \sigma_{(2)} < \sigma_{(3)}$

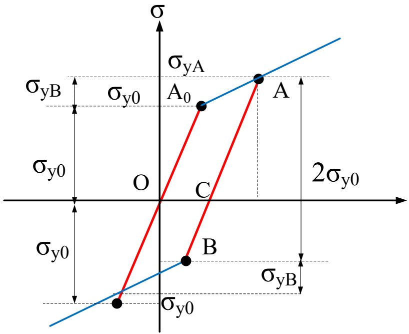
- We first define a mean normal stress as $\sigma = p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{ii} = \frac{1}{3}\text{tr}\,\boldsymbol{\sigma}$
- The stress tensor can be written as the sum of two tensors a hydrostatic one and a deviatoric one:
  Hydrostatic stress in which each normal stress is equal to $-p$ and the shear stresses are zero. The hydrostatic stress produces volume change without change in shape

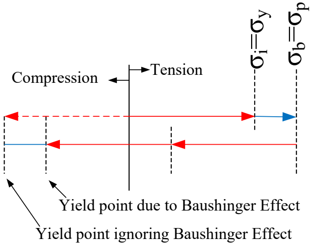
$$\boldsymbol{\sigma}_{hyd} = p\mathbf{I} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

- Deviatoric Stress: which causes the change in shape.
  $$\sigma_{dev} = \mathbf{s} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{hyd} = \begin{bmatrix} \sigma_{11} - p & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - p & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - p \end{bmatrix}$$
- The deviatoric stress Invariants are referred as (in terms of principal stresses) $J_1 = s_{(1)} + s_{(2)} + s_{(3)}$, $J_2 = -(s_{(1)}s_{(2)} + s_{(2)}s_{(3)} + s_{(3)}s_{(1)})$, and $J_3 = s_{(1)}s_{(2)}s_{(3)}$.
- It can be shown (upon substitution) that in terms of principal stresses, $J_2 = \frac{2}{3}\left[\left(\frac{\sigma_{(1)} - \sigma_{(2)}}{2}\right)^2 + \left(\frac{\sigma_{(2)} - \sigma_{(3)}}{2}\right)^2 + \left(\frac{\sigma_{(3)} - \sigma_{(1)}}{2}\right)^2\right]$
  and $J_2 = \frac{2}{3}\left(\tau^2_{(1)} + \tau^2_{(2)} + \tau^2_{(3)}\right)$
- We note that J_2 is really associated with shear stresses and distortional deformation.



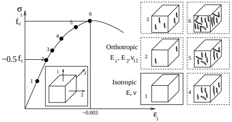
- Up to A, linearly elastic unloading follows initial loading path. O-A behavior is load path independent.
- At A we reach the elastic limit, material becomes plastic and behaves irreversibly. First yielding (A-D) and then hardening.
- Unloading: permanent strain or plastic strain ε^p. Thus only part of the total strain ε_B at B is recovered upon unloading, that is the elastic strain ε_A^e.



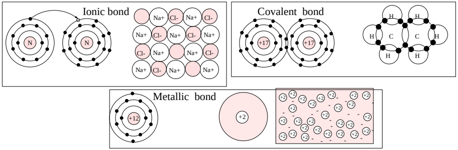


- Strain hardening in one direction, followed by reversed loading in the other, ⇒ stress-strain curve will be different from the one obtained from pure tension or compression.

- New yield point in compression at B corresponds to stress σ_y^B smaller than σ_y^0 and much smaller than the previous yield stress at A. This phenomena is called Bauschinger effect, or kinematic hardening (as opposed to isotropic hardening).
- Stress-strain behavior in the plastic range is path dependent, i.e. strain will not depend on the the current stress state, but also on the entire loading history, i.e. stress history and deformation history.



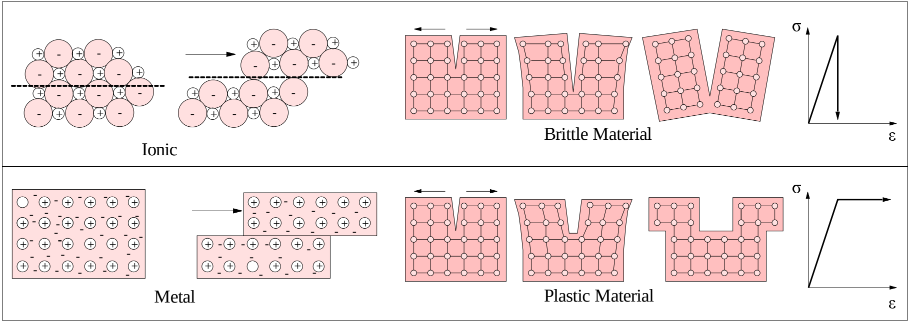
- Concrete contains microcracks due to shrinkage, and is originally isotropic.
- As the stress reaches $\simeq 0.5 f_c'$, interface cracks around the aggregates propagate, and tend to align themselves with the compressive stress.
- At peak stress, a mechanism is formed (coalescence of the micro-cracks).
- If a load is applied, sudden failure at peak.
- if displacement is imposed, post-peak softening



- Ionic Bond Atoms held together by electrostatic attraction as electrons are transferred from one atom to a neighbouring one. The atom giving up the electron, becomes positively charged and the atom receiving it becomes negatively charged.

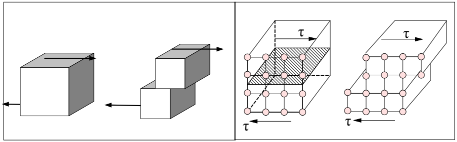
- Covalent Bond: electrons are shared more or less equally between neighboring atoms. Although the electrostatic force of attraction between adjacent atoms is less than it is in ionic bonds, covalent bonds tend to be highly directional, meaning that they resist motion of atoms past one another. Diamond has covalent bonds.

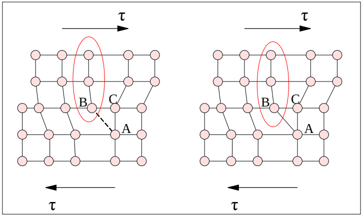
- Metallic Bond: electrons are delocalized or distributed equally through a metallic crystal, bond is not localized between two atoms. Best described as positive ions in a sea of electrons.

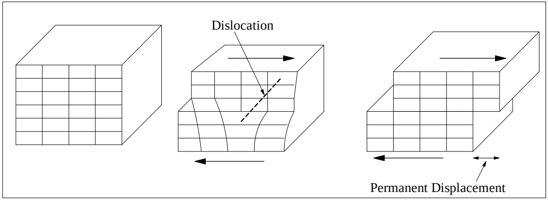


ionic solid: each ion is surrounded by oppositely charged ions, $\Rightarrow$ slipping much more difficult to achieve, and the material responds by breaking in a brittle behavior. When a force of sufficient magnitude displaces atoms from one equilibrium position to another we have a plastic deformation along the slip plane.

Shear stress applied on a metal bond: atoms can slip and slide past one another without regard to electrical charge constraint, and thus it gives rise to a ductile response.

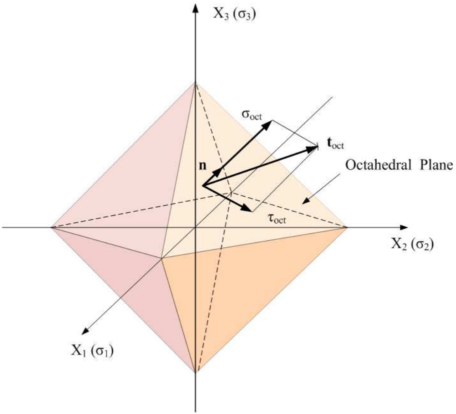






- Theoretical strength of perfect crystals (stress it takes to separate two adjacent atoms) is about $E/10$. Never achieved due to presence of random imperfections.

- Edge dislocation: internal flaws in atomic plane. Due to τ there is a driving force for breaking atomic bonds between atoms A and C until the dislocation passes entirely out of the crystal: dislocation glide.
- When the dislocation leaves the crystal: permanent offset.
- Yield stress: the applied **shear** stress necessary to provide the dislocations with enough energy to overcome short range forces exerted by the obstacles.
- Work-Hardening: With plastic deformation dislocations multiply and greater stresses are needed to overcome this resistance and strain hardening occurs.
- Bauschinger Effect: With deformations, dislocations accumulate, $\Rightarrow$ dislocation pile-ups. Since strain hardening is related to increased dislocation density, reducing the number of dislocations (through stress reversal) reduces strength.

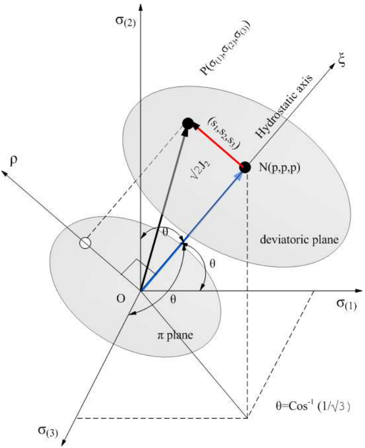


- Octahedral plane is one which makes equal angles with respect to each of the principal-stress directions, the normal to this plane is given by
$$\mathbf{n} = \frac{1}{\sqrt{3}}\left\{\begin{array}{c}1\\1\\1\end{array}\right\}$$
- The vector of traction on this plane is
$$\mathbf{t}_{oct} = \frac{1}{\sqrt{3}}\left\{\begin{array}{c}\sigma_1\\\sigma_2\\\sigma_3\end{array}\right\}$$
and the normal component of the stress on the octahedral plane is given by $\sigma_{oct} = \mathbf{t}_{oct}\cdot\mathbf{n} = \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{1}{3}\mathbf{I}_1 = \sigma_{hyd}$

- The octahedral shear stress is obtained from $\tau_{oct}^2 = |\mathbf{t}_{oct}|^2 - \sigma_{oct}^2 = \frac{\sigma_1^2}{3} + \frac{\sigma_2^2}{3} + \frac{\sigma_3^2}{3} - \frac{(\sigma_1+\sigma_2+\sigma_3)^2}{9}$ Upon algebraic manipulation, it can be shown that $\tau_{oct} = \sqrt{\frac{2}{3}J_2}$ and finally, the direction of the octahedral shear stress is given by $\cos 3\theta = \sqrt{2}\frac{J_3}{\tau_{oct}^3}$

- The elastic strain energy (total) per unit volume can be decomposed into two parts: $U = U_1 + U_2$, where

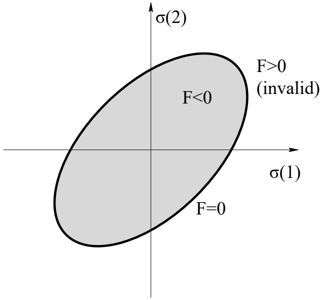
$$\begin{aligned} U_1 &= \frac{1-2\nu}{E} I_1^2 && \text{Dilational energy} \\ U_2 &= \frac{1+\nu}{E} J_2 && \text{Distortional energy} \end{aligned}$$



- Using the three principal stresses $\sigma_{(1)}$, σ_2, and σ_3, as the coordinates, a three-dimensional stress space can be constructed. This stress representation is known as the Haigh-Westergaard stress space.
- $\mathbf{OP} = \mathbf{ON} + \mathbf{NP}$. The former is along the direction of the unit vector $(1\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, and $\mathbf{NP} \perp \mathbf{ON}$.

- $\mathbf{NP}$ represents the deviatoric component of the stress state (s_1, s_2, s_3) and is perpendicular to the ξ axis. Any plane perpendicular to the hydrostatic axis is called the deviatoric plane and is expressed as $\frac{1}{\sqrt{3}}(\sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)}) = \xi$

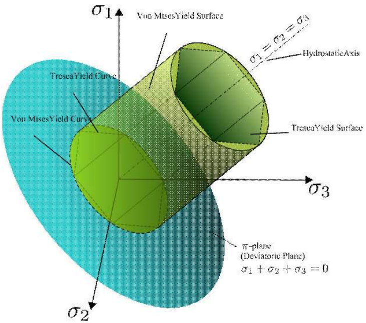
- Plane which passes through the origin is called the π plane and is represented by $\xi = 0$. Any plane containing the hydrostatic axis is called a meridian plane. The vector $\mathbf{NP}$ lies in a meridian plane and has $\rho = \sqrt{s_1^2 + s_2^2 + s_3^2} = \sqrt{2J_2}$

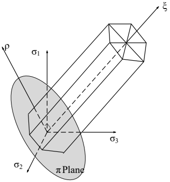


- Yielding in a uniaxially loaded structural element can be easily determined from $|\frac{\sigma}{\sigma_{yld}}| \geq 1$. But what about a general three dimensional stress state?
- Yield function F is a function of all six stress components of the stress tensor and a (or multiple) uniaxial yield stress.
- In biaxial or triaxial state of stresses, the elastic limit is defined mathematically by a certain yield criterion which is a function of the stress state σ_{ij} expressed as $F(\sigma_{ij}) = 0$
- *F* can not be greater than zero.

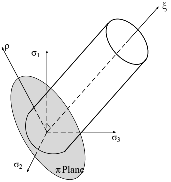
$$F = F(\sigma_{(1)}, \sigma_{(2)}, \sigma_{(3)}, \sigma_y) \begin{cases} < 0 & \text{Elastic} & \left|\frac{d\varepsilon^P}{dt}\right| = 0 \\ = 0 & \text{Plastic} & \left|\frac{d\varepsilon^P}{dt}\right| \geq 0 \\ > 0 & \text{Impossible} & \end{cases}$$

- For isotropic materials, the stress state is usually defined by $F(\sigma_{(1)}, \sigma_{(2)}, \sigma_{(3)}) = 0$ or $F(I_1, J_2, J_3) = 0$ which define the yield surface.
- We will distinguish between pressure independent and pressure dependent models.





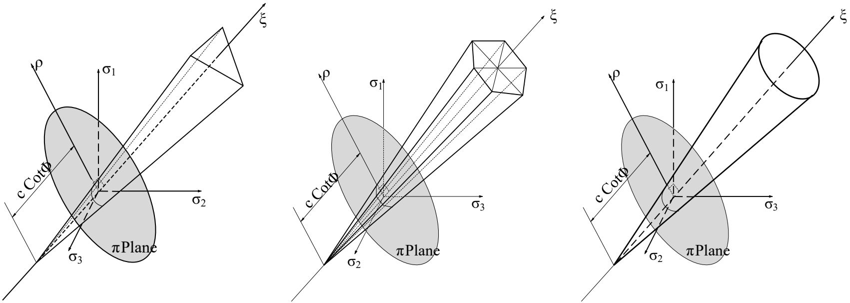
Tresca



Von Mises

- For hydrostatic pressure independent yield surfaces (e.g. steel) shearing stress (and not its direction) is the major cause of yielding $\Rightarrow$ elastic-plastic behavior in tension and in compression should be equivalent for hydrostatic-pressure independent materials.
- Tresca criterion: yielding occurs when the maximum shear stress reaches a limiting value k, or $F\left(\max\left(\frac{1}{2}|\sigma_{(1)} - \sigma_{(2)}|, \frac{1}{2}|\sigma_{(2)} - \sigma_{(3)}|, \frac{1}{2}|\right.\right.$ 0 from uniaxial tension test, we determine that $k = \sigma_y/2$ and from pure shear test $k = \tau_y$. Hence, in Tresca, tensile strength and shear strength are related by $\sigma_y = 2\tau_y$

- von Mises: Material will yield when the second deviatoric stress invariant reaches a critical value
$F(J_2) = J_2 - k^2 = \frac{(\sigma_{(1)}-\sigma_{(2)})^2+(\sigma_{(2)}-\sigma_{(3)})^2+(\sigma_{(1)}-\sigma_{(3)})^2}{2} - \sigma_y^2 = 0$ or when the maximum distorsional (shear) energy reaches the same critical value as for yield as in uniaxial tension.



Rankine | Mohr-Coulomb | Drucker Prager

- Pressure sensitive frictional materials (such as soil, rock, concrete) need to consider the effects of both the first and second stress invariants.
- The cross-sections of a yield surface are the intersection curves between the yield surface and the deviatoric plane (perpendicular to the hydrostatic axis ξ) and with $\xi =$ constant. Threefold symmetry.

- Rankine criterion postulates that yielding occur when the maximum principal stress reaches the tensile strength; $\sigma_{(1)} = \sigma_y$; $\sigma_{(2)} = \sigma_y$; $\sigma_{(3)} = \sigma_y$
- Mohr-Coulomb: extension of the Tresca criterion. The maximum shear stress is a constant plus a function of the normal stress acting on the same plane; $|\tau| = c - \sigma \tan \phi$ where c is the cohesion, and ϕ the angle of internal friction.
- Both c and ϕ are material properties which can be calibrated from uniaxial tensile and uniaxail compressive tests; $\sigma_t = \frac{2c\cos\phi}{1+\sin\phi}$ and $\sigma_c = \frac{2c\cos\phi}{1-\sin\phi}$.
- Drucker-Prager postulates is a simple extension of the von Mises criterion to include the effect of hydrostatic pressure on the yielding of the materials through I_1: $F(I_1, J_2) = \alpha I_1 + J_2 - k$ The strength parameters α and k can be determined from the uni axial tension and compression tests $\sigma_t = \frac{\sqrt{3}k}{1+\sqrt{3}\alpha}$; and $\sigma_c = \frac{\sqrt{3}k}{1-\sqrt{3}\alpha}$

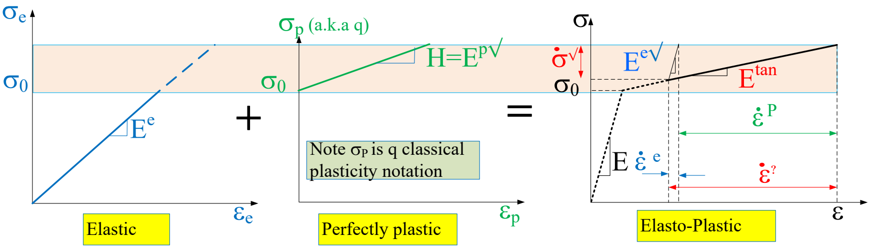
- There are two major theories for elastoplasticity

  Deformation Theory (or Total) of Hencky and Nadai, where the total strain ε_{ij} is a function of the current stress. $\varepsilon = \varepsilon_e + \varepsilon_p$ leads to a secant-type formulation of plasticity that is based on the additive decomposition of total strain into elastic and plastic components (Hencky).

  Rate Theory (or incremental) of Prandtl-Reuss, defined by $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_p$

  We will use the rate theory in this course.

- Though most of the formulations in this course will be uniaxial, for the sake of completeness multi-axial failure models will be briefly presented



- We can experimentally determine E^e and $E^p = H$ (elastic and plastic moduli):

$$\dot{\sigma}^e = E^e \dot{\varepsilon}^e \tag{1}$$

$$\dot{\sigma}^p = E^p \dot{\varepsilon}^p \tag{2}$$

i.e. taken separately we know how to compute the incremental stress from the incremental elastic or plastic strains.

- The incremental (total) strain $\dot{\varepsilon}$, has two components: elastic and plastic ones:

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \tag{3}$$

  Hence, the incremental strain has a component characterized by E^e and another by E^p

- We seek to determine the total incremental stress-strain relationship such that

$$\dot{\sigma} = E^{tan}\dot{\varepsilon} \tag{4}$$

  where E^{tan} is the tangent modulus yet to be determined.

- Thus, we can rewrite Eq. 3

$$\dot{\varepsilon}^e = \dot{\varepsilon} - \dot{\varepsilon}^p \tag{5}$$

  and

$$\dot{\sigma} = E^e(\dot{\varepsilon} - \dot{\varepsilon}^p) \tag{6}$$

- The plastic strain and corresponding plastic stress increment:

$$\dot{\sigma_p} = \dot{q} = H\dot{\varepsilon_p} \tag{7}$$

- The total strain ($\dot{\varepsilon}_{ij}$)is usually known because it is incrementally defined, thus we seek to determine the incremental plastic one $\dot{\varepsilon}_{ij}^{p}$ such that

$$\dot{\sigma} = E^{tan}\dot{\varepsilon} \tag{8}$$

- Do not confuse H and E^{tan}
- Two Approaches:
  1. Generalized, 3D, Mechanics; this will require
     1. Yield function
     2. Flow rule
     3. Consistency condition
  2. Engineering 1D

Each will be separately addressed.

- Rate theory (Eq. 3): $\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$
- if $\sigma \leq \sigma_y$(elasticity), then $\dot{\varepsilon} = \dot{\varepsilon}^e = \frac{\dot{\sigma}}{E^e}$
- if $\sigma > \sigma_y$ (plasticity), then from Eq. 1, 2, and 4:

$$\begin{aligned} \dot{\varepsilon} &= \dot{\varepsilon^e} + \dot{\varepsilon}^p \\ &= \frac{\dot{\sigma}}{E^e} + \frac{\dot{\sigma}}{E^p} = \frac{\dot{\sigma}}{E^{tan}} \\ \Rightarrow E^{tan} &= \frac{E.E^p}{E + E^p} \end{aligned}$$

- Note

$$E^{tan} = \begin{cases} > 0, & \textit{Hardening} \\ = 0, & \textit{Perfectly Plastic} \\ < 0, & \textit{Softening} \end{cases}$$

andE^{tan} is independent of the type of hardening (isotropic or kinematic).

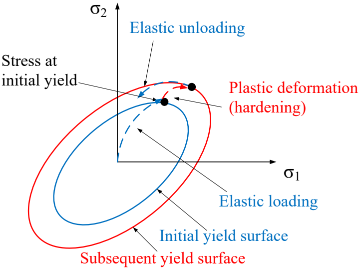
Note: There is a counterpart in nonlinear structural analysis where we seek to determine the Tangent stiffness matrix of an element at a given time step. Procedure is conceptually identical to what was done analytically in the first part.

- Yield function denotes the current level of stress minus the initial yield stress to which one may add a function of α which describes the type of hardening. In 1D, it can be defined as

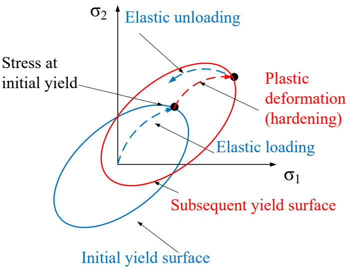
$$h(\sigma, q) = |\sigma| - q \leq 0 \tag{9}$$

- h determines the motion and deformation of the yield surface (Hardening or Softening).
- if $h < 0$, then the stress is within the elastic domain, and if $h = 0$, the stress has reached its plastic limit.
- First the strain reaches yielding ($\sigma_0 = \sigma_y$), and then at that point further increase in strain results in an expansion of q this will in turn expand the elastic domain.

- When the stress is inside the yield surface, it is elastic, Hooke's law is applicable, strains are recoverable, and there is no dissipation of energy
- When the load on the structure pushes the stress tensor to be beyond the yield surface, the stress tensor locks up on the yield surface, and the structure deforms plastically.
- If the material exhibits hardening as opposed to elastic-perfectly plastic response, then the yield surface expands or moves with the stress point still on the yield surface.
- The crucial question is what will be direction of the plastic flow (that is the relative magnitude of the components of ε^P. This question is addressed by the flow rule, or normality rule.



Isotropic Hardening



Kinematic Hardening

- Flow rule assumes that the plastic strain increment and deviatoric stress tensor have the same principal directions and it defines the evolution of plastic strain

$$\dot{\varepsilon}^p := \lambda \cdot \frac{\partial g(\sigma, \varepsilon^p, q)}{\partial \sigma} = \lambda \cdot h(\sigma,\ q) \tag{10}$$

where, $g(\sigma, \varepsilon^p, q)$ is the plastic potential and λ is a plastic multiplier that measures the magnitude of the plastic deformation (as we shall see later, this term will drop and lead to E^{tan} (which is what we are ultimately seeking).

- If $g \equiv h$, we have associated flow rule (usually in metals). If $g \neq h$ we have non associated flow rule, (concrete and geomaterials exhibiting dilatancy).
- For isotropic hardening models and associated flow rule,

$$g(\sigma, \varepsilon^p, q) = h(\sigma, q) = |\sigma| - q \Rightarrow \frac{\partial g}{\partial \sigma} = \text{sign}(\sigma) \tag{11}$$

  where sign(σ) is 1 if $\sigma \geq 0$ or -1 if $\sigma < 0$.
- Then, from Eq. 10

$$\dot{\varepsilon}^p = \lambda \cdot \text{sign}(\sigma) \tag{12}$$

  or

$$\lambda = |\dot{\varepsilon}^p| \geq 0 \tag{13}$$

- For simplicity and under the assumption of elasto-plastic hardening material with bilinear curve,

$$h(\sigma,\ q) = -\frac{\partial g}{\partial q} = 1 \tag{14}$$

- Therefore, we can express the stress (Eq. 6) as

$$\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p) \Rightarrow \dot{\sigma} = E \cdot (\dot{\varepsilon} - \lambda \cdot \frac{\partial g}{\partial \sigma}) = E \cdot (\dot{\varepsilon} - \lambda \cdot \text{sign}(\sigma)) \tag{15}$$

and from Eq. 7

$$\dot{q} = H\dot{\varepsilon_p} \Rightarrow \dot{q} = H \cdot \lambda \cdot h(\sigma,\ q) = H \cdot \lambda \tag{16}$$

During plastic loading the stress path is constrained to move along the yield surface, thus this consistency condition precludes us from going beyond the yield surface and is mathematically expressed as $\dot{f}(\sigma, q) = 0$, or

$$\begin{aligned} \dot{f}(\sigma,\ q) &= \frac{\partial f}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial t} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial t} \\ &= \frac{\partial f}{\partial \sigma} \cdot \dot{\sigma} + \frac{\partial f}{\partial q} \cdot \dot{q} = 0 \end{aligned} \tag{17}$$

- Now, we can finally solve Eq. 8

$$\dot{\sigma} = E^{tan}\dot{\varepsilon} \tag{18}$$

where the tangent modulus is the slope of the stress-strain curve at any specified stress or strain. Below the proportional limit the tangent modulus is equivalent to Young's modulus

- Not to be confused with $\dot{\sigma_p} = \dot{q} = H\dot{\varepsilon_p}$
- Substituting

$$\begin{array}{ll}\text{Flow Rule Eq. 15} & \dot{\sigma} = E \cdot (\dot{\varepsilon} - \lambda \cdot \frac{\partial g}{\partial \sigma}) \\ \text{Eq. 16} & \dot{q} = H \cdot \lambda \cdot h(\sigma,\ q) \\ \text{Consistency Eq. 17} & \dot{f}(\sigma,\ q) = \frac{\partial f}{\partial \sigma} \cdot \dot{\sigma} + \frac{\partial f}{\partial q} \cdot \dot{q} = 0\end{array}$$

- we obtain

$$\frac{\partial f}{\partial \sigma} \cdot E \cdot \left(\dot{\varepsilon} - \lambda \cdot \frac{\partial g}{\partial \sigma}\right) + \frac{\partial f}{\partial q} \cdot H \cdot \lambda \cdot h(\sigma,\ q) = 0$$

- Therefore, $\lambda = \frac{\frac{\partial f}{\partial \sigma} \cdot E \cdot \dot{\varepsilon}}{\frac{\partial f}{\partial \sigma} \cdot E \cdot \frac{\partial g}{\partial \sigma} - \frac{\partial f}{\partial q} \cdot H \cdot h(\sigma,\ q)}$

- Substituting back into $\dot{\sigma} = E \cdot (\dot{\varepsilon} - \dot{\gamma} \cdot \text{sign}(\sigma))$, we obtain an explicit expression for the incremental stress,

$$\begin{aligned}\dot{\sigma} &= E \cdot \left(\dot{\varepsilon} - \lambda \cdot \frac{\partial g}{\partial \sigma}\right) \\ &= \underbrace{\left(E - \frac{\frac{\partial f}{\partial \sigma} \cdot E^2 \cdot \frac{\partial g}{\partial \sigma}}{\frac{\partial f}{\partial \sigma} \cdot E \cdot \frac{\partial g}{\partial \sigma} - \frac{\partial f}{\partial q} \cdot H \cdot h(\sigma,\ q)}\right)}_{(\textit{by definition})\ E^{tan}} \cdot \dot{\varepsilon}\end{aligned}$$

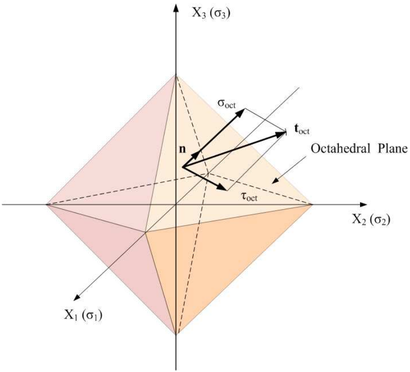
- For elasto-plastic hardening material with bilinear curve in one dimension,

$$\lambda = \frac{\text{sign}(\sigma) \cdot E \cdot \dot{\varepsilon}}{E + H}$$

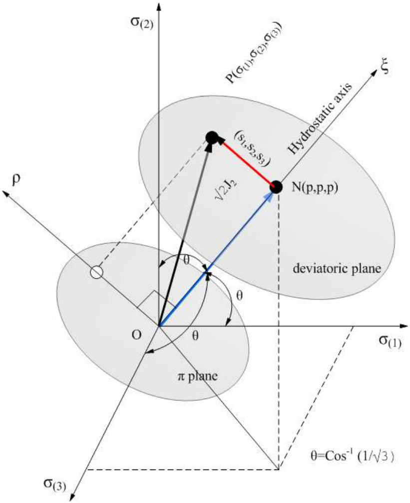
and the tangent modulus reduces to

$$\begin{aligned}E^{tan} &= E - \frac{\text{sign}(\sigma) \cdot E^2 \cdot \text{sign}(\sigma)}{\text{sign}(\sigma) \cdot E \cdot \text{sign}(\sigma) - (-1) \cdot H \cdot (1)} \\ &= E - \frac{E^2}{E + H} = \frac{E \cdot H}{E + H}\end{aligned}$$

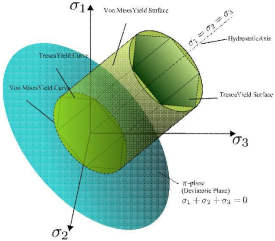
- Note that if $H = E/9$, then $E^{tan} = E/10$

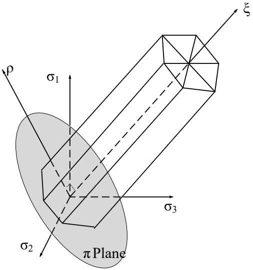


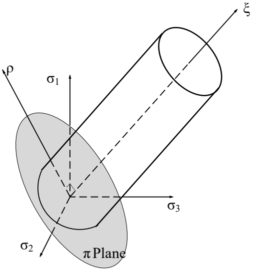
X3 (σ3)
σoct
toct
n
Octahedral Plane
τoct
X2 (σ2)
X1 (σ1)



σ(2)
P(σ(1),σ(2),σ(3))
ξ
Hydrostatic axis
(s1,s2,s3)
ρ
√2J2
N(p,p,p)
deviatoric plane
θ
O
σ(1)
π plane
θ=Cos⁻¹ (1/√3)
σ(3)


σ_1
Von Mises Yield Surface
$\sigma_1 = \sigma_2 = \sigma_3$
Hydrostatic Axis
Tresca Yield Curve
Von Mises Yield Curve
Tresca Yield Surface
σ_3
π-plane
(Deviatoric Plane)
$\sigma_1 + \sigma_2 + \sigma_3 = 0$
σ_2


ξ
ρ
σ_1
σ_3
σ_2
π Plane


ξ
ρ
σ_1
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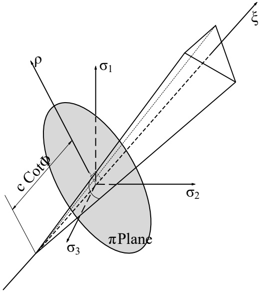
σ_2
π Plane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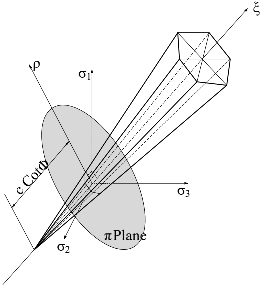
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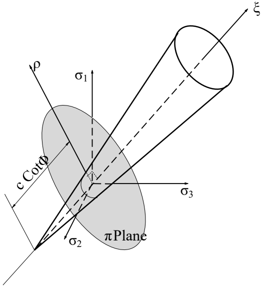
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c CotΦ
σ3
σ2
πPlane

Non Linear Structural Analysis

Plasticity II; Sections

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Fall 2020

Table of Contents I

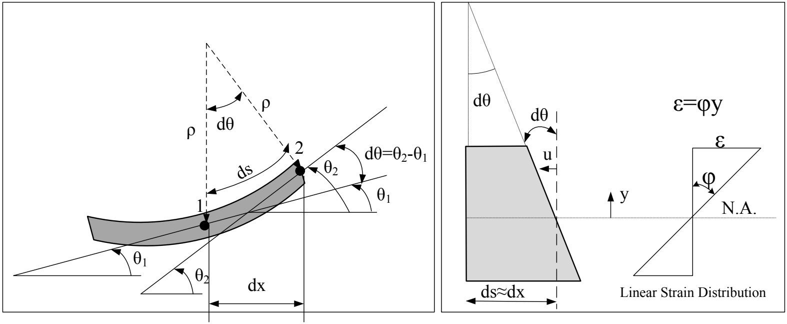
Contents

There are two types of element formulation for material nonlinearity:

- Lumped plasticity:
  - Inelastic behavior of beam-column concentrated at end of members (adequate for horizontal load, not so for vertical ones).
  - Use zero-length plastic hinges through nonlinear spring elements.
  - Requires stiffness calibration to determine the nonlinear $M - \theta$
- Distributed plasticity
  - Sectional ($M - \Phi$) constitutive behavior of cross-section formulated in terms of moment and axial forces . Does not capture gradual spread of plasticity.
  - Layered/Fiber ($\sigma - \varepsilon$), the cross-section is discretized into section fibers, stress-strain formulation for each fiber, captures gradual spread of plasticity over the cross section.
- Two formulations
  - Euler-Bernoulli, linear strain distribution, does not account for shear deformation.
  - Timoshenko accounts for shear displacement, non-linear strain distribution.

- Basic Assumptions:
  - Forces: Axial, N and Moment, M
  - Plane section remains plane (Bernouilli)
  - For R/C perfect bond between steel and concrete.
- Governing equations:
  - Equilibrium of force and moment at cross section
  - Compatibility of strain-curvature $\varepsilon = \Phi . y = \frac{M}{E.I} y$
  - Constitutive Laws
    - Moment-Curvature $M - \Phi$
    - Stress-Strain $\sigma - \varepsilon$
- Curvature: rotation per unit length $\theta_{AB} = \int_A^B \Phi dx$
- Outcome
  - Moment-Curvature $M - \Phi$
  - Load-Deflection (pushover $P - \Delta$)
  - Interaction diagram ($M - N$)

- Note: Often use non-layered approach based on $M - \Phi$ for steel, and layered for concrete.
- Ductility may be defined as the ability to undergo deformations without a substantial reduction in the flexural capacity of the member.
- The ductility ratio is defined as $\xi = \frac{\Phi_u}{\Phi_y}$ where Φ_u is the curvature at ultimate when the concrete compression strain reaches a specified limiting value, Φ_y is the curvature when the tension reinforcement first reaches the yield strength. This is very important for seismic design.



- The slope is denoted by θ, the change in slope per unit length is the curvature ϕ, the radius of curvature is ρ. From *Strength of Materials* we have the following relations

$$\phi = \frac{1}{\rho} = \frac{d\theta}{ds} \tag{1}$$

- For small displacements, and as a first order approximation, with $ds \approx dx$ and $\theta = \frac{dy}{dx}$ Eq. 1 becomes

$$\phi = \frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2} \quad (2)$$

- A positive $d\theta$ at a positive y (upper fibers) will cause a *shortening* of the upper fibers $du = -yd\theta$, Dividing both sides by dx,

$$\underbrace{\frac{du}{dx}}_{\varepsilon} = -y\frac{d\theta}{dx}$$

- Combining this with Eq. 2

$$\frac{1}{\rho} = \phi = -\frac{\varepsilon}{y} \quad \text{or} \quad \varepsilon = -\Phi y \quad (3)$$

This is the fundamental relationship between curvature (ϕ), elastic curve (i.e. displacement) (y), and linear strain (ε).

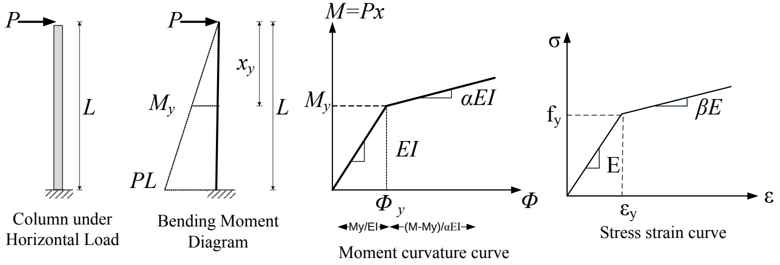
- Note that so far we made no assumptions about material properties, i.e. it can be elastic or inelastic.
- For the elastic case:

$$\left.\begin{array}{rcl} \varepsilon & = & \frac{\sigma}{E} \\ \sigma & = & -\frac{My}{I} \end{array}\right\} \varepsilon = -\frac{My}{EI} \tag{4}$$

- Combining this last equation with Eq. 1 yields

$$\phi = \frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2} = \frac{M}{EI}$$

This fundamental differential equation governing for beam. Similar equations will be derived later for cables and beam-columns.



- Yielding occurs at x_y such that $x_y = \frac{M_y}{P} \leq L$
- The curvatures are given by
$$\phi = \begin{cases} \frac{P.x}{E.I} & \text{if } \; x < x_y \\ \frac{M_y}{E.I} + \frac{P.x - M_y}{\alpha.E.I} & \text{if } \; x > x_y \end{cases}$$
- Using the principle of complementary virtual work (virtual force), where $\delta\overline{M} = x$
$$\delta\overline{P}\Delta = \int_0^L \delta\overline{M}.\phi dx = \int_0^L x\,.\phi.dx$$
- $$\delta\overline{P}\Delta = \int_0^{x_y} x\frac{P.x}{E.I}dx + \int_{x_y}^{L} x\left(\frac{M_y}{E.I} + \frac{P.x - M_y}{\alpha.E.I}\right)\cdot dx$$
or $\delta\overline{P}.\Delta =$
$$\frac{P.{x_y}^3}{3.E.I} + \frac{3(-1+\alpha)M_y\left(L^2 - {x_y}^2\right) + 2.P\left(L^3 - {x_y}^3\right)}{6E.I.\alpha}$$
- Substituting with $x_y = M_y/P$ gives Force-displacement or Pushover
$$\Delta = \frac{2L^3P^3 - (\alpha - 1)M_y\left(M_y^2 - 3L^2P^2\right)}{6\alpha EIP^2}$$

```
clear all
clc
close all
fprintf('============================\n')
%% Initialization for figures
fs = 28;
scrsz = get(0,'ScreenSize');
figpos = [2 2 scrsz(3)/2 scrsz(4)/2];
%% units: kips in
E=29000;fy=36;L=6*12;alpha=0.01;
%% Consider W16x100
I=1490; S=175; EI=E*I; My=S*fy; Phiy=My/EI;
%
%% MOMENT CURVATURE PLOT
M(1) = 0; Phi(1) = 0.;
M(2) = My; Phi(2) = Phiy;
M(3) = 1.2*M(2); Phi(3) = (M(3) My)/(alpha*EI)
h = figure('Position',figpos); set(gca,'FontSize',28);
plot(Phi,M/12,'LineWidth',2)
xlabel('Curvature'); ylabel('Moment [k.ft]'); title('W16x100');
grid minor; set(gcf,'PaperPositionMode','auto');
print depsc2 NonLayeredMomentCurvature.eps
% LOAD DISPLACEMENT CURVE
%%
DeltaP = 0.5;% kip increment
P(1) = 0.0; Delta(1) = 0;
for i=2:190
    P(i) = P(i 1)+DeltaP;
    xy = My/P(i);
    if xy>L
        Delta(i) = P(i)*L^3/(3*E*I);
```

```
    else
        Delta(i) = (2*L^3*P(i)^3   My*(My^2   3*L^2*P(i)^2)*(1 + alpha))/...
            (6.*EI*P(i)^2*alpha);
    end
end
%%
h = figure('Position',figpos); set(gca,'FontSize',28);
plot(Delta,P,'LineWidth',2)
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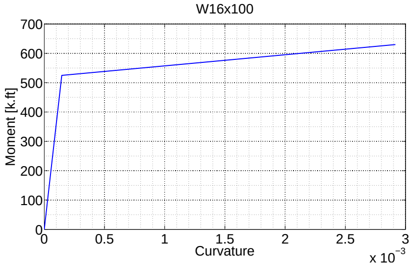
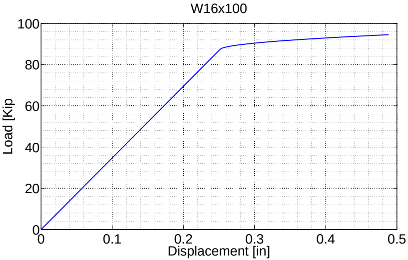
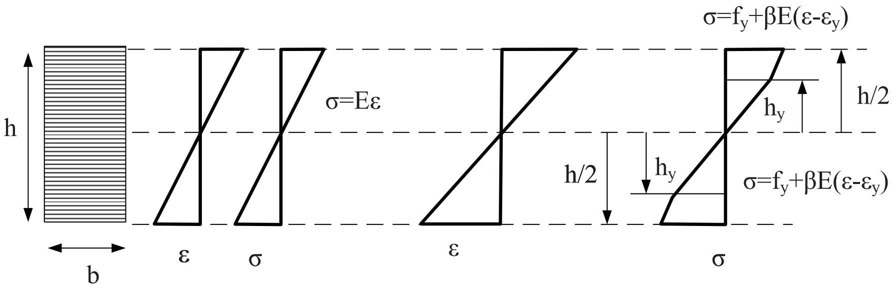






- We analyze a rectangular section and account for the nonlinear stress distribution across the section (as will be done later in layered fiber elements).
- Yielding occurs first at the outer fibers at x_y such that (and recalling that $\phi = \varepsilon/y$): $P.x_y = E.I.\phi_y = E.I\frac{\varepsilon_y}{h/2} \Rightarrow x_y = \frac{2.f_y.I}{P.h} \leq L$.
- The maximum curvature below which all the stresses are elastic is given by $\phi_y = \frac{2.f_y}{E.h}$

- If $h \leq h_y$, then we have a linear elastic stress distribution and $\phi = \frac{M}{E.I} = \frac{P.x}{E.I}$
- If $h > h_y$, then at a distance h_y from the neutral axis hardening begins. and $h_y = \frac{f_y}{E.\phi} \leq \frac{h}{2}$.
- The stresses are given by $\sigma_{xx} = \begin{cases} E.\varepsilon & \text{if} \quad h < h_y \\ f_y + \beta E.(\varepsilon - \varepsilon_y) & \text{if} \quad h_y < h \end{cases}$
- Recalling that $\varepsilon = \phi y$, the internal resisting moment will thus be given by $M = \int_{-h/2}^{h/2} b.\sigma_{xx}.y.dy$ or

$$M = 2.b.\int_0^{h_y} E.(\phi.y).ydy + 2b\int_{h_y}^{h/2} [f_y + \beta.E.(\phi y - \varepsilon_y)].y.dy =$$

$$2E.b.\phi\int_0^{h_y} y^2.dy + 2b\int_{h_y}^{h} \left[f_y.y + \beta.E.\phi.y^2 - \beta f_y.y\right] dy =$$

$$\frac{-\left(b\left(3.f_y\left(h^2 - 4.{h_y}^2\right)(-1+\beta) + E\phi\left(8{h_y}^3(-1+\beta) - h^3\beta\right)\right)\right)}{12}$$

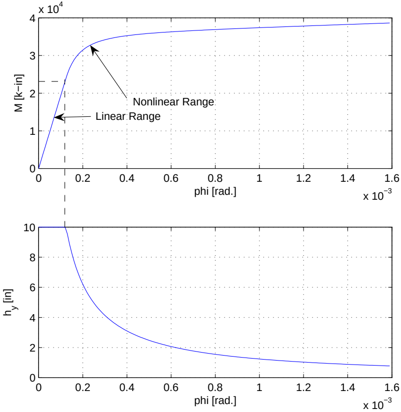
- Substituting (using Mathematica) the value of h_y, this reduces to Moment Curvature relation

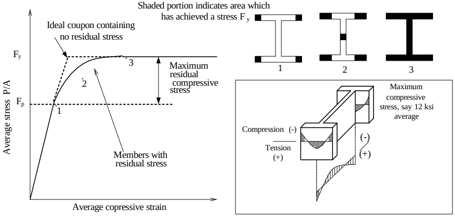
$$M = \frac{b\left(4{f_y}^3(-1+\beta) - 3E^2 f_y h^2 \phi^2(-1+\beta) + E^3 h^3 \phi^3 \beta\right)}{12E^2\phi^2} \tag{5}$$

which is essentially an expression for the moment in terms of the curvature $\phi > \phi_y$.

- Armed with this relationship, we can repeat the procedure used in the previous example and apply the principle of virtual force to determine the displacement.
- Numerical solution: Increment Φ

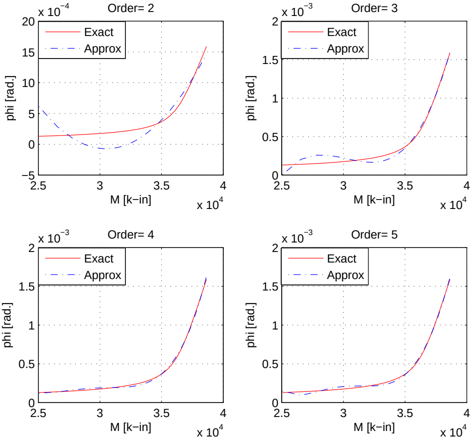
```
clear all
fprintf("===========================\n")
E=29000;fy=36;h=20;h2=h/2;b=10;beta=0.01;I=b*h^3/12;
epsiy=E/fy; phi_y=fy/(E*h2);delta=10^ 5;phi(1)=0.;
M(1)=0.;hy(1)=h2;k=0;
for i=2:80
    phi(i)=phi(i 1)+delta; hy(i)=fy/(E*phi(i));
    if hy(i)>h2
        M(i)=phi(i)*E*I; hy(i)=h2;
    else
        M(i)=(b*(4*fy^3*( 1 + beta)   3*E^2*fy*h^2*phi(i)^2*( 1 + beta)...
            + E^3*h^3*phi(i)^3*beta))/(12.*E^2*phi(i)^2);
        k=k+1;
        nlphi(k)=phi(i); nlM(k)=M(i);
    end
    fprintf("i %6.0f hy %10.4e phi %10.4e phi_y %10.4e\n",i,hy(i),phi(i),phi_y);
end
subplot(2,1,1);plot(phi,M);xlabel("phi [rad.]");ylabel("M [k in]");grid;
subplot(2,1,2);plot(phi,hy);xlabel("phi [rad.]");ylabel("h_y [in]");grid;
ylim([0.,10])
%========================
% Fit polynomial in nonlinear portion of the curve
figure
for i=2:5
    clear p
    [p]=polyfit(nlM,nlphi,i); f=polyval(p,nlM);
    subplot(2,2,i 1);plot(nlM,nlphi,"r",nlM,f," .b");
    asc=num2str(i);legend("Exact","Approx",2);ylabel("phi [rad.]");
    xlabel("M [k in]");grid;title(["Order= " asc])
end
```





- We captured spread of plasticity (note analogy with effect of residual strains on steel $\sigma - \varepsilon$ curve), $\Rightarrow$ nonlinear shape, as contrasted with the bilinear $M - \Phi$ earlier assumed.

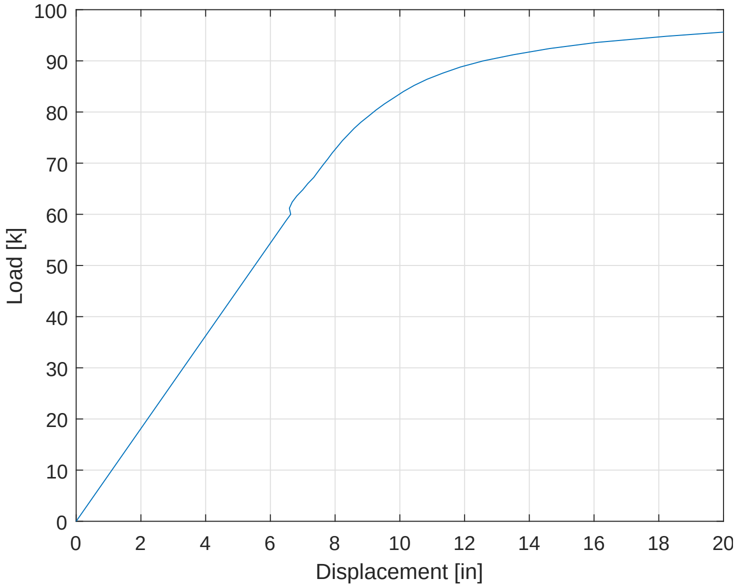
- Since we will need the curvature-moment relationship ($\phi = \phi(M)$), a polynomial is then fitted to the nonlinear portion of the moment curvature,



- Opting for the one of order 5, the coefficients are:
  $\phi = 3.6396 \times 10^{-23} M^5 - 5.4368 \times 10^{-18} M^4 + 3.2411 \times 10^{-13} M^3 - 9.6360 \times 10^{-9} M^2 + 1.4285 \times 10^{-4M} - 8.4445 \times 10^{-1}$

We now apply the principle of virtual force to solve for the displacement of the cantilevered beam. This is solved by Matlab:

```
%===========================
% Compute the deflection of a beam of length L using virtual force method
%P_y is when x_y is equal to the length (all the element is linear)
L=20*h;P_y=2*fy*I/(h*L);P_max=2*P_y;np=100;delta_P=P_max/np;E.I=E*I;delta_L=0.001*L;
%
P(1)=0.;delta(1)=0;
for i=2:np
    P(i)=P(i 1)+delta_P; x_y(i)=2*fy*I/(h*P(i));
    if x_y(i)>L
        delta_1=P(i)*L^3/(3*E.I);
        % Entire Beam is in the linear range
        delta_2=0.;
    else
        delta_1=P(i)*x_y(i)^3/(3*E.I);
        delta_2=0.;
        for x=x_y(i):delta_L:L
            M=P(i)*x; phi=polyval(p,M); delta_2=delta_2+phi*x*delta_L;
        end
    end
    delta(i)=delta_1+delta_2;
end
%
figure; subplot(2,1,1);
plot(delta,P);grid;xlabel("Displacement [in]");ylabel("Load [k]");
subplot(2,1,2);plot(delta,P);grid;
xlabel("Displacement [in]");ylabel("Load [k]");xlim([0.,50])
```



Load [k]
Displacement [in]
0
10
20
30
40
50
60
70
80
90
100
0
2
4
6
8
10
12
14
16
18
20

As an Engineer questioning the validity of the ACI equation for the ultimate flexural capacity of R/C beams, you determined experimentally the following stress strain curve for concrete (Desayi and Krishnan other equations have been proposed by Saenz, Kent-Park and Mander):

$$\sigma = \frac{2\frac{f'_c}{\varepsilon_{max}}\varepsilon}{1 + \left(\frac{\varepsilon}{\varepsilon_{max}}\right)^2} \tag{6}$$

where f'_c corresponds to ε_{max}.

1. Determine the exact balanced steel ratio for a R/C beam with $b = 10$", $d = 23$", $f'_c = 4,000$ psi, $f_y = 60$ ksi, $\varepsilon_{max} = 0.003$.
    1. Determine the equation for the exact stress distribution on the section.
    2. Determine the total compressive force C, and its location, in terms of the location of the neutral axis c.
    3. Apply equilibrium
2. Using the ACI equations, determine the:
    1. Ultimate moment capacity.

2. Balanced steel ratio.

3. For the two approaches, compare:
    1. Balanced steel area.
    2. Location of the neutral axis.
    3. Centroid of resultant compressive force.
    4. Ultimate moment capacity.

- Stress-Strain:

$$\sigma = \frac{2\frac{4{,}000}{.003}\varepsilon}{1+\left(\frac{\varepsilon}{.003}\right)^2} = \frac{2.667\times 10^6\varepsilon}{1+1.11\times 10^5\varepsilon^2} \tag{7}$$

- Compatibility:Assume crushing at failure, hence strain distribution will be given by

$$\varepsilon = \frac{0.003}{c}y \tag{8}$$

- Combine those two equation:

$$\sigma = \frac{8{,}000\frac{y}{c}}{1+\left(\frac{y}{c}\right)^2} \tag{9}$$

- The total compressive force is given by

$$\begin{aligned}
F &= \int_0^c dF = b\int_0^c \sigma dy = b\int_0^c \frac{8,000\frac{y}{c}}{1+\left(\frac{y}{c}\right)^2}dy = b\frac{8,000}{c}\int_0^c \frac{y}{1+\left(\frac{y}{c}\right)^2}dy && (10)\\
&= 8,000\frac{b}{c}\frac{1}{2\left(\frac{1}{c}\right)^2}\ln\left[1+\left(\frac{y}{c}\right)^2\right]\Bigg|_0^c = 8,000\frac{b}{c}\frac{c^2}{2}\ln\left[1+\left(\frac{y}{c}\right)^2\right]\Bigg|_0^c && (11)\\
&= 4,000bc\ln(2) = 2,773bc && (12)
\end{aligned}$$

- Equilibrium requires that $C = T$

$$2,773bc = A_s f_y \quad (13)$$

From the strain diagram:

$$\begin{aligned}
\frac{.003}{c} &= \frac{\varepsilon_y + .003}{d} \Rightarrow c = \frac{(.003)d}{\varepsilon_y + .003} && (14)\\
c &= \frac{(.003)(23)}{\frac{60}{29,000} + .003} = 13.6\text{in.} && (15)
\end{aligned}$$

- Combining Eq. 13 with Eq. 15

$$A_s = \frac{(2,773)(10)(13.6)}{60,000} = 6.28\text{in.}^2 \tag{16}$$

- To determine the moment, we must first determine the centroid of the compressive force measured from the neutral axis

$$\begin{aligned}
\overline{y} &\stackrel{\text{def}}{=} \frac{\int y dA}{A} = \frac{b\int y \overbrace{\sigma dy}^{dC}}{C} = \frac{b\int_0^c \frac{8,000\frac{y^2}{c}}{1+(y/c)^2}dy}{2,773bc} = \frac{8,000b}{2,773bc^2}\int_0^c \frac{y^2}{1+\left(\frac{1}{c}\right)^2 y^2}dy \\
&= \frac{2.885}{c^2}\int_0^c \frac{y^2}{1+\left(\frac{1}{c}\right)^2 y^2}dy = \frac{2.885}{(13.61)^2}\left[\frac{y}{\left(\frac{1}{c}\right)^2} - \frac{1}{\left(\frac{1}{c}\right)^2}\int_0^c \frac{dy}{1+\left(\frac{1}{c}\right)^2 y^2}\right] \\
&= .01557\left[yc^2 - c^2\left[\frac{1}{\sqrt{\frac{1}{c^2}}}\tan^{-1} y\sqrt{\frac{1}{c^2}}\right]\right]\Bigg|_0^c \\
&= .01557\left[c^3 - c^3\tan^{-1}(1)\right] = (.01557)(13.61)^3(1-\tan^{-1}(1)) = 8.43\text{in.}
\end{aligned} \tag{17}$$

- Next we solve for the moment

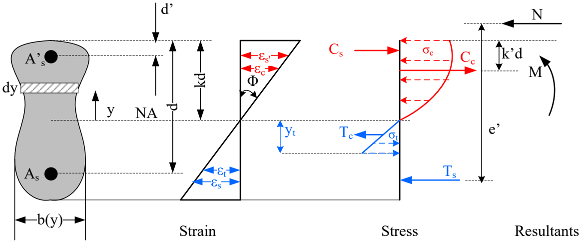
$$M = A_s f_y(d - c + \overline{y}) = (6.28)(60)(23 - 13.61 - 8.43) = 6,713 \text{ k.in} \tag{18}$$

- Using the ACI Code

$$\begin{aligned}
\rho_b &= .85\beta_1 \frac{f_c'}{f_y}\frac{87}{87+60} = (.85)^2\frac{4}{60}\frac{87}{147} = .0285 \\
A_s &= \rho_b bd = (.0285)(10)(23) = 6.55\text{in.}^2 \\
a &= \frac{A_s f_y}{.85 f' cb} = \frac{(6.55)(60)}{(.85)(4)(10)} = 11.57\text{in.} \\
M &= A_s f_y\left(d - \frac{a}{2}\right) = (6.55)(60)\left(23 - \frac{11.57}{2}\right) = 6,765\text{k.in} \\
c &= \frac{a}{\beta_1} = \frac{11.57}{.85} = 13.61\text{in.}
\end{aligned}$$

- We summarize

		Exact	ACI
A_s	in^2	6.28	6.55
c	Kip	13.6	13.6
$\overline{y}'$	in.	5.18	5.78
M	K-in	6,713	6,765



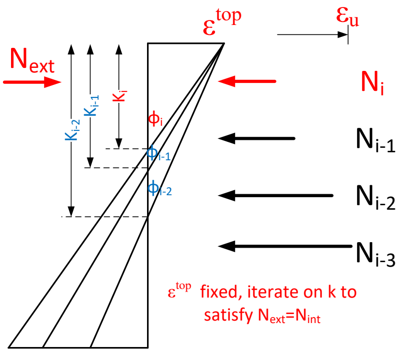
- External axial force N_{ext} is applied with an eccentricity e' and is fixed.
- Internal equilibrium of forces requires $N_{ext} = N_{int}$ and $M_{int} = N_{int}.e'$
- We seek to determine the $M - \Phi$ relation

$$N_{int} = \underbrace{\int_0^{k.d} \sigma_c b_c dy}_{C_c} + \underbrace{A'_s f'_s}_{C_s} - \underbrace{\int_0^{y_t} \sigma_t b_t dy}_{T_c} - \underbrace{A_s f_s}_{Ts} \quad (19)$$

$$N_{int} e' + M = C_c(d - k'd) + C_s(d - d') - T_c(d - kd - \frac{2}{3} y_t) \quad (20)$$

Note: $\sigma_c = \sigma_c(\varepsilon_c)$, $b_c = b_c(y_c)$, $\sigma_t = \sigma_t(\varepsilon_t)$, $b_t = b_t(y_t)$.

- For a given (and fixed) N_{ext}, we gradually increase ε_c^{top} (i.e. Φ indirectly), and solve for k and corresponding M_{int}. This will result in the $M - \Phi$ diagram.
- The problem is nonlinear as there is only one value of k which will ensure equilibrium of axial forces.
- Caution: If $N_{ext} \neq 0$, then must add the initial strain due to the constant force.



1. Increment top strain $\varepsilon_{n+1}^{top} = \varepsilon_n^{top} + \Delta\varepsilon^{top} < \varepsilon_u$ (N_{ext} fixed).
2. Assume k, Determine forces
3. Steel stress from

$$\begin{aligned}
\Phi &= \frac{\varepsilon_c^{top}}{k.d} = \frac{\varepsilon_s'}{k.d - d'} = \frac{\varepsilon_s}{d - kd} \\
\varepsilon_s &= \Phi(d - k.d) \\
f_s &= E_s \varepsilon_s
\end{aligned}$$

4. Compute N_{int} from Eq. 19
5. If $|N_{ext} - N_{int}| > \epsilon$ correct k and iterate, otherwise exit.

- Solve for the corresponding M from Eq. 20
- Determine corresponding curvature from

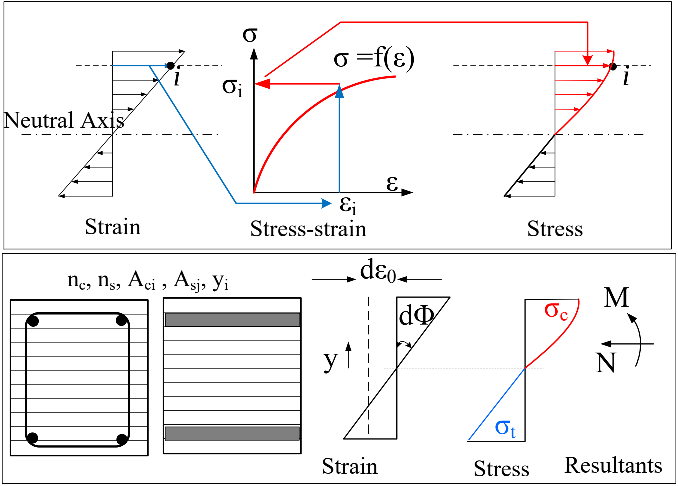
$$\Phi_{i+1} = -\frac{\varepsilon_{i+1}^{top}}{k_{i+1}d} \tag{21}$$

and stiffness

$$EI_{i+1} = \frac{dM}{d\Phi} \tag{22}$$

- Once completed, plot $M-\Phi$, and identify Φ_{cr}, Φ_y and Φ_u, ductility ratio $\xi = \frac{\Phi_u}{\Phi_y}$
- if we repeat analysis for different N, we could then generate the beam-column interaction diagram (corresponding to M_u).

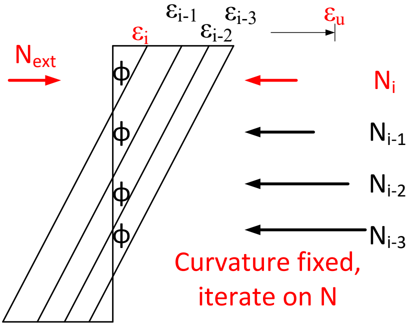
Analytical evaluation of $M - Phi$ was presented; Next a numerical procedure is described. It begins with the discretization of the cross-section into layers.



- Break concrete section into n_c layers (index i), and steel into n_s layers (index j).
- Later, we will differentiate between confined (inside the steel cage) and unconfined as they have different properties.
- A_{ci} and $A_{sj} = A_s$ area of each concrete and steel layer
- y_i distance of fiber i from NA
- For a given (and fixed) N_{ext}, we will gradually increase Φ by $\Delta\Phi$ (note that in the analytical approach we increased ε_c^{top}), and solve for N_{int} (Previously k). We seek to determine M_{ext} for a given Φ.
- At any section $\varepsilon(y) = \varepsilon_0 + y\Phi$ where ε_0 is the axial strain caused by N_{ext}.
- Assume that strain is given, resulting internal force:

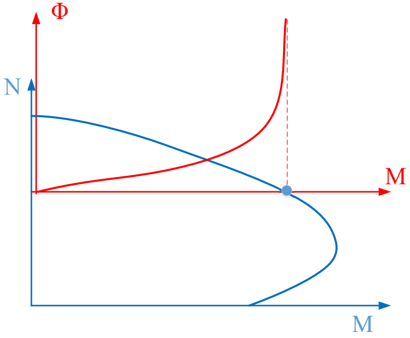
$$N_{int} = \int E d\varepsilon dA = \int EdAd\varepsilon_0 + \int EdAyd\Phi \quad (23)$$

$$= \underbrace{\left[\sum_{i=1}^{n_c} E_{ci}A_{ci} + \sum_{j=1}^{n_s} E_{sj}A_{sj}\right]\varepsilon_0}_{\text{Initial strain}} + \underbrace{\left[\sum_{i=1}^{n_c} E_{ci}A_{ci}y_i + \sum_{j=1}^{n_s} E_{sj}A_{sj}y_i\right]\Phi}_{\text{curvature } \Phi} \quad (24)$$



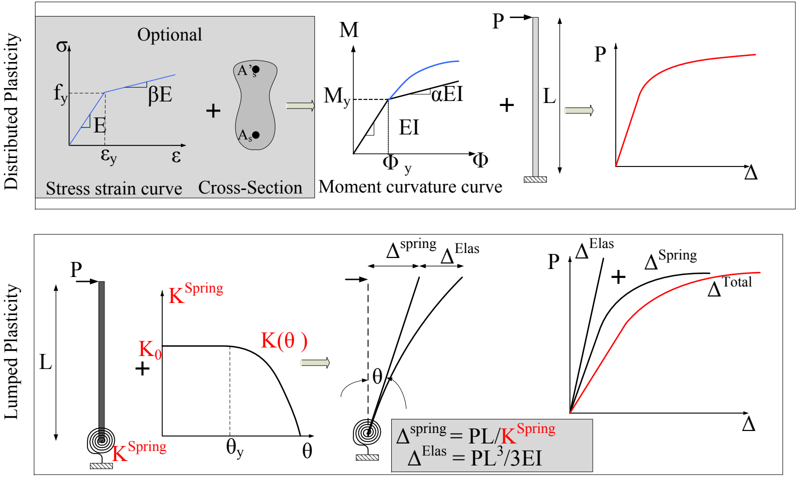
1. Increment curvature $\Phi^{n+1} = \Phi^n + \Delta\phi$ (N_{ext} fixed).
2. Assume neutral axis location (k) and update strain profile $\varepsilon(y_i) = d\varepsilon_0 + y_i d\Phi$ where $d\varepsilon_0 = (\Delta N - E_x \Delta\Phi)/E_a$.
3. If $\varepsilon \geq \varepsilon_u$ (usually 0.003 for concrete), exit.
4. Determine N_{int} from Eq. 24
5. If $|N_{ext} - N_{int}| > \epsilon$ adjust k and iterate, otherwise exit.
6. Compute

$$M_{int} = \sum_{i=1}^{n_c} \sigma_{ci} A_{ci} y_i + \sum_{j=1}^{n_s} \sigma_{sj} A_{sj} y_i$$

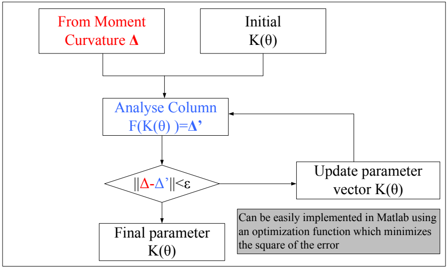


Repeating previous procedure for various N_{ext} we can derive the beam column interaction diagram.

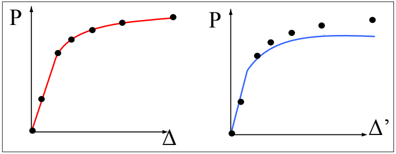
- Single zero length end nonlinear spring, typically linear elements in between.
- Spring must capture effects of
  - Bond
  - Bond-Slip
  - Cracked moment of inertia
  - Diagonal tension



Calibration: Solve for spring stiffness $K(\theta)$ for the two force displacements to be nearly equal



From Moment Curvature Δ
Initial K(θ)
Analyse Column F(K(θ))=Δ'
||Δ-Δ'||<ε
Update parameter vector K(θ)
Final parameter K(θ)
Can be easily implemented in Matlab using an optimization function which minimizes the square of the error



P
Δ
P
Δ'

Non Linear Structural Analysis

Push-Over Analysis

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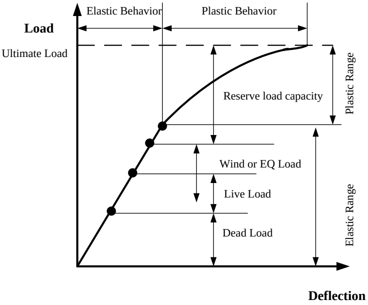
Fall 2020

Draft

Table of Contents I

- In the context of PBEE, Engineering Demand Parameters (EDP) are essential. They are the results of response prediction (i.e analysis). Those include interstory drift, plastic hinge rotations, and member forces. Analysis procedures in FEMA 356/ASCE 41, and commonly used in Performance based design approach, are:
  1. Linear static
  2. Linear time history (LTH)
  3. Nonlinear static: Pushover analysis (POA).
  4. Nonlinear time history (NTH)
- Why is POA relevant?
  - A linear elastic based design would have a much higher reserve strength beyond the elastic, i.e. the ultimate strength is much higher.



- This is due to the structural redundancy and the ability of structural members to deform inelastically without major loss of strength (i.e., ductility).
- In this context, we must differentiate between localized failure and structural failure. The former relates to the failure of one single member, the later to the collapse of the entire structure.

- Before POA is initiated, one must ensure that:

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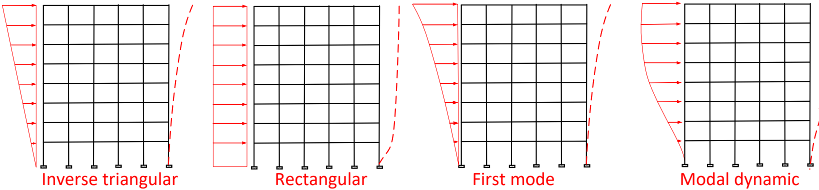
1. Structure is well "understood",
2. Identify element properties and strengths.
3. Concrete: use effective cracked section properties
4. Prepare $M - \Phi$ relations, identify nonlinear concrete model.

- Select lateral load pattern distributed along the height based on first mode response. This can be



Inverse triangular where the force is linearly distributed with height.

Rectangular where we approximate the first mode with very soft or post-yield response with weak first story.

First Mode obtained from a modal analysis.

Modal dynamic deformed shape is based on combining modes.

Force or Displacment Typically in a high seismic zone like California and Washington, use displacement-based design because the requirements for ductility and displacement capacity are more rigorous in seismic than in LRFD,

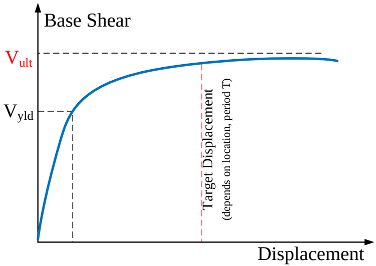
- First the static vertical load is applied, and then the pushover load.
- Though it may be more accurate to imposed displacements, most pushover analysis impose forces instead.
- The objective is to push the structure to the displacement expected under design earthquake, the target displacement (or drift).
- Target displacement can be determined from the response spectrum

$$\Delta = g\left(\frac{T}{2\pi}\right)^2 C_s \tag{1}$$

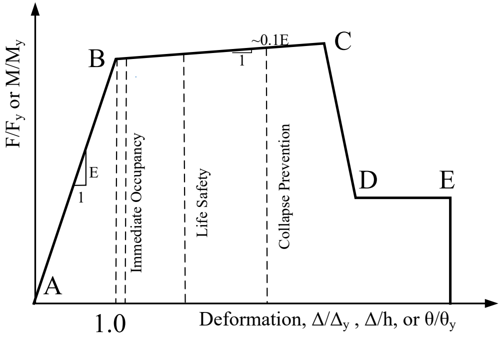
where Δ is the spectral displacement, T is the period, and C_s the elastic seismic coefficient.

- With the increase in the magnitude of the loading, weak links (plastic joint) and failure modes of the structure are found.
- Loading is monotonic with the effects of the cyclic behavior and load reversals being estimated by using a modified monotonic force-deformation criteria and with damping approximations.
- Careful POA may provide very misleading results for force and overturning moments.
- ATC-40 and FEMA-273 documents have developed modeling procedures, acceptance criteria and analysis procedures for pushover analysis. These documents define force-deformation criteria for hinges used in pushover analysis.
- It seeks to determine the collapse mechanism of a structure through a static analysis with increasing load.

- The method allows tracing the sequence of yielding and failure on the member and the structure levels as well as the progress of the overall capacity curve of the structure.
- This essentially defines the Demand on the structure

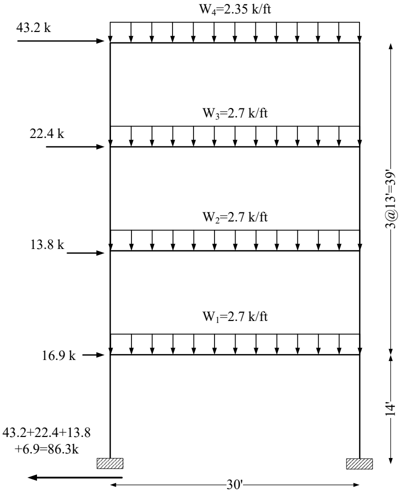


- Prior to a pushover analysis, the moment curvature of the sections must be identified.
- Moment curvatures are often nonlinear, and can be idealized as follows



- This is essentially the Capacity of an individual member.

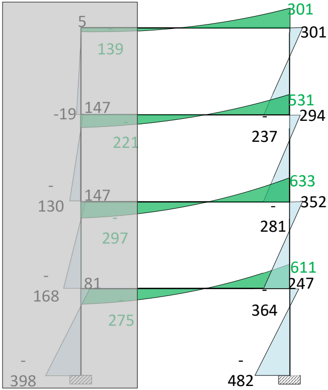
Adapted from *Performance-Based Plastic Design: Earthquake-Resistant Steel Structures* by Goel and Chao.



F_y= 50 ksi, E=30,000 ksi, axial and shear deformation are neglected.
Elastic-perfectly plastic Moment curvature assumed.

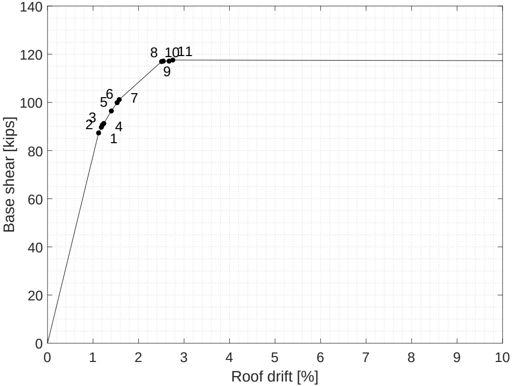
- First an elastic analysis is performed, and all members satisfied $M_{max} < M_p = F_y Z_x$.

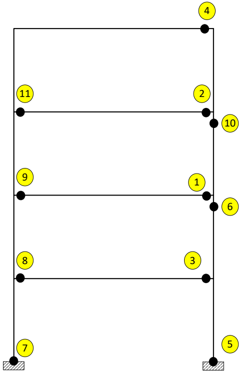
Floor	Beam $M_{u,req}$ (k-ft)	Beam Section	Beam M_p (k-ft)	Column $M_{u,req}$ (k-ft)	Column Section	Column M_p (k-ft)
4	301	W16x40	304	301	W16x40	304
3	531	W24x55	558	294	W16x40	304
2	633	W24x62	638	364	W21x44	398
1	611	W24x62	638	482	W21x55	525



- Base shear (total lateral force) versus roof drift ratio (roof displacement/height) and the location and sequence of formation of the plastic hinges are shown below.

- The lateral force at the elastic limit when the first plastic hinge forms is 87.6 kips, slightly above the design value of 86.3 kips.
- From that point onward, redistribution of moments occurs with plastic hinges forming sequentially, and the frame reaches its ultimate strength of 117.5 kips at a roof drift ratio of 2.7%.
- The yield mechanism turned out to be a partial sway mechanism over 3 stories with plastic hinges at the beam ends and the base of the columns and at the top of the second and third stories.
- The ductility ratio (V_{ult}/V_y) is 1.36.


Base shear [kips]
Roof drift [%]
0
20
40
60
80
100
120
140
0
1
2
3
4
5
6
7
8
9
10
1
2
3
4
5
6
7
8
9
10
11


4
11
2
10
9
1
6
8
3
7
5

Non Linear Structural Analysis

Plasticity III; Limit Analysis of Structure

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Table of Contents I

Table of Contents II

- Elastic solution fulfills:
  1. Equilibrium
  2. "Elasticity" condition ($|M| < M_{yld}$)
  3. Compatibility (continuity)
- Plastic solution fulfills:
  1. Equilibrium
  2. Yield condition ($|M| < M_{pl}$)
  3. Mechanism (additional deformations are possible without load increase)
- Frames typically fail after a sufficient number of plastic hinges form, and the structures turns into a mechanism, and thus collapse (partially or totally).
- At times, it is sufficient to capture the failure mechanism (and corresponding load) and not worry about deflections (strength and not stiffness)
- Limit loads can be determine from two approaches:

- [Upper Bound; Kinematic: A load computed on the basis of an assumed mechanism will always be greater than, or at best equal to, the true ultimate load. We do not seek to simulate the order in which hinges formed, we simply assume the simultaneous presence of all possible hinges. Easier. Assumes equilibrium, fulfills plasticity but may violate formation of mechanism.
- Lower Bound; Statics: A load computed on the basis of an assumed moment distribution, which is in equilibrium with the applied loading, and where no moment exceeds M_p is less than, or at best equal to the true ultimate load. We seek to simulate the order in which hinges formed. Assumes mechanism, fulfills equilibrium but may violate plasticity.

Method	Bound	Assumes	fulfills	Violates
Kinematic	Upper	Mechanism	Plasticity	formation mechanism
Statics	Lower	Equilibrium	Mechanism	Plasticity

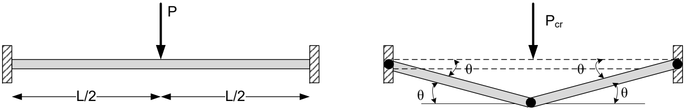
- We shall examine each one separately.

- Theorem: Any set of loads in equilibrium with an assumed kinematically admissible field is larger than or at least equal to the set of loads that produces collapse of the structure. The safety factor is the smallest kinematically admissible multiplier.
- Note similarly with principle of Virtual Work (or displacement).
- A kinematically admissible field is one where the external work W_e done by the forces $\mathbf{F}$ on the deformation $\mathbf{\Delta}_F$ and the internal work W_i done by the moments $\mathbf{M}_p$ on the rotations θ are positives.
- The collapse of a structure can be determined by equating the external and internal work during a virtual movement. Considering a possible mechanism, i, equilibrium requires that $U_i = \lambda_i W_i$
- W_i is the external work of the applied service load a, λ_i is a kinematic multiplier, U_i is the total internal energy dissipated by plastic hinges $U_i = \sum_{j=1}^{n} M_{p_j}\theta_{ij}$, M_{p_j}: plastic moment, θ_{ij} the hinge rotation, and n the number of potential plastic hinges or critical sections.
- Assumptions:

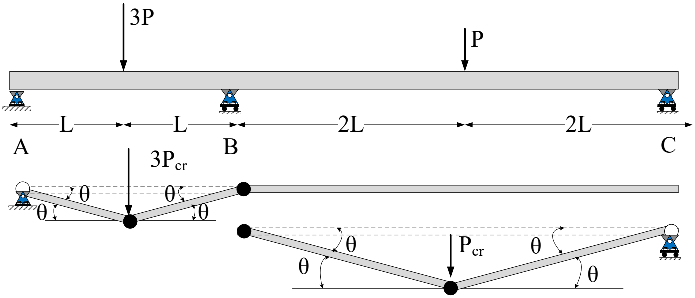
- Response of a member is elastic perfectly plastic.
- Plasticity is localized at specific points.
- Only the plastic moment capacity M_p of a cross section is governing.

- Number of independent mechanisms n is related to the number of possible plastic hinge locations h and number of degree of redundancy r

$$n = h - r \tag{1}$$

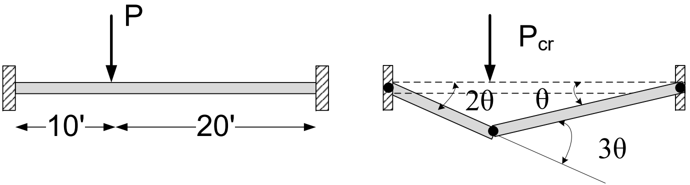


- $n = 3 - 2 = 1$
- Internal work done $M_p\theta + M_p\theta + 2M_p\theta = 4M_p\theta$
- External work done by the point load $P_{cr}\theta L/2$
- Equating the two, we obtain $P_{cr} = 8M_p/L$.

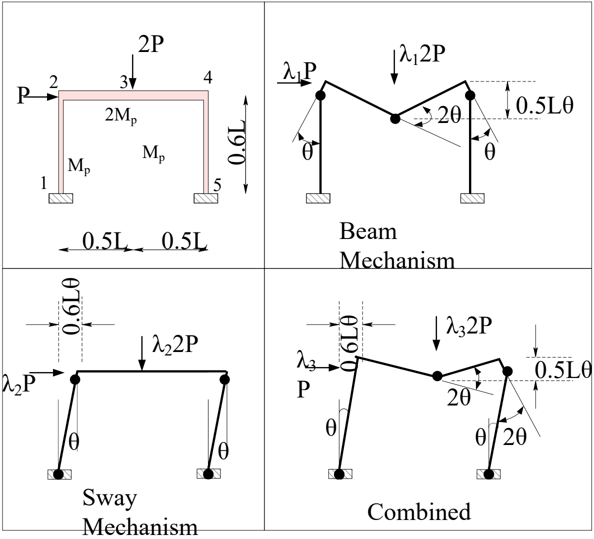


Equating external work done by the vertical forces to the internal work:

- Span AB: $3P_{cr}L\theta = 3M_p\theta$ or $P_{cr} = \frac{M_p}{L}$.
- Span BC: $2P_{cr}L\theta = 3M_p\theta$ or $P_{cr} = \frac{3}{2}\frac{M_p}{L}$



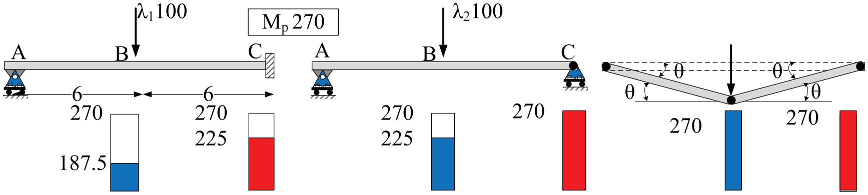
$$
\begin{aligned}
W_{int} &= W_{ext} \\
M_p(\theta + 2\theta + 3\theta) &= P_{cr}\Delta \\
6M_p\theta &= P_{cr}\theta(20) \Rightarrow P_{cr} = 6\frac{M_P}{20} \\
P_{cr} &= 0.3M_p
\end{aligned}
$$


2P
P
2
3
4
$2M_p$
M_p
M_p
1
5
0.6L
0.5L
0.5L
$\lambda_1 2P$
$\lambda_1 P$
$0.5L\theta$
2θ
θ
θ
Beam Mechanism
$0.6L\theta$
$\lambda_2 2P$
$\lambda_2 P$
θ
θ
Sway Mechanism
$0.6L\theta$
$\lambda_3 2P$
λ_3 P
$0.5L\theta$
2θ
θ
θ
2θ
Combined

- $n = 5 - 3 = 2$ independent modes.
- The total number of possible mechanisms is three (one is dependent on the other two).
- To verify that λ is indeed the lowest bound, we may draw the corresponding moment diagram, and verify that at no section is the moment greater than M_p.

Beam Mechanism	$M_p(\theta + \theta) + 2M_p(2\theta) = \lambda_1(2P)(0.5L\theta)$	$\Rightarrow \lambda_1 = 6\dfrac{M_p}{PL}$
Sway Mechanism	$M_p(\theta + \theta + \theta + \theta) = \lambda_2(P)(0.6L\theta)$	$\Rightarrow \lambda_2 = 6.67\dfrac{M_p}{PL}$
Combined Mechanism	$M_p(\theta + \theta + 2\theta) + 2M_p(2\theta) = \lambda_3\left(P(0.6L\theta) + 2P(0.5L\theta)\right)$	$\Rightarrow \lambda_3 = 5\dfrac{M_p}{PL}$

- Assumptions
  - The applied loads must be in equilibrium with the internal forces.
  - There must be a sufficient number of plastic hinges for the formation of a mechanism.
- load computed on the basis of an assumed moment distribution, which is in equilibrium with the applied loading, and where no moment exceeds M_p is less than, or at best equal to the true ultimate load.
- Note similarly with principle of complementary virtual work.
- The statics method of solution is as follows:
  1. Select redundant moments.
  2. Draw the statically determinate moment diagram.
  3. Superimpose the redundant moments on the determinate moment diagram and determine the peak moments.
  4. Set peak moments equal to M_p and check that the number of plastic hinges is sufficient to form a mechanism.
  5. Compute the corresponding ultimate load by statics.

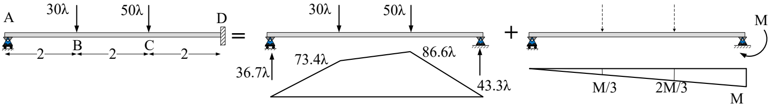


- Perform an incremental analysis.
- Stage 1: We have a statically indeterminate structure subjected to P=100 (magnitude is irrelevant, could have been 1, 10, etc.), following analysis, we obtain;

$$
\begin{array}{rclcrcl}
M_C & = & \frac{3}{16}PL = 225 & \Rightarrow & \lambda_1^C & = & \frac{270}{225} = 1.2 \\
M_B & = & \frac{5}{32}PL = 187.5 & \Rightarrow & \lambda_1^B & = & \frac{270}{187.5} = 1.44
\end{array}
$$

Thus, the first hinge forms at C, and $P = 1.2(100) = 120$, $M_C^{total} = 1.2(225) = 270$, and $M_B^{total} = 1.2(187.5) = 225$

- Stage 2: A hinge has formed at C ($M_C = 0$), we now have a statically determinate beam with a point middle load of again P=100 and $M_B = PL/4$=300. The remaining plastic moment capacity at B is 270-225=45, $\Rightarrow \lambda_2 = 45/300$=0.15
- Adding the two loads (by now M_C^{total}=270, and M_B^{total}=225+0.15(300)=270), and the total $\lambda_C = \lambda_1 + \lambda_2 = 1.2 + 0.15 = 1.35$, thus the collapse load is $1.35(100) = 135$



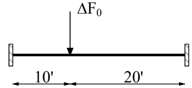
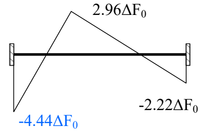
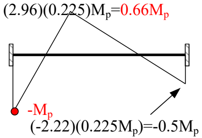
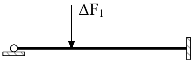
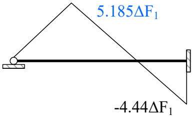
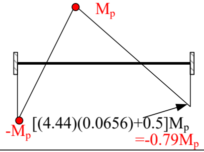
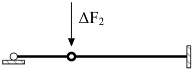
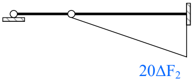
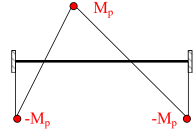
- Statically indeterminate structure. Rather than performing an analysis to determine where first hinge occurs, we make two assumptions
- First consider the formation of hinges at C and D

$$\left\{ \begin{array}{lcccc} M^D & & & = & M_p \\ M^C & = & 86.6\lambda - \frac{2}{3}M^D & = & M_p \end{array} \right. \Rightarrow \lambda = \frac{M_p}{52}$$

- The other possibility hinges at B and D

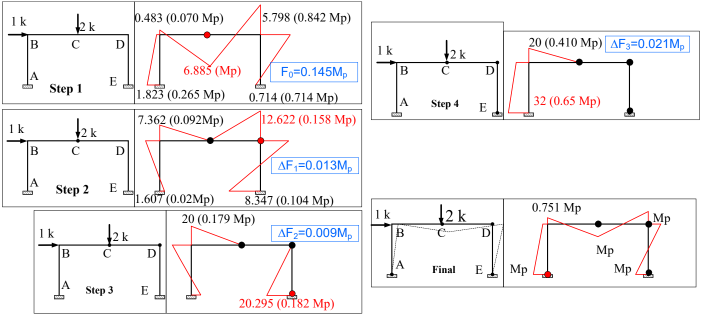
$$\left\{ \begin{array}{lcccc} M^D & & & = & M_p \\ M^B & = & 73.4\lambda - \frac{1}{3}M^D & = & M_p \end{array} \right. \Rightarrow \lambda = \frac{M_p}{55}$$

- Hence, the second case governs.

MODEL	INCREMENTAL M		TOTAL M
		$-4.44\Delta F_0 = M_p$ $\downarrow$ $\Delta F_0 = 1/4.44 M_p$ $= 0.225 M_p$	
		$5.185\Delta F_1 + 0.66M_p = M_p$ $\downarrow$ $\Delta F_1 = 0.0656M_p$ $\downarrow$ $F_1 = \Delta F_0 + \Delta F_1$ $= (0.225+0.0656)M_p$	
		$-20\Delta F_2 - 0.79M_p = -M_p$ $\downarrow$ $\Delta F_2 = 0.0105Mp$ $\downarrow$ $F_2 = (0.225+0.0656$ $+0.0105)M_p = \mathbf{0.30M_p}$	

1. First we consider the original structure
   1. Apply a load F_0., determine the corresponding moment diagram.
   2. Identify the largest moment (-4.44F_0) and set it equal to M_P. This is the first point where a plastic hinge will form.
   3. We redraw the moment diagram in terms of M_P.
2. Next we consider the structure with a plastic hinge on the left support.
   1. Apply an incremental load ΔF_1.
   2. Draw the corresponding moment diagram in terms of ΔF_1.
   3. Identify the point of maximum **total** moment as the point under the load $5.185\Delta F_1 + 0.666 M_P$ and set it equal to M_P.
   4. Solve for ΔF_1, and determine the total externally applied load.
   5. Draw the updated total moment diagram. We now have two plastic hinges, we still need a third one to have a mechanism leading to collapse.
3. Finally, we analyze the revised structure with the two plastic hinges.
   1. Apply an incremental load ΔF_2.
   2. Draw the corresponding moment diagram in terms of ΔF_2.
   3. Set the total moment node on the right equal to M_P.

4. Solve for ΔF_2, and determine the total external load. This load will correspond to the failure load of the structure.

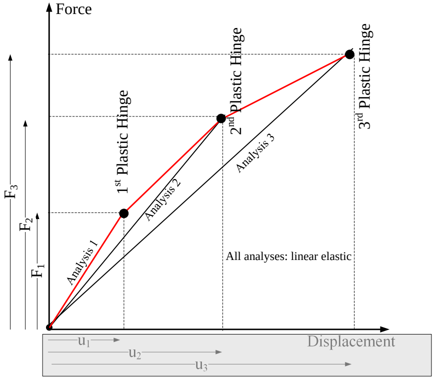


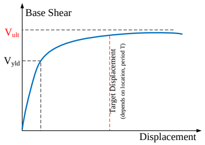
NEED TO CHECK PROCEDURE CORRECT NUMBERS FOR MP DO NOT

1. First plastic hinge will occur at C: $6.885F_0 = M_p \Rightarrow F_0 = 0.145M_p$.
2. Second hinge at D: $M_{max} = M_p - 0.842M_p = 0.158M_p$, and $\Delta F_1 = \frac{0.158}{12.633}M_p = 0.013M_p$

3. Third hinge at E: $M_{max} = M_p - (0.714 + 0.104)M_p = 0.182M_p$ and $\Delta F_2 = \frac{0.182}{20.295}M_p = 0.009M_p$
4. Fourth hinge at A $M_{max} = M_p - 0.344M_p = 0.656M_p$ and $\Delta F_3 = \frac{0.656}{32}M_p = 0.021M_p$
5. Hence the final collapse load is $F_0 + \Delta F_1 + \Delta F_2 + \Delta F_3 = (0.145 + 0.013 + 0.009 + 0.021)M_p = 0.188M_p$ or $F_{max} = 3.76\frac{M_p}{L}$

- The statics approach to determine the failure mechanism/load bears great similarity with the Push Over analysis that will be covered later.
- Major difference: in the approach followed in the preceding examples, linear elastic analyses are performed, and the procedure is akin of a nonlinear solution using the Secant method.
- Note that in our analysis, we do not need to keep track of the displacements (whereas in a Push Over analysis, those are essential to determine the Capacity Curve.

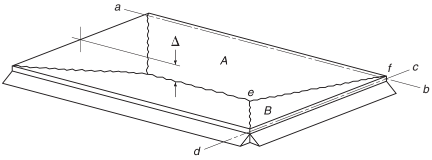




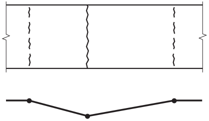
- ACI code: The design of the slab may be achieved through the combined use of classic solutions based on a linearly elastic continuum, numerical solutions based on discrete elements, or yield-line analyses.
- Yield line theory (YLT) investigates failure mechanisms at the ultimate limit state. It is simple, but demands familiarity with the failure patterns (i.e. knowledge of how slabs may fail).
- When a slab is on the verge of collapse (sufficient number of real or plastic hinges to form a mechanism) axes of rotation will be located along lines of support or over point supports such as columns. The slab segments can be considered to rotate as rigid bodies in space about these axes of rotation.
- Two type of YL: positive (crack below) and negative (crack on top).
- Guidelines for establishing YL patterns:
  1. YL are straight lines (intersections of two planes).
  2. YL are axes of rotation.
  3. Supported edges of a slab will also establish axes of rotation. If fixed: -ve YL; if free: no restraint.

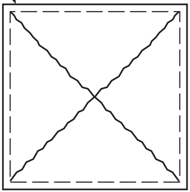
4. Continuous supports repel and simple supports attract positive or sagging YL.
5. Axis of rotation passes over any column support. Orientation depends on other considerations.
6. YL form under concentrated loads radiating outward.
7. YL between two slab segments must pass through the point of intersection of the axes of rotation of the adjacent slab segments.

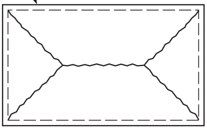
- The aim of investigating YL patterns is to find the one pattern that gives the least load capacity).

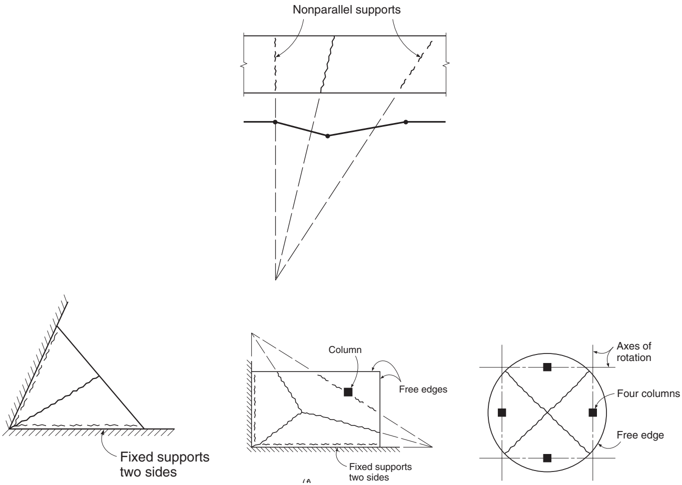


- Slab simply supported along its four sides.
- Rotation of slab segments *A* and *B* is about *ab* and *cd*.
- YL ef is a straight line passing through *f* (point of intersection of the axes of rotation).

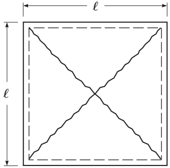







Nonparallel supports
Fixed supports
two sides
Column
Free edges
Fixed supports
two sides
Axes of
rotation
Four columns
Free edge

This method is seldom used for slab analysis.



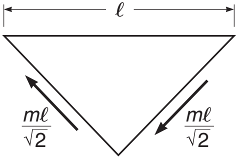
A square slab is simply supported along four sides and is isotropically reinforced. Given the plastic moment per linear foot m_P, determine the uniform w_{ult}.

Due to symmetry, the YL pattern is as shown. Considering equilibrium of moment of any of the four slab segments about its support:

$$M_{ext} = w \underbrace{\frac{l^2}{4}}_{\text{area}} \underbrace{\frac{L}{6}}_{\text{moment arm}}$$

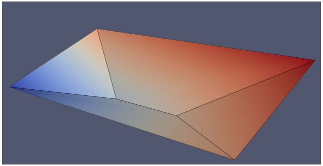
$$M_{int} = 2 \underbrace{\frac{m_p l}{\sqrt{2}}}_{\text{moment}} \underbrace{\frac{1}{\sqrt{2}}}_{\text{Hor. component}}$$

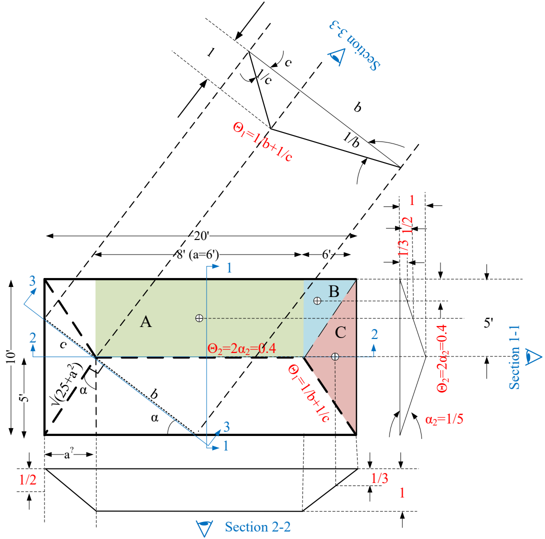
$$M_{ext} = M_{int} \quad \Rightarrow \quad w_{ult} = \frac{24 m_p}{l^2}$$



- External work: $W_{ext} = \sigma(N_i\delta_i)$ where N is the resultant force, δ corresponding vertical displacement.
- Internal work: $W_{int} = \sum ml\theta$ where m is the internal moment in the slab per meter run, l is the length of the YL or its projected length onto the axis os the rotation for the corresponding region; θ rotation of the region about its axis.

The two way slab is simply supported on all four sides and supports a uniform load w. Determine the required resistance for the slab.





- +ve YL form as shown, dimension a is unknown.
- From geometry: length of diagonal $=\sqrt{25+a^2}$, and form similar triangles

$$\frac{b}{5} = \frac{\sqrt{25+a^2}}{a} \Rightarrow b = 5\frac{\sqrt{25+a^2}}{a} \tag{2}$$

$$\frac{c}{a} = \frac{\sqrt{25+a^2}}{5} \Rightarrow c = a\frac{\sqrt{25+a^2}}{5} \tag{3}$$

- Corresponding to a unit deflection, the rotation of the plastic hinge at the diagonal YL is
  $\theta_1 = \frac{1}{b} + \frac{1}{c} = \frac{a}{5\sqrt{25+a^2}} + \frac{5}{a\sqrt{25+a^2}} = \frac{1}{\sqrt{25+a^2}}\left(\frac{a}{5} + \frac{5}{a}\right)$
- The rotation of the yield line parallel to the long edges of the slab
  $\theta_2 = \frac{1}{5} + \frac{1}{5} = 0.40$
- Assume $a = 6$ ft, then the length of the diagonal YL is $\sqrt{25+36} = 7.81$ ft.
- Corresponding rotation of the diagonal YL is: $\theta_1 = \frac{1}{7.81}\left(\frac{6}{5} + \frac{5}{6}\right) = 0.261$
- Rotation angle at the central YL: $\theta_2 = 0.4$.

- $W_{int} = 4(m_p \times 7.81 \times 0.261) + (m_p \times (20-6-6) \times 0.4) = 11.36 m_p$
- $W_{ext} = \underbrace{2\left(10 \times 6 \times \frac{1}{2} w \times \frac{1}{3}\right)}_{A} + + \underbrace{4\left(6 \times 5 \times \frac{1}{2} w \times \frac{1}{3}\right)}_{B} + \underbrace{2\left(8 \times 5 w \times \frac{1}{2}\right)}_{C} =$ $80w$
- Equating $W_{int} = W_{ext}$ gives $w_{ult} = \frac{11.36 M_P}{80} = 0.14 m_P$
- Successive trials

a	W_{int}	W_{ext}	w_{ult}
6.0	11.36 m_P	80.0 w	0.142m_P
6.5	11.08 m_P	78.4 w	0.141 m_P Controls
7.0	10.87 m_P	76.6 w	0.142 m_P
7.5	10.69 m_P	75.0 w	0.143m_P

- Note that if a unit width strip was considered instead, then $m_p = \frac{w_{ult} L^2}{8} \Rightarrow w_{ult} = \frac{8 m_p}{L^2} = \frac{8 m_p}{10^2} = 0.08 m_p$ instead of $0.141 m_p$.

Non Linear Structural Analysis

Nonlinear Analysis; Introduction: Numerical Methods

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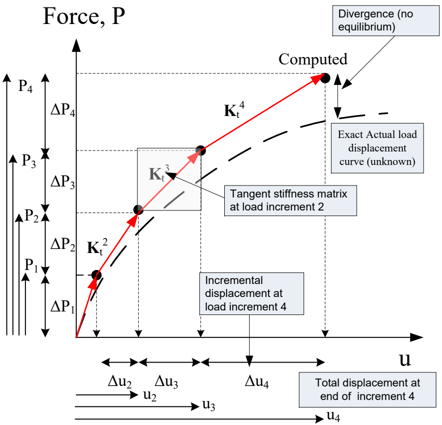


- In an explicit integration scheme (also known as “step by step”), load is applied incrementally and at the end of each increment:
  1. Compute the tangent stiffness on the basis of the current displacements, this is the slope of the load displacement curve);
  2. Invert the stiffness matrix and multiply it by the incremental load to get the corresponding incremental displacement;
  3. Add the incremental displacement to the sum of the previous ones to obtain the actual displacement corresponding to the actual load (sum of all previous incremental loads).

- Major advantage a solution will always be found
- Major disadvantage: at the end of each increment we do not verify that equilibrium between internal and external forces is satisfied. This may result in a diverging solution as the load increases.
- A partial palliative to this problem, is the adoption of very small load increments to minimize errors.
- This method should be used extremely carefully, as a solution will always be obtained no matter how good or bad the model and its parameters are.
- Unfortunately, there are some constitutive models which are very fragile when run within an implicit integration scheme, and as a result they are used (or misused) in an explicit one.

- Linear problems: unique solution; Nonlinear problems: can not ensure the existence of a solution, nor ensure the uniqueness of one.
- At best we can say that an approximate numerical solution of the problem is given, or that an approximation does not exist (typically this implies local or global failure).
- Most widely used class of numerical solution: "Newton Methods", or "Quasi Newton". Other methods may include the bisection method (only linearly convergent).
- Essence of the method which seeks to solve $f(x) = 0$, is to linearize the equation about the current approximation x_n and solve for the resulting linear equation for the next approximation x_{n+1}

- Traditionally, Newton's method start with Taylor's series where we express a function as an infinite series with respect to point $\overline{x}$:

$$
\begin{aligned}
f(x) &= f(\overline{x}) + (x-\overline{x})f'(\overline{x}) + \frac{(x-\overline{x})^2}{2!}f''(\overline{x}) + \ldots \\
\Rightarrow &= f(\overline{x}) + (x-\overline{x})f'(\overline{x}) + \mathcal{O}(\underbrace{|x-\overline{x}|^2|}_{\xi^2})
\end{aligned}
$$

- Ignoring the higher order terms, gives a linear function

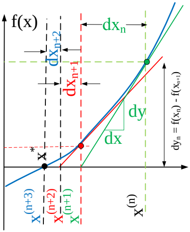
$$L(x) = f(\overline{x}) + (x-\overline{x})f'(\overline{x})$$

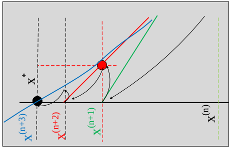
- If $\mathtt{f}(\mathtt{x})$ is a function of two variables, then $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{x} = \lfloor\ x_1 \quad x_2\ \rfloor^T$. The Taylor series expansion about the fixed point $(\overline{x}_1, \overline{x}_2)$ will be

$$\mathtt{f}(x_1, x_2) = f(\overline{x}_1, \overline{x}_2) + \frac{\partial f(\overline{x}_1, \overline{x}_2)}{\partial x_1}(x_1 - \overline{x}_1) + \frac{\partial f(\overline{x}_1, \overline{x}_2)}{\partial x_2}(x_2 - \overline{x}_2)$$

Ignoring the higher order terms, we have again linearized the equation.

- If we set $f(x) = 0 \Rightarrow x \simeq \overline{x} - \frac{f(\overline{x})}{f'(\overline{x})}$
- This is an approximate solution, at $\overline{x}$, which presumes that we also have $f'(x)$.
- In an iterative procedure, this equation can be rewritten as





$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= f'(x_n) \\
\Rightarrow \mathrm{d}x &= \frac{\mathrm{d}y}{f'(x_n)} = \frac{\overbrace{f(x_{n+1})}^{0} - f(x_n)}{f'(x_n)} \\
x_{n+1} &\simeq x_n \underbrace{- \frac{f(x_n)}{f'(x_n)}}_{\delta x_n}
\end{aligned}
$$

- Convergence will be ensured when $|\delta x_n| \leq \varepsilon_\delta$ or $|f(x_{n+1})| \leq \varepsilon_f$

Solve $f(x) = Tan(x) - x = 0$

```
clear
xn = 4.3;
n = 0;
epsi=1e-4;
maxiter = 20;
disp("                    ")
disp("     n            xn                             norm")
xn_m1=0.;
for i = 1:maxiter
    f_x=tan(xn)-xn;df_dx=sec(xn)^2-1;
    xn = xn - f_x/df_dx;
    my_norm = abs(xn-xn_m1);
    disp(sprintf( "%5i   %16.15e   %16.15e",i,  xn,my_norm"))
    if my_norm <epsi
        break
    end
    xn_m1=xn;
end
```

Note that this is a particularly sensitive problem, because `tan` x is discontinuous, a small change in the initial guess may yield to divergence of the solution.

- For the single variable

$$x_{n+1} \simeq x_n \underbrace{- \frac{f(x_n)}{f'(x_n)}}_{\delta x_n} \Rightarrow f(x) + f'(x)(x_2 - x_1) \simeq 0$$

- If we want to solve two equations with two unknowns, then we should linearize $f_1(\mathtt{x})$ and $f_2(\mathtt{x})$ and

$$L(\mathtt{x}) = \mathrm{f}(\overline{\mathtt{x}}) + J(\overline{\mathtt{x}})(\mathtt{x} - \overline{\mathtt{x}}) = 0 \Rightarrow$$

or

$$L(\mathtt{x}) = \begin{bmatrix} f_1(\overline{\mathtt{x}}) + \frac{\partial f_1(\mathtt{x})}{\partial x_1}(x_1 - \overline{x}_1) + \frac{\partial f_1(\mathtt{x})}{\partial x_2}(x_2 - \overline{x}_2) \\ f_2(\overline{\mathtt{x}}) + \frac{\partial f_2(\mathtt{x})}{\partial x_1}(x_1 - \overline{x}_1) + \frac{\partial f_2(\mathtt{x})}{\partial x_2}(x_2 - \overline{x}_2) \end{bmatrix} = 0$$

where $J(\overline{\mathtt{x}})$ is the Jacobian matrix

$$J(\overline{\mathtt{x}}) = \begin{bmatrix} \frac{\partial f_1(\mathtt{x})}{\partial x_1} & \frac{\partial f_1(\mathtt{x})}{\partial x_2} \\ \frac{\partial f_2(\mathtt{x})}{\partial x_1} & \frac{\partial f_2(\mathtt{x})}{\partial x_2} \end{bmatrix}$$

and $J(\overline{\mathtt{x}})(\mathtt{x} - \overline{\mathtt{x}})$ is a matrix vector product; Note that the i^{th} row corresponds to the gradient of the i^{th} component function f_i (∇f_i.).

- Scalar derivative has been replaced by the 2x2 Jacobian matrix of partial derivatives.
- We can further generalize the problem to one of m nonlinear equations with m unknowns

$$\mathtt{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{Bmatrix} \text{ and } \mathtt{f}(\mathtt{x}) = \begin{Bmatrix} f_1(\mathtt{x}) \\ f_2(\mathtt{x}) \\ \vdots \\ f_m(\mathtt{x}) \end{Bmatrix}$$

- The Jacobian matrix $J(\mathtt{x})$ for this matrix will be an $m \times m$ matrix with (i, j) entries corresponding to the partial derivative of the function i with respect to the variable j or $\frac{\partial f_i(\mathtt{x})}{\partial x_j}$
- At each step of the Newton method, we have an approximate value of $\mathtt{x}_n$ to the exact solution $\mathtt{x}^*$ of the nonlinear equations $\mathtt{f}(\mathtt{x}) = 0$. We thus determine $\mathtt{x}_{n+1}$ by solving the linearized system of equations $L^{(n)}(\mathtt{x}_{n+1}) = 0$.

$$\mathtt{f}(\mathtt{x}_n) + J(\mathtt{x}_n)(\mathtt{x}_{n+1} - \mathtt{x}_n) = 0 \Rightarrow \mathtt{x}_{n+1} = \mathtt{x}_n - J(\mathtt{x}_n)^{-1}\mathtt{f}(\mathtt{x}_n)$$

- Note the similarity with $J(\mathtt{x}) = f'(x)$ for $m = 1$.
- At each step we should evaluate the Jacobian matrix at a new point, and then solve a linear system of equations using this new updated matrix.
- Again convergence will be ensured when

$$\parallel \delta x_n \parallel \leq \varepsilon_\delta \ \text{ or } \ \parallel f(x_{n+1}) \parallel \leq \varepsilon_f$$

where $\parallel \mathtt{v} \parallel$ is the Euclidian norm of $\mathtt{v}$ (strictly speaking, it should be written as $\parallel \mathtt{v} \parallel_2$) computed as the square root of the sum of the vector components square,

$$\parallel \mathtt{v} \parallel = \sqrt{\sum_{i=1}^{N} v_i^2}$$

in an N dimensional space.

- Given an initial $\mathtt{x}$, a required tolerance $\varepsilon > 0$
  Repeat
  1. Evaluate $g = f(\mathtt{x})$ and $H = J(\mathtt{x})$
  2. If $\| g \| \leq \varepsilon$, **return** $\mathtt{x}$
  3. $\mathtt{v} = \mathtt{x}_n - \mathtt{x}_{n-1} = \frac{f(\mathtt{x})}{J(\mathtt{x})}$
  4. Solve $H\mathtt{v} = -g$
  5. $\mathtt{x} := \mathtt{x} + \mathtt{v}$

  until maximum number of iterations is exceeded
- Each iteration requires the evaluation of $(\mathtt{x})$ (n scalar functions evaluation in terms of $\mathtt{x}$) and $J(\mathtt{x})$ (n^2 derivatives).

- Must ensure convergence of the method, and the order of the error
- Given $g(x) = x - \frac{f(x)}{f'(x)}$ define a convergence factor $\rho^{(n)}$ as the ratio of the error in x_{n+1} to the error in x_n. Near the exact solution (x^*, where $g(x^*) = x^*$) $\rho^{(n)} \simeq g'(x^*)$ and is called the asymptotic convergence factor. Determining $g'(x)$

$$g'(x) = \frac{f(x)f''(x)}{f'(x)^2}$$

if $f'(x^*) \neq 0$ and $f''(x^*)$ is finite, then $g'(x^*)$=0 and we conclude that the convergence factor tends to zero when and if $x_n \to x^*$.

- To determine the error $x^* - x_n$,

$$x^* - x_{n+1} = x^* - x_n - \frac{f(x^*) - f(x_n)}{f'(x_n)} \Rightarrow x^* - x_{n+1} = -\frac{1}{2}(x^* - x_n)^2 \frac{f''(\xi_n)}{f'(x_n)}$$

Thus if the iteration converges to x^*, there follows

$$\underbrace{x^* - x_{n+1}}_{\text{error at } n+1} \simeq -\frac{f''(x^*)}{2f'(x^*)} \underbrace{(x^* - x_n)^2}_{\text{error at } n} \quad \text{as} \quad n \to \infty$$

as long as $f'(x^*)$ and $f''(x^*)$ are both finite and nonzero.

- Hence we note that the error tends to be proportional to the square of the error in x_n as n tends to infinity.
- Other methods (such as the bisection) have a linear convergence.

The Newton method:

1. Requires an analytical expression of the derivative
2. If the initial value is too far from the correct value, convergence may not be ensured (which is why one must place an upper limit on the number of iterations).
3. Fails if the slope is close to zero (such as around the peak load).
4. Works best with curves with low curvatures.
5. Convergence is often quadratic.

$$\| x_{n+1} - x^* \| \leq c \| x_n - x^* \|^2 \tag{1}$$

however, in practice we do not know what c is

Solve $f(\mathbf{x}) = \left\{ \begin{array}{c} x_1^2 + x_2^2 + x_3^2 - 9 \\ x_3 - x_2 \sin(x_1) \\ 3x_2 + 4x_3 \end{array} \right\} \Rightarrow J(\mathbf{x}) = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \\ -x_2 \cos(x_1) & -\sin(x_1) & 1 \\ 0 & 3 & 4 \end{bmatrix}$

```
f = @(x) [x(1)^2+x(2)^2+x(3)^2-9
    x(3)-x(2)*sin(x(1))
    3*x(2)+4*x(3)] ;
% The Jacobian matrix:
J = @(x) [2*x(1) 2*x(2) 2*x(3)
    -x(2)*cos(x(1)) -sin(x(1)) 1
    0 3 4] ;
% initial guess:
x = [-1;-2;1] ;
maxiter = 10;
tol = 1e-12;
disp(' ')
disp('iteration x(1) x(2) x(3) norm(delta) ')
for n=1:maxiter
    delta = -J(x)\f(x);
    x = x + delta ;
    disp(sprintf( '%5i %10.5e %10.5e %10.5e %8.3e',...
    n, x(1), x(2) , x(3) ,norm(delta, inf))) ;
    if norm(delta, inf) < tol
        break
    end
end
if n==maxiter
    disp("Warning: may not have converged  tolerance not satisfied")
end
```

- Quasi-Newton; Secant Method: In many instances, it is nearly impossible to compute $J(\mathtt{x})$, as $\mathtt{f}(\mathtt{x})$ may not be analytically defined. In such cases, we will numerically determine the Jacobian based on the simple approximation

$$\mathtt{f}'(\mathtt{x}_{n+1}) \simeq \frac{\mathtt{f}(\mathtt{x}_n) - \mathtt{f}(\mathtt{x}_{n-1})}{\mathtt{x}_n - \mathtt{x}_{n-1}}$$

  Substituting for $J(\mathtt{x}_n)^{-1}$

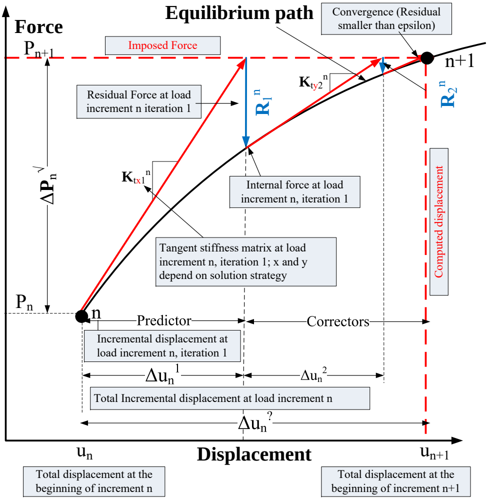
$$\mathtt{x}_{n+1} = \mathtt{x}_n - \underbrace{\frac{\mathtt{x}_n - \mathtt{x}_{n-1}}{\mathtt{f}(\mathtt{x}_n) - \mathtt{f}(\mathtt{x}_{n-1})}}_{\simeq J^{-1}(\mathtt{x}_n)} \mathtt{f}(\mathtt{x}_n)$$

- Modified Newton In some applications, the evaluation of the Jacobian is computationally expensive, and in such case, J^{-1} is kept constant throughout the analysis (this will be referred to the initial stiffness method) or the load increment (modified Newton).

- Objective go from n to $n+1$.
- Jacobian corresponds to the tangent stiffness matrix of the structure which in turn depends on the tangent of the constitutive matrix (D_T). $\left(\mathrm{K}_T = \int_\Omega \mathrm{B}^T \mathrm{D}_T \mathrm{B} d\Omega\right)$.
- So far: $\mathrm{f}(x) = 0$, we know how to handle it.
- In structural analysis must satisfy within an increment $\mathrm{P}^R_{t,n} = \mathrm{P}^{ext}_{t,n} - \mathrm{P}^{int}_{t,n} = 0$, superscript R refers to the residual.
- Internal nodal force vector $\mathrm{P}^{int}_{t,n}$ is a function of nodal displacements $\mathrm{u}_{t,n}$, thus we have a nonlinear problem. (Recall $\mathrm{P}^{int} = \int \mathrm{B}^T \sigma d\Omega$ or $\mathrm{K}\Delta$)
- Within each iteration we determine the residual nodal force vector, and set it to zero: $\mathrm{P}^R_{t,n} = 0$
- It is an iterative procedure that continues until the residual nodal force vector or the incremental nodal displacement vector, is sufficiently small (i.e. convergence is satisfied).

- Newton's methods hinge on our ability to linearize (through a truncated Taylor series) the problem as follows
  $\mathrm{P}_{t,n}^{R,k} = \mathrm{P}_{t,n}^{ext} - \mathrm{P}_{t,n}^{int,k} \qquad \delta\mathrm{u}_{t,n}^{k} = [\mathrm{K}_{tt,n}^{k-1}]^{-1} \cdot \mathrm{P}_{t,n}^{R,k};$
  and
  $\mathrm{u}_{t,n}^{k} = \mathrm{u}_{t,n}^{k-1} + \delta\mathrm{u}_{t,n}^{k}$ where, $\mathrm{u}_{t,n}^{k=0} = \mathrm{u}_{t,n-1}$ and $\mathrm{P}_{t,n}^{int,k=0} = \mathrm{P}_{t,n-1}^{int}$
  and subscript *n* refers to the load increment, and subscript *k* to the iteration number within a load increment.
- Assume equilibrium to have been reached at increment *n*, we then apply an increment of external force $\Delta\mathrm{P}^{ext}$, and we seek to determine the corresponding incremental displacement $\Delta\mathrm{u}_{n+1}$.
- The internal forces and corresponding displacements will then be in (near) equilibrium.
- We distinguish between load increment, and iteration number within an increment to reach equilibrium.

- At each iteration, we determine the residual $\mathrm{R}_i^{(n+1)}$ which corresponds to $\mathrm{P}_{ext} - \mathrm{P}_{int}$, and seek to minimize this residual. At each iteration, we update (in the Newton method) the tangent stiffness matrix which corresponds to the jacobian.
- At the heart of all of them, is the determination of the internal nodal force vector $\mathrm{P}_{t,n}^{int,k}$, and the tangent stiffness matrix $\mathrm{K}_{tt,n}^{k-1}$.

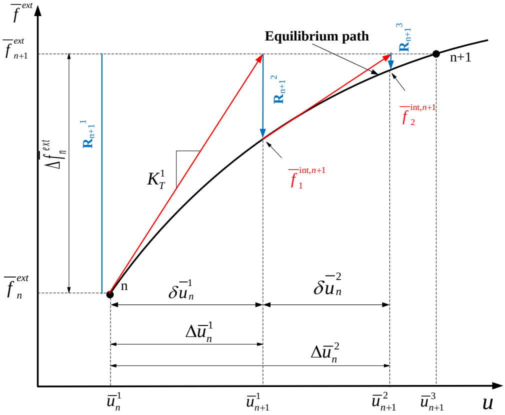

Force
Equilibrium path
Convergence (Residual smaller than epsilon)
Imposed Force
P_{n+1}
n+1
Residual Force at load increment n iteration 1
R_1^n
K_{ty2}^n
R_2^n
ΔP_n
K_{tx1}^n
Internal force at load increment n, iteration 1
Tangent stiffness matrix at load increment n, iteration 1; x and y depend on solution strategy
Computed displacement
P_n
n
Predictor
Correctors
Incremental displacement at load increment n, iteration 1
Δu_n^1
Δu_n^2
Total Incremental displacement at load increment n
$\Delta u_n^?$
u_n
Displacement
u_{n+1}
Total displacement at the beginning of increment n
Total displacement at the beginning of increment n+1

- Newton's method can be redescribed in terms of Predictor-Corrector
- I Predictor (associated with an increment of load) Determine incremental displacement
- II Corrector to check equilibrium iteratively.
  1. Compute the corresponding internal forces (not evaluated in the explicit method);
  2. Compute the residual forces
  3. If residual is larger than user specified convergence criteria, update the displacement by multiplying the inverse of the tangent stiffness matrix by the residual force;
  4. Update the total displacement vector.

  Hence to each load increment, we would have multiple iterations until equilibrium is satisfied within a numerical tolerance.

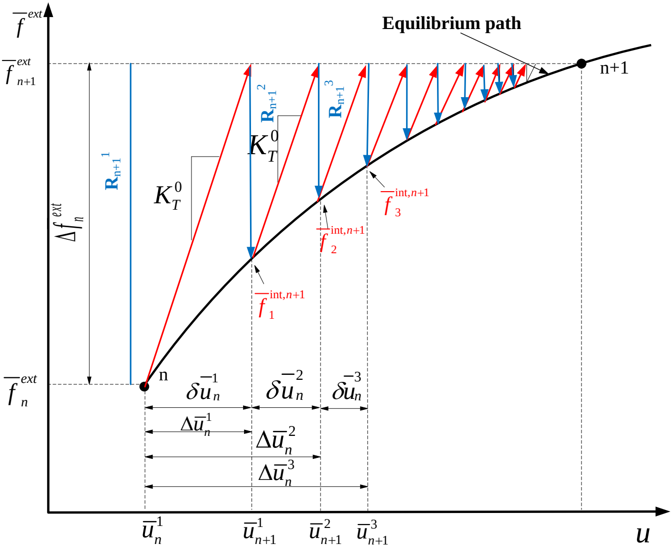
- There are different flavors of this so-called Newton technique. Those are associated with the tangent stiffness matrix to be considered

Method	Tangent stiffness matrix computed at			
	Predictor (x in K^n_{txi})		Corrector (y in K^n_{tyi})	
	Increment	Iteration	Increment	Iteration
Newton-Raphson	n	1	n	i
Modified Newton-Raphson	n	1	n	1
Initial Stiffness	1	1	1	1



- Need to solve $\mathtt{f}(\mathtt{u}^*) = \mathtt{P}^{ext}_{t,n}(\mathtt{u}^*) - \mathtt{P}^{int}_{t,n}(\mathtt{u}^*) = 0$ and $\mathtt{f}(\cdot)$ is the function of internal state value $(\cdot)$. In the preceding equation it is often, but not exclusively, the vector of nodal displacement $\mathtt{u}$.

- Assuming that $\mathtt{u}_{t,n}^{k-1}$ is known, then a Taylor series expansion gives $\mathtt{f}(\mathtt{u}^*) = f(\mathtt{u}_{t,n}^{k-1}) + \frac{\partial \mathtt{f}}{\partial \mathtt{u}}|_{\mathtt{u}_{t,n}^{k-1}} \cdot (\mathtt{u}^* - \mathtt{u}_{t,n}^{k-1}) +$ High-order terms Substituting we obtain $\frac{\partial \mathtt{P}_t^{int}}{\partial \mathtt{u}}|_{\mathtt{u}_{t,n}^{k-1}} \cdot (\mathtt{u}^* - \mathtt{u}_{t,n}^{k-1}) +$ High-order terms $= \mathtt{P}_{t,n}^{ext} - \mathtt{P}_{t,n}^{int,k-1} = \mathtt{P}_{t,n}^{R,k}$ where we assume that the external nodal forces are displacement-independent.
- Since an incremental analysis is driven by external force steps (or time steps Δt), the initial conditions are given by $\mathtt{K}_{tt,n}^{k=0} = \mathtt{K}_{tt,n-1}$, $\mathtt{u}_{t,n}^{k=0} = \mathtt{u}_{t,n-1}$, $\mathtt{P}_{t,n}^{int,k=0} = \mathtt{P}_{t,n-1}^{int}$. Again, the iterations continue until an appropriate convergence criteria is satisfied.
- A characteristic of this iterative method is that an updated tangent stiffness matrix must be determined at each iteration, as such this method is often referred to as the full Newton-Raphson iterative method.


Equilibrium path
n+1
n
$\bar{f}^{ext}$
$\bar{f}^{ext}_{n+1}$
$\bar{f}^{ext}_{n}$
Δf^{ext}_n
$\mathbf{R}^1_{n+1}$
$\mathbf{R}^2_{n+1}$
$\mathbf{R}^3_{n+1}$
K^0_T
K^0_T
$\bar{f}^{int,n+1}_1$
$\bar{f}^{int,n+1}_2$
$\bar{f}^{int,n+1}_3$
$\delta\bar{u}^1_n$
$\delta\bar{u}^2_n$
$\delta\bar{u}^3_n$
$\Delta\bar{u}^1_n$
$\Delta\bar{u}^2_n$
$\Delta\bar{u}^3_n$
$\bar{u}^1_n$
$\bar{u}^1_{n+1}$
$\bar{u}^2_{n+1}$
$\bar{u}^3_{n+1}$
u

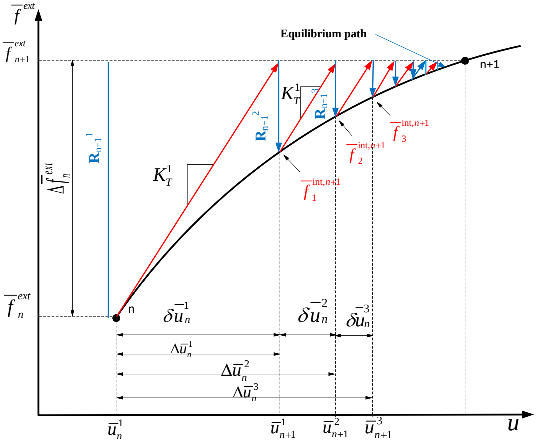
- In the Newton-Raphson iterative method most of the computational effort is associated with the factorization of the tangent stiffness matrix. For large systems, it is often more convenient to modify the approach by reducing the number of such factorizations albeit at the cost of increased number of iterations to reach proper convergence.
- Initial stiffness algorithm

$$\delta \mathrm{u}_{t,n}^{k} = [\mathrm{K}_{tt}]^{-1} \cdot \mathrm{P}_{t,n}^{R,k}$$

with the initial conditions defined by

$$\mathrm{u}_{t,n}^{k=0} = \mathrm{u}_{t,n-1}$$
$$\mathrm{P}_{t,n}^{int,k=0} = \mathrm{P}_{t,n-1}^{int}$$

In this method, only the initial $\mathrm{K}_{tt,n=0}^{k=0}$ needs to be factorized, thus avoiding the expense of recalculating and factorizing many times the tangent stiffness matrix. This initial stiffness iterative method corresponds to a linearization of the response about the initial configuration of the finite element system and will converge very slowly and may even diverge.



$\overline{f}^{ext}$
$\overline{f}^{ext}_{n+1}$
$\overline{f}^{ext}_{n}$
$\Delta\overline{f}^{ext}_{n}$
$\mathbf{R}^{1}_{n+1}$
$\mathbf{R}^{2}_{n+1}$
$\mathbf{R}^{3}_{n+1}$
K^1_T
K^1_T
Equilibrium path
n+1
n
$\overline{f}^{\text{int},n+1}_{1}$
$\overline{f}^{\text{int},n+1}_{2}$
$\overline{f}^{\text{int},n+1}_{3}$
$\delta\overline{u}^{1}_{n}$
$\delta\overline{u}^{2}_{n}$
$\delta\overline{u}^{3}_{n}$
$\Delta\overline{u}^{1}_{n}$
$\Delta\overline{u}^{2}_{n}$
$\Delta\overline{u}^{3}_{n}$
$\overline{u}^{1}_{n}$
$\overline{u}^{1}_{n+1}$
$\overline{u}^{2}_{n+1}$
$\overline{u}^{3}_{n+1}$
u

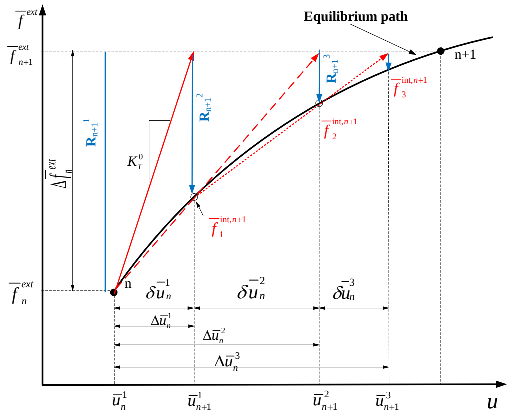
- Modified Newton-Raphson iterative method is an approach somewhat in between Newton-Raphson iterative method and the initial stiffness iterative method.

$$\delta \mathrm{u}_{t,n}^{k} = [\mathrm{K}_{tt,n-1}]^{-1} \cdot \mathrm{P}_{t,n}^{R,k}$$

with the initial conditions

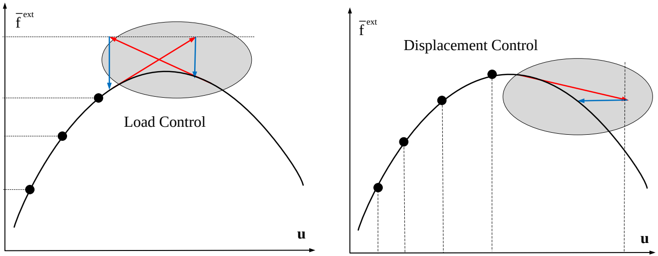
$$\mathrm{u}_{t,n}^{k=0} = \mathrm{u}_{t,n-1}$$
$$\mathrm{P}_{t,n}^{int,k=0} = \mathrm{P}_{t,n-1}^{int}$$

- The modified Newton-Raphson iterative method involves fewer stiffness decompositions than the Newton-Raphson iterative method. The choice of external force steps or time steps when the stiffness matrix should be updated depends on the degree of nonlinearity in the system response; i.e. the more nonlinear the response, the more often the updating should be performed.



we do not explicitly invert the Jacobian (or need to invert $\mathtt{K}_T$), but rather compute $\mathtt{K}_T$ through finite difference.

- In the load control method, we may have oscillation of the solution, or even divergence. This implies that equilibrium was not restored. This is often associated with “failure” at peak load.
- Failure may be structural or just localized in a subregion.
- In most engineering problems, we only seek the peak-load.
- In softening materials (such as concrete), we may be seek to capture the ductility or post-peak response when an imposed displacement is applied (such as thermal or even seismic). However, this is impossible under load control because the determination of the residual is impossible close to a peak.
- Under displacement control (as further explained below), the restoring force can always be determined (unless there is a snap-back).

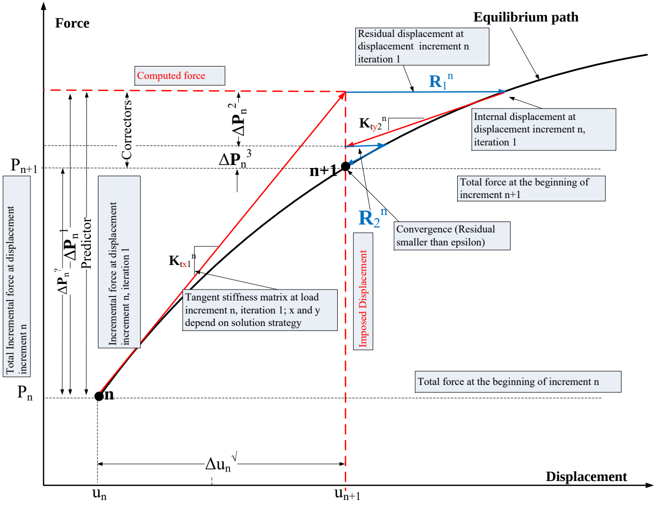


- For displacement control, we define the vector of residual displacements R_n^k as

$$\mathrm{R}_{n+1}^k \equiv \mathrm{R}_n^k(\bar{\mathrm{f}}_{n+1}) = \bar{\mathrm{u}}^{int}(\bar{\mathrm{f}}_{n+1}) - \bar{\mathrm{u}}^{ext} = 0 \tag{2}$$

  note the difference with Equation 2.
- Hence to capture the post-peak response we need to numerically adopt a displacement control algorithm.
- At the beginning of each step $n+1$, we start from the forces $\bar{\mathrm{f}}_n$ that were computed in the previous step through equilibrium, $\mathrm{R}_n^k \approx 0$ or $\bar{\mathrm{u}}_n^{int} \approx \bar{\mathrm{u}}_n^{ext}$.

- The external displacement are now increased from $\overline{\mathrm{u}}_n^{ext}$ to $\overline{\mathrm{u}}_{n+1}^{ext} = \overline{\mathrm{u}}_{n+1}^{ext} + \Delta\overline{\mathrm{u}}^{ext}$, and we seek to determine the corresponding forces $\bar{\mathrm{f}}_{n+1}$ through equilibrium, $\mathrm{R}_{n+1}^{k} \approx 0$ or $\overline{\mathrm{u}}_{n+1}^{int} \approx \overline{\mathrm{u}}_{n+1}^{ext}$.
- Within the current step (identified through the subscript n), we will be iterating (through superscript k) in order to achieve equilibrium, Figure **??**



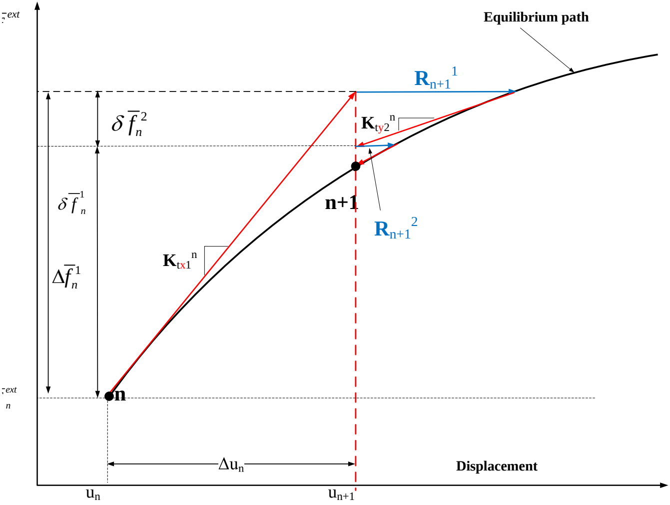
Force
Equilibrium path
Residual displacement at displacement increment n iteration 1
Computed force
Correctors
Predictor
Internal displacement at displacement increment n, iteration 1
Total force at the beginning of increment n+1
Convergence (Residual smaller than epsilon)
Imposed Displacement
Tangent stiffness matrix at load increment n, iteration 1; x and y depend on solution strategy
Incremental force at displacement increment n, iteration 1
Total Incremental force at displacement increment n
Total force at the beginning of increment n
Displacement

- As an initial guess for $\bar{\mathrm{f}}^0_{n+1}$ we take it to be $\bar{\mathrm{f}}_n$, and based on the linearization around this initial state, we have

$$\bar{\mathrm{u}}_{int}(\bar{\mathrm{u}}^0_{n+1}) + \mathrm{K}_T(\bar{\mathrm{f}}^0_{n+1})\Delta\bar{\mathrm{f}}^1_{n+1} = \bar{\mathrm{u}}^{ext}_{n+1} \tag{3}$$

  where $\Delta\bar{\mathrm{f}}^1_{n+1}$ is the first approximation for the unknown displacement increment, $\Delta\bar{\mathrm{f}}_{n+1} = \bar{\mathrm{f}}_{n+1} - \bar{\mathrm{f}}_n$.

- Alternatively, we begin from a linearization of Equation 2, Figure **??**


Equilibrium path
$\mathbf{R}_{n+1}^{1}$
$\mathbf{K}_{ty2}^{n}$
$\delta \bar{f}_n^2$
$\delta \bar{f}_n^1$
$\Delta \bar{f}_n^1$
n+1
$\mathbf{R}_{n+1}^{2}$
$\mathbf{K}_{tx1}^{n}$
n
Δu_n
Displacement
u_n
u_{n+1}

$$\mathrm{R}_n^k(\bar{\mathrm{f}}_{n+1}^{i+1}) \approx \mathrm{R}_n^k(\bar{\mathrm{f}}_{n+1}^{i}) + \left(\frac{\partial \mathrm{R}_n^k}{\partial \bar{\mathrm{f}}}\right)_{n+1}^{i} \delta\bar{\mathrm{f}}_n^i = 0 \tag{4}$$

where i is a counter starting from $\bar{\mathrm{f}}_{n+1}^1 = \bar{\mathrm{f}}_n$. We observe that

$$\frac{\partial \mathrm{R}_n^k}{\partial \bar{\mathrm{f}}} = \frac{\partial \bar{\mathrm{u}}^{int}}{\partial \bar{\mathrm{f}}} = \mathrm{K}_T \tag{5}$$

- Assuming that $\bar{\mathrm{u}}^{ext}$ is constant and K_T is the tangent stiffness matrix, Equation 4 yields

$$\mathrm{K}_T^i \delta\bar{\mathrm{f}}_n^i = -\mathrm{R}_{n+1}^i \tag{6}$$

or

$$\delta\bar{\mathrm{f}}_n^i = -(\mathrm{K}_T^i)^{-1}\mathrm{R}_{n+1}^i \tag{7}$$

- Thus, a series of successive approximations yields

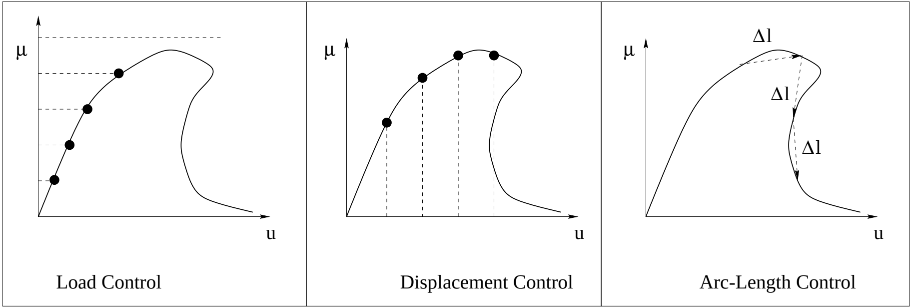
$$\bar{\mathrm{f}}_{n+1}^{i+1} = \bar{\mathrm{f}}_n + \Delta\bar{\mathrm{f}}_n^i = \bar{\mathrm{f}}_{n+1}^i + \delta\bar{\mathrm{f}}_n^i \tag{8}$$

  with

$$\Delta\bar{\mathrm{f}}_n^i = \sum_{k\leq i} \delta\bar{\mathrm{f}}_n^k \tag{9}$$

  very rapidly.

- It should be noted that each iteration involves three computationally expensive steps:
  1. Evaluation of internal displacements $\bar{\mathrm{u}}^{int}$
  2. Evaluation of the global tangent stiffness matrix, K_T
  3. Solution of a system of linear equations



- Displacement control should be used when softening is present; arc length should be used if snap-back is anticipated.
- Arch-length method hinges on our ability to define an arc length in terms of both displacement and force, and then seek a multiplier.

- An appropriate termination criteria of the iteration should be adopted for any incremental solution strategy based on iterative methods. At the end of each iteration, the solution obtained should be checked to see whether it has converged within defined tolerances or whether the iteration may be diverging.
- If the convergence tolerances are too loose, inaccurate results are obtained, and if the tolerances are too tight, much computational effort is spent to obtain needless accuracy.

Some commonly used convergence criteria include:

Displacement criteria $\|\delta \mathbf{u}_n^k\| < \epsilon_D$ where ϵ_D is a displacement convergence tolerance and $\|\cdot\|$ is the Euclidian norm defined as the square root of the sum of the vector components squared.

Force criteria $\mathbf{P}_{t,n}^{R,k}$ and $\|\mathbf{P}_{t,n}^{R,k}\| < \epsilon_F$ where ϵ_F is a force convergence tolerance.

Energy criteria A difficulty with the force criterion is that the displacement solution does not introduce the termination criterion. As an illustration, consider an elasto-plastic truss with a very small strain-hardening modulus entering the plastic region. In this case, the residual force vector may be very small while the displacements may still be much in error. Hence, the convergence criteria may have to be used with very small values of ϵ_D and ϵ_F. Also, the expressions must be modified appropriately when quantities of different units are measured. In order to provide some indication of when both the displacements and the forces are near their equilibrium values, the energy criteria can be used

$$\left|\frac{1}{2} \cdot \mathrm{P}_{t,n}^{R,k} \cdot \delta \mathrm{u}_n^k\right| < \epsilon_E$$

Non Linear Structural Analysis
Element Formulation Notation
Victor E. Saouma

1 Nodal Quantities

$\mathbf{P}_S^{int}$	Internal nodal force vector
$\mathbf{P}_t^{ext}$	External nodal force vector at free degrees of freedom at structural level
$\mathbf{P}_t^{int}$	Internal nodal force vector at free degrees of freedom at structural level
$\mathbf{P}_u^{int}$	Internal nodal force vector at constraint degrees of freedom at structural level
$\mathbf{P}_t^{R}$	Residual nodal force vector at free degrees of freedom at structural level
$\mathbf{u}_t$	Nodal displacement vector at free degrees of freedom at structural level
$\mathbf{F}_e$	Element nodal force vector in global reference; $\lfloor N_{X1},\ V_{Y1},\ M_{Z1},\ N_{X2},\ V_{Y2},\ M_{Z2}\rfloor^T$
$\mathbf{F}_e^{int}$	Internal element nodal force vector in global reference
$\boldsymbol{\delta}_e$	Element nodal displacement vector in global reference; $\lfloor u_{X1},\ v_{Y1},\ \theta_{Z1},\ u_{X2},\ v_{Y2},\ \theta_{Z2}\rfloor^T$
$\overline{\mathbf{f}}_e$	Element nodal force vector in local reference with rigid body modes; $\lfloor \overline{N}_{x1},\ \overline{V}_{y1},\ \overline{M}_{z1},\ \overline{N}_{x2},\ \overline{V}_{y2},\ \overline{M}_{z2}\rfloor^T$
$\overline{\mathbf{f}}_e^{int}$	Internal element nodal force vector in local reference with rigid body modes
$\overline{\mathbf{d}}_e$	Element nodal displacement vector in local reference with rigid body modes; $\lfloor \overline{u}_{x1},\ \overline{v}_{y1},\ \overline{\theta}_{z1},\ \overline{u}_{X2},\ \overline{v}_{Y2},\ \overline{\theta}_{Z2}\rfloor^T$
$\tilde{\mathbf{f}}_e$	Element nodal force vector in local reference without rigid body modes; $\lfloor \tilde{M}_{z1},\ \tilde{M}_{z2},\ \tilde{N}_{x2}\rfloor^T$
$\tilde{\mathbf{f}}_e^{int}$	Internal element nodal force vector in local reference without rigid body modes
$\tilde{\mathbf{f}}_e^{R}$	Residual element nodal force vector in local reference without rigid body modes
$\tilde{\mathbf{d}}_e$	Element nodal displacement vector in local reference without rigid body modes; $\lfloor \tilde{\theta}_{z1},\ \tilde{\theta}_{z2},\ \tilde{u}_{x2}\rfloor^T$
$\tilde{\mathbf{d}}_e^{R}$	Residual element nodal displacement vector in local reference without rigid body modes
$\delta\overline{\mathbf{d}}_e$	Virtual element nodal displacement vector in local reference

2 Section Quantities

$\mathbf{d}_s(x)$	Section displacement vector; $\lfloor u(x),\ v(x)\rfloor^T$
Φ	Curvature
$\boldsymbol{\sigma}_s(x)$	Section force vector; $\lfloor N(x),\ M(x)\rfloor^T$
$\boldsymbol{\sigma}_s^{int}(x)$	Internal section force vector
$\boldsymbol{\sigma}_s^{R}(x)$	Residual section force vector
$\boldsymbol{\varepsilon}_s(x)$	Section deformation vector; $\lfloor \varepsilon_x(x),\ \phi_z(x)\rfloor^T$
$\boldsymbol{\varepsilon}_s^{int}(x)$	Residual section deformation vector
$\delta\boldsymbol{\varepsilon}_s(x)$	Virtual section deformation vector
κ	Plastic stress

3 Fiber Quantities

σ	Uniaxial stress
ε	Uniaxial strain
σ_r	Uniaxial stress of layer/fiber
ε_r	Uniaxial strain of layer/fiber

4 Stiffness Matrices

$\mathbf{N}_d(x)$	Shape function on displacement field
$\mathbf{B}_d(x)$	The matrix derived from the derivatives of $\mathbf{N}_d(x)$
$\mathbf{N}_f(x)$	Shape function on force field
$\mathbf{K}_S$	Augmented stiffness matrix at structural level
$\mathbf{K}_{tt}$	Stiffness matrix associated natural boundary conditions
$\mathbf{K}_{tu}$	Stiffness matrix associated natural and essential boundary conditions
$\mathbf{K}_{ut}$	Stiffness matrix associated essential and natural boundary conditions
$\mathbf{K}_{uu}$	Stiffness matrix associated essential boundary conditions
$\mathbf{K}_e$	Element stiffness matrix in global reference
$\overline{\mathbf{k}}_e$	Element stiffness matrix in local reference with rigid body modes
$\overline{\mathbf{k}}_e^{tan}$	Element tangent stiffness matrix in local reference with rigid body modes
$\tilde{\mathbf{k}}_e$	Element stiffness matrix in local reference without rigid body modes
$\tilde{\mathbf{c}}_e$	Element flexibility matrix in local reference without rigid body modes
$\mathbf{k}_s(x)$	Section stiffness matrix
$\mathbf{k}_s^{tan}(x)$	Section tangent stiffness matrix
$\mathbf{c}_s(x)$	Section flexibility matrix

5 Misc.

$E(x)$	Elastic modulus
$A(x)$	Section area
$I_z(x)$	Moment of inertia on section area
L_e	Element length
$\mathbf{\Gamma}_e$	Transformation matrix between local and global coordinate system
$\tilde{\mathbf{\Gamma}}_e$	Transformation matrix between rigid body modes and no rigid body modes

6 Subscripts

t	Known traction
u	Known displacement
S	Structural level
e	Element level or e^{th} element at element state determination
r	Layer/fiber level or r^{th} layer/fiber at layer/fiber state determination
s	Section level or s^{th} section at section state determination
d	Displacement field
f	Force field
n	Current step of External force/displacement vector

7 Superscripts

int	Internal
ext	External
R	Residual
k	k^{th} iteration at structural level
j	j^{th} iteration at element level

Non Linear Structural Analysis

Element Formulations

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Table of Contents I

Table of Contents II

- Element formulations, and constitutive models are at the heart of our nonlinear analysis.
- Element formulation of beam-column is more complex than the one of solid elements (except for plates and shells).
- We will review "standard" (stiffness based) element formulation, but will also review formulation of "modern elements", such as
  - Fiber sections
  - zero length elements/sections
  - Flexibility based elements
- At time coverage of some of those elements is quite complex, brace yourself.
- Careful with the notation.

- In the context of the classical stiffness method, derivation of the truss stiffness matrix is simple. We hereby re-derive it as a mean to "gently" introduce new notation that you should familiarize yourself with.
- As with all finite elements, stiffness matrix derivation hinges on three requirements.

1. Compatibility
    - Displacements generalized relationship between section displacement vector $\mathbf{d}_s(x)$ and element nodal displacement vector $\overline{\mathbf{d}}_e$ is expressed through the displacement interpolation functions (shape functions), $\mathbf{N}_d(x)$ as

$$\mathbf{d}_s(x) = \left\{ \begin{array}{c} u(x) \end{array} \right\} = \underbrace{\left[\begin{array}{cc} -\frac{x}{L}+1 & \frac{x}{L} \end{array} \right]}_{\mathbf{N}_d(x)} \cdot \underbrace{\left\{ \begin{array}{c} \overline{u}_{x1} \\ \overline{u}_{x2} \end{array} \right\}}_{\overline{\mathbf{d}}_e} \tag{1}$$

- Deformation of displacements: Under the assumption that displacements are small, the section deformation vector $\boldsymbol{\varepsilon}_s(x)$ is related to the element nodal displacement vector by

$$\boldsymbol{\varepsilon}_s(x) = \left\{ \begin{array}{c} \varepsilon_x(x) \end{array} \right\} = \underbrace{\left[\begin{array}{cc} -\frac{1}{L} & \frac{1}{L} \end{array} \right]}_{\mathbf{B}_d(x)} \cdot \overline{\mathbf{d}}_e$$

where $\mathbf{B}_d(x)$ is the matrix which relates displacement to strain through the derivatives of $\mathbf{N}_d(x)$.

2. Constitutive law is expressed as

$$\underbrace{\left\{ \begin{array}{c} N_x(x) \end{array} \right\}}_{\boldsymbol{\sigma}_s(x)} = \mathbf{k}_s(x) \cdot \boldsymbol{\varepsilon}_s(x)$$

where $\boldsymbol{\sigma}_s(x)$ is the section[1] force vector, and $\mathbf{k}_s(x)$ is the section stiffness matrix. For linear elastic analysis $\mathbf{k}_s(x)$ is simply a scalar equal to $\mathbf{k}_s(x) = \left[\begin{array}{c} E(x) \cdot A(x) \end{array} \right]$ where, $E(x)$ and $A(x)$ are elastic modulus and cross sectional area.

3. Equilibrium (weak form) through the principle of virtual work (displacement) which is expressed as

$$\underbrace{\delta\overline{\mathbf{d}}_e^T \cdot \overline{\mathbf{f}}_e}_{\text{External}} = \underbrace{\int_0^{L_e} \delta\boldsymbol{\varepsilon}_s(x)^T . \boldsymbol{\sigma}_s(x)\, \mathrm{d}x}_{\text{Internal}}$$

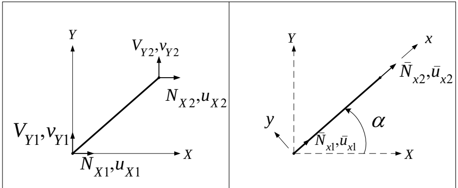
Substitution leads to

$$\begin{aligned} \delta\overline{\mathbf{d}}_e^T \cdot \overline{\mathbf{f}}_e &= \int_0^{L_e} \delta\overline{\mathbf{d}}_e^T \cdot \mathbf{B}_d(x)^T \cdot \mathbf{k}_s(x) \cdot \boldsymbol{\varepsilon}_s(x)\, \mathrm{d}x \\ \Rightarrow \overline{\mathbf{f}}_e &= \int_0^{L_e} \mathbf{B}_d(x)^T \cdot \mathbf{k}_s(x) \cdot \boldsymbol{\varepsilon}_s(x)\, \mathrm{d}x = \underbrace{\int_0^{L_e} \mathbf{B}_d(x)^T \cdot \mathbf{k}_s(x) \cdot \mathbf{B}_d(x)\, \mathrm{d}x}_{\overline{\mathbf{k}}_e} \cdot \overline{\mathbf{d}}_e \end{aligned}$$

or $\overline{\mathbf{f}}_e = \overline{\mathbf{k}}_e \cdot \overline{\mathbf{d}}_e$ The element stiffness matrix in local reference is thus given by

$$\overline{\mathbf{k}}_e = \int_0^{L_e} \mathbf{B}_d(x)^T \cdot \mathbf{k}_s(x) \cdot \mathbf{B}_d(x)\, \mathrm{d}x$$

[1] The notion of section is not essential to understand the formulation of the truss element stiffness matrix. It is nevertheless introduced to be consistent with the subsequent formulation of beam-column



(a) Global reference in an element

(b) Local reference in an element

- Element nodal forces and displacements are expressed with respect to the global reference $\mathbf{F}_e = \lfloor N_{X1},\ V_{Y1},\ N_{X2},\ V_{Y2} \rfloor^T$; $\boldsymbol{\delta}_e = \lfloor u_{X1},\ v_{Y1},\ u_{X2},\ v_{Y2} \rfloor^T$

- Element nodal forces and displacements can also be expressed with respect to the local reference,

$$\overline{\mathbf{f}}_e = \lfloor \overline{N}_{x1},\ \overline{N}_{x2} \rfloor^T; \qquad \overline{\mathbf{d}}_e = \lfloor \overline{u}_{x1},\ \overline{u}_{x2} \rfloor^T$$

- Rotation matrix which transform global reference to local reference, is given by $\boldsymbol{\Gamma}_e$ such that

$$\overline{\mathbf{f}}_e = \boldsymbol{\Gamma}_e \cdot \mathbf{F}_e; \qquad \overline{\mathbf{d}}_e = \boldsymbol{\Gamma}_e \cdot \boldsymbol{\delta}_e; \qquad \mathbf{K}_e = \boldsymbol{\Gamma}_e^T \cdot \overline{\mathbf{k}}_e \cdot \boldsymbol{\Gamma}_e$$

where, $\mathbf{K}_e$ is the element stiffness matrix in global reference and the rotation matrix is

$$\boldsymbol{\Gamma}_e = \left[\begin{array}{c|cccc} & N_{X1} & V_{Y1} & N_{X2} & V_{Y2} \\ \hline \overline{N}_{x1} & \cos\alpha & \sin\alpha & 0 & 0 \\ \overline{N}_{x2} & 0 & 0 & \cos\alpha & \sin\alpha \end{array} \right]$$

- In thermodynamic the current state of the material can be uniquely characterized by a suitably selected set of state variables, i.e. what we need to predict future states of the system.
- In the context of structural analysis, state determination is the process of determining for a set of element nodal displacements:

Tangent stiffness matrix
: to apply Newton's method to solve the nonlinear system (Tangent stiffness matrix corresponding to the Jacobian).

Internal forces
: to then determine the residual forces which should be nearly equal to zero.

needed to make prediction for state $n+1$ from state n.

1. Compatibility of
   - Displacement Section displacements are determined from the element nodal displacements through the shape functions.

$$\mathbf{d}_s(x) = \left\{ \begin{array}{c} u(x) \\ v(x) \end{array} \right\} = \mathbf{N}_d(x) \cdot \underbrace{\lfloor \ \overline{u}_{x1}, \quad \overline{v}_{y1}, \quad \overline{\theta}_{z1}, \quad \overline{u}_{x2}, \quad \overline{v}_{y2}, \quad \overline{\theta}_{z2} \ \rfloor^T}_{\overline{\mathbf{d}}_e}$$

where $\mathbf{N}_d(x)$ is the matrix of displacement interpolation functions which can be expressed as

$$\mathbf{N}_d(x) = \begin{bmatrix} \psi_1(x) & 0 & 0 & \psi_2(x) & 0 & 0 \\ 0 & \phi_1(x) & \phi_2(x) & 0 & \phi_3(x) & \phi_4(x) \end{bmatrix}$$

where ψ_1, ψ_2, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are the interpolation functions for axial and transverse displacements respectively and are given by

$$\begin{aligned} \psi_1(x) &= -\frac{x}{L_e} + 1 & \psi_2(x) &= \frac{x}{L_e} \\ \phi_1(x) &= 2\frac{x^3}{L_e{}^3} - 3\frac{x^2}{L_e{}^2} + 1 & \phi_2(x) &= \frac{x^3}{L_e{}^2} - 2\frac{x^2}{L_e} + x \\ \phi_3(x) &= -2\frac{x^3}{L_e{}^3} + 3\frac{x^2}{L_e{}^2} & \phi_4(x) &= \frac{x^3}{L_e{}^2} - \frac{x^2}{L_e} \end{aligned}$$

We note the uncoupling between axial and transverse displacements since geometric nonlinearity is ignored.
Again note that $\mathbf{d}_s$ is nonlinear

- Deformation Under the assumptions of small displacements and plane sections remaining plane (Euler Bernouilli as opposed to Timoshenko), the section deformation vector $\boldsymbol{\varepsilon}_s(x)$ (axial strain $\varepsilon_x(x)$ and curvature $\phi_z(x)$) is related to the element nodal displacement vector

$$\boldsymbol{\varepsilon}_s(x) = \left\{ \begin{array}{c} \varepsilon_x(x) \\ \phi_z(x) \end{array} \right\} = \mathbf{B}_d(x) \cdot \overline{\mathbf{d}}_e \tag{2}$$

where $\mathbf{B}_d(x)$ is the matrix obtained from the appropriate derivatives of the displacement interpolation functions

$$\mathbf{B}_d(x) = \begin{bmatrix} \psi_1'(x) & 0 & 0 & \psi_2'(x) & 0 & 0 \\ 0 & \phi_1''(x) & \phi_2''(x) & 0 & \phi_3''(x) & \phi_4''(x) \end{bmatrix} \text{ with}$$

$$\begin{aligned} \psi_1'(x) &= -\frac{1}{L_e} & \psi_2'(x) &= \frac{1}{L_e} \\ \phi_1''(x) &= \frac{12x}{L_e{}^3} - \frac{6}{L_e{}^2} & \phi_2''(x) &= \frac{6x}{L_e{}^2} - \frac{4}{L_e} \\ \phi_3''(x) &= -\frac{12x}{L_e{}^3} + \frac{6}{L_e{}^2} & \phi_4''(x) &= \frac{6\,x}{L_e{}^2} - \frac{2}{L_e} \end{aligned}$$

- Note $\mathbf{B}_{d,e}(x)$ is an approximation since we are approximating the displacement field.

2. Constitutive law Section constitutive law relates axial strain and curvature to axial force and moment

$$\underbrace{\left\{ \begin{array}{c} N_x(x) \\ M_z(x) \end{array} \right\}}_{\boldsymbol{\sigma}_s(x)} = \mathbf{k}_s(x)\, \boldsymbol{\varepsilon}_s(x) \tag{3}$$

where $\boldsymbol{\sigma}_s(x)$ is the section force vector, and $\mathbf{k}_s(x)$ is the section stiffness matrix.

If $\mathbf{k}_s(x)$ is not derived from layer/fiber discretization of the cross section, then we assume a moment-curvature relation

$$\mathbf{k}_{s,}(x) = \left[\begin{array}{cc} E(x) \cdot A(x) & 0 \\ 0 & E(x) \cdot I_z(x) \end{array} \right] \tag{4}$$

where, $E(x)$, $A(x)$, and $I_z(x)$ are elastic modulus at increment n, cross sectional area, and section moment of inertia. Note that $\mathbf{k}_s$ is nonlinear as the elastic modulus E varies in a nonlinear formulation.

3. Equilibrium will be satisfied only in the weak sense through the principle of virtual displacement expressed as

$$\underbrace{\delta\overline{\mathbf{d}}_e^T \cdot \overline{\mathbf{f}}_e}_{\text{External}} = \underbrace{\int_0^{L_e} \delta\boldsymbol{\varepsilon}_s(x)^T \cdot \boldsymbol{\sigma}_s(x)\, \mathrm{d}x}_{\text{Internal}}$$

Substituting and since the latter must hold for any arbitrary $\delta\overline{\mathbf{d}}_e$, the principle of virtual work leads to

$$\begin{aligned}
\delta\overline{\mathbf{d}}_e^T \cdot \overline{\mathbf{f}}_e &= \int_0^{L_e} \delta\overline{\mathbf{d}}_e^T \cdot \mathbf{B}_d(x)^T \cdot \mathbf{k}_s(x) \cdot \boldsymbol{\varepsilon}_s(x)\, \mathrm{d}x \\
\Rightarrow \overline{\mathbf{f}}_e &= \int_0^{L_e} \mathbf{B}_d(x)^T \cdot \mathbf{k}_s(x) \cdot \boldsymbol{\varepsilon}_s(x)\, \mathrm{d}x \\
&= \underbrace{\int_0^{L_e} \mathbf{B}_d(x)^T \cdot \mathbf{k}_s(x) \cdot \mathbf{B}_d(x)\, \mathrm{d}x}_{\overline{\mathbf{k}}_e} \cdot \overline{\mathbf{d}}_e
\end{aligned}$$

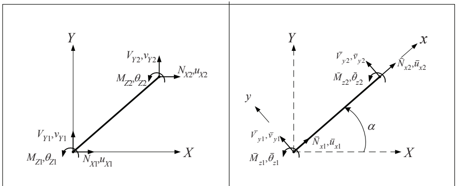
or

$$\overline{\mathbf{f}}_e = \overline{\mathbf{k}}_e \cdot \overline{\mathbf{d}}_e \tag{5}$$

The element stiffness matrix in local reference is thus given by

$$\overline{\mathbf{k}}_e = \int_0^{L_e} \mathbf{B}_d(x)^T \cdot \mathbf{k}_s(x) \cdot \mathbf{B}_d(x)\,\mathrm{d}x \tag{6}$$

Note analogy with $\mathbf{k}_e = \int \mathbf{B}^T \mathbf{D}(\Omega)\mathbf{B} d\Omega$ where $\mathbf{D}(\Omega)$ is now replaced by $\mathbf{K}_s(x)$ and Ω by L_e.



(a) Global reference in an element
(b) Local reference in an element

Element nodal forces and displacements are expressed with respect to the global reference

$$\mathbf{F}_e = \lfloor N_{X1},\ V_{Y1},\ M_{Z1},\ N_{X2},\ V_{Y2},\ M_{Z2} \rfloor^T$$
$$\boldsymbol{\delta}_e = \lfloor u_{X1},\ v_{Y1},\ \theta_{Z1},\ u_{X2},\ v_{Y2},\ \theta_{Z2} \rfloor^T$$

- Element nodal forces and displacements of the element can be expressed with respect to the local reference

$$\overline{\mathbf{f}}_e = \lfloor \overline{N}_{x1},\ \overline{V}_{y1},\ \overline{M}_{z1},\ \overline{N}_{x2},\ \overline{V}_{y2},\ \overline{M}_{z2} \rfloor^T;\ \overline{\mathbf{d}}_e = \lfloor \overline{u}_{x1},\ \overline{v}_{y1},\ \overline{\theta}_{z1},\ \overline{u}_{x2},\ \overline{v}_{y2},\ \overline{\theta}_{z2} \rfloor^T$$

- Rotation matrix which transforms from global reference

$$\overline{\mathbf{f}}_e = \boldsymbol{\Gamma}_e \cdot \mathbf{F}_e; \qquad \overline{\mathbf{d}}_e = \boldsymbol{\Gamma}_e \cdot \boldsymbol{\delta}_e; \qquad \mathbf{K}_e = \boldsymbol{\Gamma}_e^T \cdot \overline{\mathbf{k}}_e \cdot \boldsymbol{\Gamma}_e$$

where, $\mathbf{K}_e$ is the element stiffness matrix in global reference and the rotation matrix is

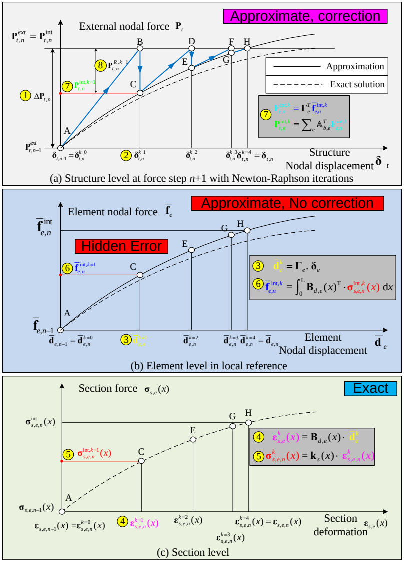
$$\mathbf{\Gamma}_e = \left[\begin{array}{c|cccccc} & N_{X1} & V_{Y1} & M_{Z1} & N_{X2} & V_{Y2} & M_{Z2} \\ \hline \overline{N}_{x1} & \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ \overline{V}_{y1} & -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ \overline{M}_{z1} & 0 & 0 & 1 & 0 & 0 & 0 \\ \overline{N}_{x2} & 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ \overline{V}_{y2} & 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ \overline{M}_{z2} & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

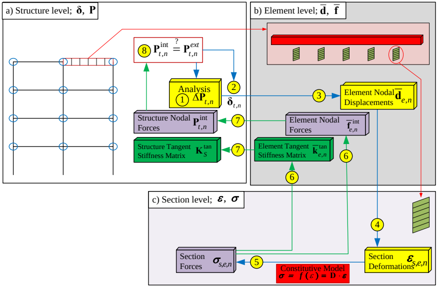
- We operate at three different levels in the structural analysis: a) structure level, b) element level, and c) section level.
- State determination (internal forces and tangent stiffness matrix corresponding to element nodal displacements) for
  1. Section: internal section forces are computed from section deformations which are in turn determined from element nodal displacements and the section stiffness matrix.
  2. Element tangent stiffness matrices and internal element nodal forces of each element are determined from the internal section forces for each element which are in turn computed from section deformations.
  3. Structure: element tangent stiffness matrices and internal element force vector of all the elements are assembled to form the (augmented) tangent stiffness matrix $\mathbf{K}_S^{tan}$ and internal nodal force vector $\mathbf{P}_S^{int}$ ($\mathbf{P}_S^{int} = \mathbf{P}_t^{int} + \mathbf{P}_u^{int}$) of the structure. Subscript t and u refer to free and constrained degrees of freedom respectively (that is along the natural and essential boundaries).

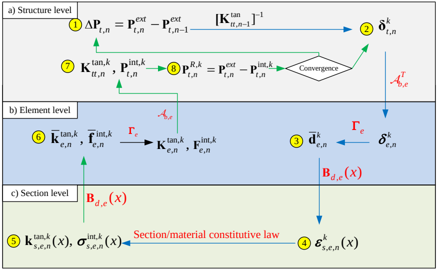
- Once the structure state determination is complete, the internal nodal force vector ($\mathbf{P}_{t,n}^{int}$) is compared with the total applied external nodal force vector ($\mathbf{P}_{t,n}^{ext}$)and the difference ($\mathbf{P}_{t,n}^{R}$), is the residual nodal force vector which is then reapplied to the structure in an iterative solution process until convergence (equilibrium) is satisfied.

Level	Internal Force	Tangent Stiffness matrix	"Displacement"
Section	$\underbrace{\left\{ \begin{array}{c} N_x(x) \\ M_z(x) \end{array} \right\}}_{\boldsymbol{\sigma}_s(x)} = \mathbf{k}_s(x)\boldsymbol{\varepsilon}_s(x)$	$\mathbf{k}_s(x) = \left[\begin{array}{cc} E(x)A(x) & 0 \\ 0 & E(x)I_z(x) \end{array} \right]$	$\boldsymbol{\varepsilon}_s(x) = \left\{ \begin{array}{c} \varepsilon_x(x) \\ \phi_z(x) \end{array} \right. = \mathbf{B}_d(x)\overline{\mathbf{d}}_e$
Element Local Element Global	$\overline{\mathbf{f}}_e^{int} = \int_0^{Le} \mathbf{B}_{d,e}(x)^T \boldsymbol{\sigma}_{s,e}^{int}(x)dx$ $\mathbf{F}_{e,n}^{int,k} = \boldsymbol{\Gamma}_e^T \overline{\mathbf{f}}_{e,n}^{int,k}$	$\overline{\mathbf{k}}_e = \int_0^{Le} \mathbf{B}_d(x)^T \mathbf{k}_s(x)\mathbf{B}_d(x)dx$ $\mathbf{K}_{e,n}^{tan,k} = \boldsymbol{\Gamma}_e^T \overline{\mathbf{k}}_{e,n}^{tan,k} \boldsymbol{\Gamma}_e$	$\overline{\mathbf{d}}_e = \boldsymbol{\Gamma}_e \cdot \boldsymbol{\delta}_e$ $\boldsymbol{\delta}_e$
Structure	$\mathbf{P}_{t,n}^{int,k} = \sum_e \mathcal{A}_{b,e}^T \mathbf{F}_{e,n}^{int,k}$	$\mathbf{K}_{S,n}^{tan,k} = \sum_e \mathcal{A}_{b,e}^T \mathbf{K}_{e,n}^{tan,k} \mathcal{A}_{b,e}$	$\boldsymbol{\delta}_e$

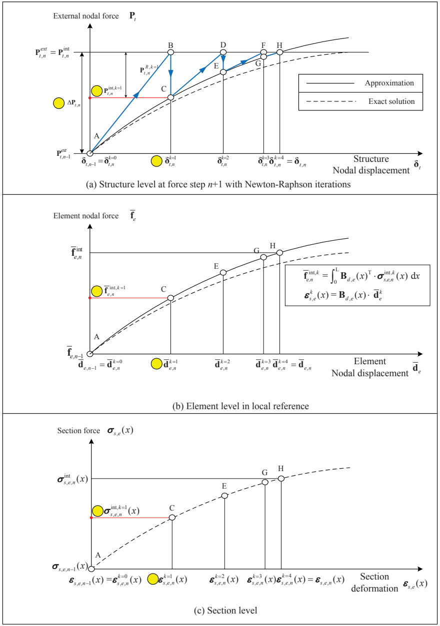
$\mathcal{A}_{b,e}^T$ is a force assembling operator, and $\mathbf{K}_{S,n}^{tan,k}$ encompasses the four submatrices, $\mathbf{K}_{tt,n}^{tan,k}$, $\mathbf{K}_{tu,n}^{tan,k}$, $\mathbf{K}_{ut,n}^{tan,k}$, and $\mathbf{K}_{uu,n}^{tan,k}$

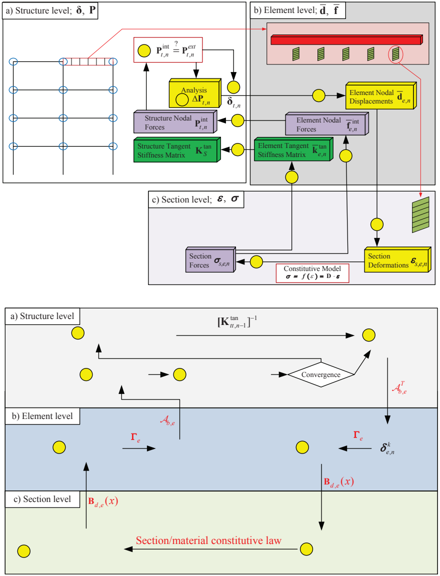




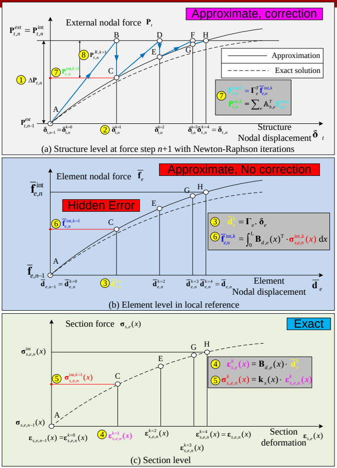


$\mathcal{A}_{b,e}$ and $\mathcal{A}_{b,e}^T$ are the displacement extracting operator and the force assembling operator.





$\mathcal{A}_{b,e}$ and $\mathcal{A}_{b,e}^{T}$ are the displacement extracting operator and the force assembling operator.



- Newton-Raphson iteration method operates in the global coordinate (structural level) system.

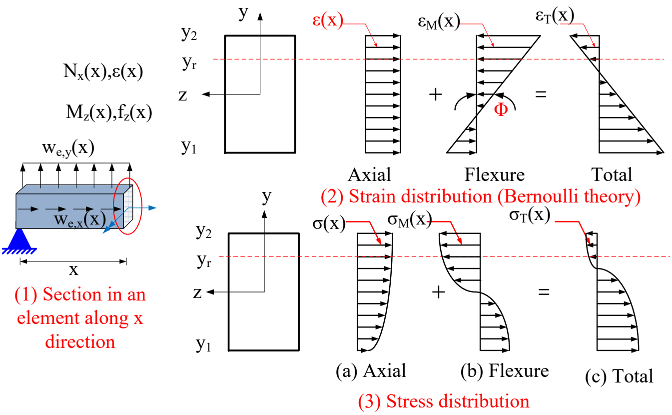
- At the k^{th} iteration: $\overline{\mathbf{d}}_{e,n} \rightarrow \mathbf{d}_{s,n}(x)$ (section displacements from element nodal displacements in local reference, Eq. 1) (③).
- For each section: $\mathbf{d}_{s,n}(x) \rightarrow \boldsymbol{\varepsilon}^{k}_{s,e,n}(x)$ (Section deformation from element displacements Eq, 2) (④), since $\mathbf{B}_{d,e}(x)$ is exact only in the linear elastic case.
- Assuming that the section constitutive law is explicitly known, $\boldsymbol{\varepsilon}^{k}_{s,e,n}(x) \rightarrow \mathbf{k}^{tan,k}_{s,e,n}(x)$ Eq. 4 (tangent stiffness matrix from section deformation); $\boldsymbol{\varepsilon}^{k}_{s,e,n}(x) \rightarrow \boldsymbol{\sigma}^{int,k}_{s,e,n}(x)$ (internal section force vectors from section deformation, Eq. 3 (⑤).
- Element stiffness matrices $\overline{\mathbf{k}}^{k}_{e,n}$ in local reference (Eq. 6) and the internal element nodal force vectors in local reference $\overline{\mathbf{f}}^{int,k}_{e,n}$ (Eq. 5 are determined next (⑥).

- During assembly of the global stiffness matrix, the structure's tangent stiffness matrix and vector of nodal internal forces are determined (⑦), before the residual is computed (⑧) for convergence.
- Since $\mathbf{B}_{d,e}(x)$ is only approximate (i.e. evaluated in terms of the estimate values of the nodal displacements at the structure level), then element stiffness matrices $\overline{\mathbf{k}}_{e,n}^{k}$ (Eq. 6) and internal element nodal force vectors $\overline{\mathbf{f}}_{e,n}^{int,k}$ (Eq. 5) are also approximate.
- The approximation of $\mathbf{B}_{d,e}(x)$ leads to stiffer solution. Note that the curve labeled "Exact solution" is only exact within the assumptions of the section constitutive law and the kinematic approximations that deformations are small and plane sections remain plane.
- Solutions: a) finer mesh discretization of the structure especially, in frame regions that undergo highly nonlinear behaviors, such as the member ends.; b) use flexibility based elements.

- So far, assumed that a section is characterized by a moment curvature relation, i.e when the moment reaches the plastic/yield moment, the whole section plastifies.
- This is only an approximation, as in reality there is a gradual plastification starting from the outer fibers, and this plastification zone gradually spreads inward until the whole section ultimately becomes plastic.
- Note analogy with what we have previously seen in terms of sectional plasticity (Moment curvature *vs* stress-strain).
- To capture this gradual spread one can either resort to continuum 2D/3D solid (finite) elements, which is computationally expensive/inefficient, or use layered elements.
- Ultimately, our objective remains the derivation of $\mathbf{k}_{s,e}^{tan}(x)$ such that

$$\left\{ \begin{array}{c} N(x) \\ M_z(x) \end{array} \right\} = \mathbf{k}_{s,e}^{tan}(x) \left\{ \begin{array}{c} \varepsilon(x) \\ \phi_z(x) \end{array} \right\}$$

- Ignoring transverse shear deformation (accounted for in the so-called Timoshenko beam), and thus assuming a linear strain distribution (Euler-Bernouilli beam), but a non linear stress-stain behavior, the stress distribution is $\sigma_t(x) = \frac{N_x(x)}{A(x)} \pm \frac{M_z(x)}{I_z(x)} y$
- At this point, from the nodal displacement, we can determine the section deformations (axial strain, $\varepsilon(x)$ and curvature, $\phi(x)$) (and thus the linear strain distribution), and since we have a nonlinear material, the exact location of the neutral axis is not yet known, and at each fiber elevation we do have a different $E_r^{tan}(x)$. r is the fiber subscript.



- We know ε and Φ, must determine N and M
- Primary Terms are those due to pure axial and flexure:

- Pure axial force due to $\sigma(x)$ is simply determined from

$$\begin{aligned} N_x(x) &= \int_{-y_1}^{y_2} \sigma(x) dA = \int_{-y_1}^{y_2} \underbrace{E_r^{tan}(x) \cdot \varepsilon(x)}_{\sigma(x)} dA \\ &\simeq \sum_r E_r^{tan}(x) \cdot A_r(x) \cdot \varepsilon(x) \end{aligned} \tag{7}$$

- Pure moment due to $\sigma_M(x)$ is considered next, and again we seek an expression of $(M(x))$ in terms of the curvature and and $E_r^{tan}(x)$, and recalling that $I = \int y^2 dA$ and $\sigma_M(x)_{@y_r} = E_r^{tan}(x) \cdot \phi_z(x) \cdot y_r$

$$\begin{aligned} M_z(x) &= \int_{-y_1}^{y_2} \sigma_M(x) \cdot y dA = \int_{-y_1}^{y_2} \underbrace{E_r^{tan}(x) \cdot \underbrace{\phi_z(x) \cdot y}_{\varepsilon}}_{\sigma} \cdot y dA \\ &= \phi_z(x) \int_{-y_1}^{y_2} E_r^{tan}(x) \cdot y^2 dA \\ &\simeq \sum_r E_r^{tan}(x) \cdot A_r(x) \cdot y_r^2 \cdot \phi_z(x) \end{aligned} \tag{8}$$

- Secondary Terms are due to coupling and will result in non-zero off diagonal terms in the stiffness matrix. Note that this cancels out in linear elastic analysis.
  - Second axial force due to curvature as there is no reason why the nonlinear flexural stress distribution will necessarily yield a summation of forces equal to zero.

$$\begin{aligned} dN_x(x) &= -E_r^{tan}(x) \cdot \varepsilon_M(x) dA = -E_r^{tan}(x) \cdot \phi_z(x) \cdot y dA \\ N_x(x) &= -\int_{-y_1}^{y_2} E_r^{tan}(x) \cdot \phi_z(x) \cdot y dA \\ &\simeq -\sum_r E_r^{tan}(x) \cdot A_r(x) \cdot y_r \cdot \phi_z(x) \end{aligned}$$

  where the strain ($\varepsilon_M(x)$) is obtained from the curvature ($\phi_z(x)$).

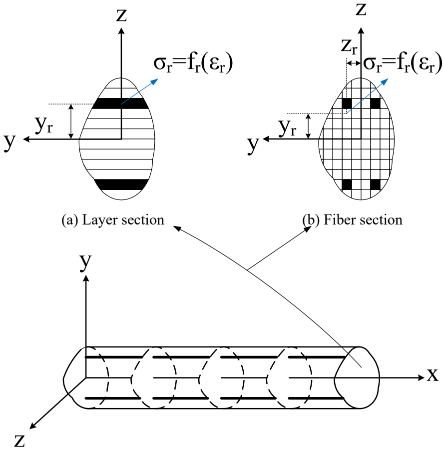
- Secondary moment due to axial strain as there is no reason why the location of the neutral axis is indeed correct resulting in a summation of moment equal to zero.

$$
\begin{aligned}
dM_z(x) &= -E_r^{tan}(x) \cdot \varepsilon(x) \cdot y \cdot dA \\
M_z(x) &= -\int_{-y_1}^{y_2} E_r^{tan}(x) \cdot \varepsilon(x) \cdot y dA \\
&\simeq -\sum_r E_r^{tan}(x) \cdot A_r(x) \cdot y_r \cdot \varepsilon(x)
\end{aligned}
$$

- Summing up within a matrix, $\mathbf{k}_{s,e}^{tan}(x)$ takes the form:

$$
\underbrace{\left\{ \begin{array}{c} N \\ M \end{array} \right\}}_{\sigma_S} = \underbrace{\sum_r \left[\begin{array}{cc} E_r^{tan}(x) \cdot A_r(x) & -E_r^{tan}(x) \cdot A_r(x) \cdot y_r \\ -E_r^{tan}(x) \cdot A_r(x) \cdot y_r & E_r^{tan}(x) \cdot A_r(x) \cdot y_r^2 \end{array} \right]}_{\mathbf{k}_S^{tan,n}} \left\{ \begin{array}{c} \varepsilon(x) \\ \phi_z(x) \end{array} \right\} \quad (9)
$$

- The implementation of this layer or fiber section will require an additional discretization of the cross section into layers or fibers


z
$\sigma_r = f_r(\varepsilon_r)$
y_r
y
(a) Layer section
z
z_r
$\sigma_r = f_r(\varepsilon_r)$
y_r
y
(b) Fiber section
y
x
z

- Noting that layer/fiber stress-strain relations are typically expressed as explicit functions of strain, state determination is given by

Layer or Fiber section

(Iteration loop: r = 1 to # of layers or fibers)

$$\varepsilon_{r,n} = \begin{bmatrix} 1 & -y_r \end{bmatrix} \cdot \boldsymbol{\varepsilon}^{k,j}_{s,e,n}$$

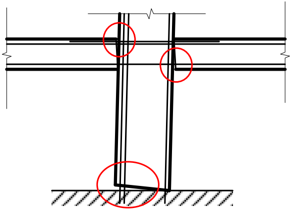
Each fiber material constitutive law

$\sigma_{r,n} = f_r(\varepsilon_{r,n})$, where f_r is function r^{th} layer/fiber with $\mathrm{E}^{\tan}_{r,n}$.

$$\sigma_{r,n}, \mathrm{E}^{\tan}_{r,n}$$

$$\boldsymbol{\sigma}^{\mathrm{int},k}_{s,e,n} = \sum_r \begin{bmatrix} 1 \\ -y_r \end{bmatrix} \cdot \sigma_{r,n}, \quad \mathbf{k}^{\tan,k}_{s,e,n}, \quad \mathbf{c}^{\tan,k}_{s,e,n} = \left[\mathbf{k}^{\tan,k}_{s,e,n}\right]^{-1}$$

- We note that this cross sectional definition allows us to easily specify longitudinal steel reinforcement. Shear reinforcement, on the other hand, can not be explicitly modeled, however, common practice is to assign modified properties to the confined concrete.
- Neutral Axis is implicitly determined.
    1. In input data, assume the neutral axis to be in the bottom layer (for ease of determining layer elevation y_r).
    2. At the global level equilibrium will not be satisfied.
    3. Displacements will be adjusted
    4. Indirectly strain distribution will be corrected by shifting the N.A.
    5. Faster convergence could be achieved if an *intelligent* guess is made for the location of the NA, and define all fibers with respect to that location.
    6. Alternatively, the program could immediately (first increment/iteration) determine the elastic neutral axis.



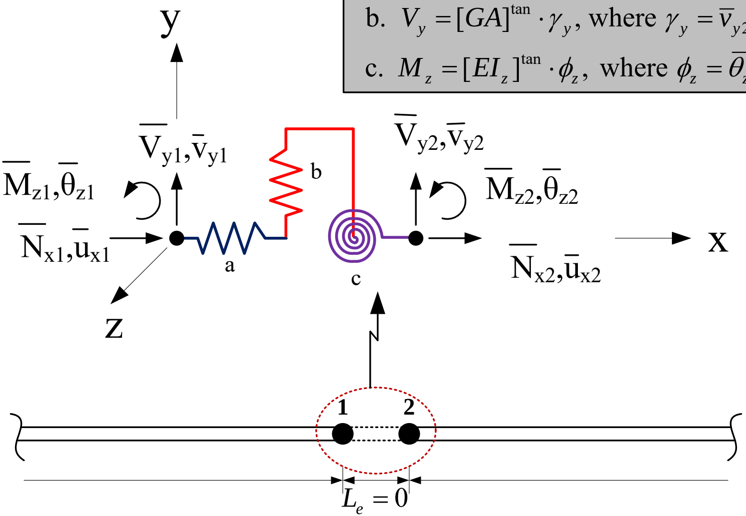
- Zero-length elements are needed for a) lumped plasticity models where plastic hinges form at the end of the element (They are more suitable for lateral loads than for vertical ones) and b) to capture bond slip.
- Element end deformations in the reinforced concrete are composed of two types:

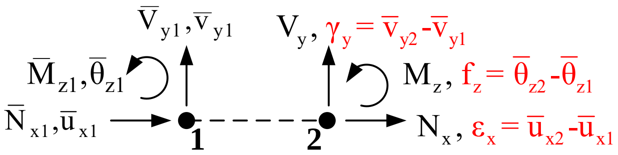
- Flexural deformation that causes inelastic strains
- Element end rotation which may be caused be the slip of longitudinal reinforcement in reinforced concrete or plastic hinges in steel members.

a. $N_x = [EA]^{\tan} \cdot \varepsilon_x$, where $\varepsilon_x = \bar{u}_{x2} - \bar{u}_{x1}$

b. $V_y = [GA]^{\tan} \cdot \gamma_y$, where $\gamma_y = \bar{v}_{y2} - \bar{v}_{y1}$

c. $M_z = [EI_z]^{\tan} \cdot \phi_z$, where $\phi_z = \bar{\theta}_{z2} - \bar{\theta}_{z1}$





1. Constitutive law Section constitutive law is expressed as

$$\underbrace{\left\{\begin{array}{c} N_x \\ V_y \\ M_z \end{array}\right\}}_{\boldsymbol{\sigma}_s} = \underbrace{\left[\begin{array}{ccc} [EA]^{tan} & 0 & 0 \\ 0 & [GA]^{tan} & 0 \\ 0 & 0 & [EI_z]^{tan} \end{array}\right]}_{\mathbf{k}_s^{tan}} \underbrace{\left\{\begin{array}{c} \overline{u}_{x2} - \overline{u}_{x1} \\ \overline{v}_{y2} - \overline{v}_{y1} \\ \overline{\theta}_{z2} - \overline{\theta}_{z1} \end{array}\right\}}_{\boldsymbol{\varepsilon}_s}$$

where, $[EA]^{tan}$, $[GA]^{tan}$ and $[EI_z]^{tan}$ are tangent stiffnesses associated with axial, shear and moment.

Note that the displaceemnets are the relative displacements between the two adjacent nodes.

2. Equilibrium Composing equilibrium equations between point A and point B

$$\overline{N}_{x1} = [EA]^{tan}\cdot(\overline{u}_{x1} - \overline{u}_{x2});\ \ \overline{V}_{y1} = [GA]^{tan}\cdot(\overline{v}_{y1} - \overline{v}_{y2});\ \ \overline{M}_{z1} = [EI_z]^{tan}\cdot(\overline{\theta}_{z1} - \overline{\theta}_{z2})$$

Likewise between point B and point C,

$$\overline{N}_{x2} = [EA]^{tan}\cdot(\overline{u}_{x2} - \overline{u}_{x1});\ \ \overline{V}_{y2} = [GA]^{tan}\cdot(\overline{v}_{y2} - \overline{v}_{y1});\ \ \overline{M}_{z2} = [EI_z]^{tan}\cdot(\overline{\theta}_{z2} - \overline{\theta}_{z1})$$

Rewriting in matrix form, the relationship between element nodal force and displacement vector is given by

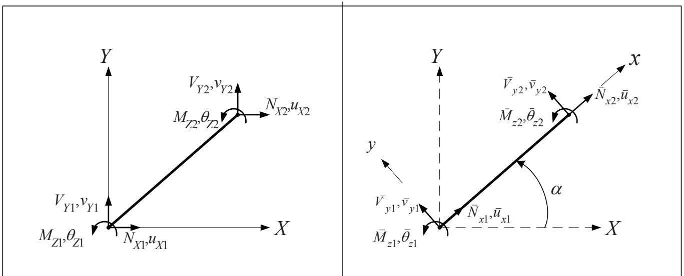
$$\underbrace{\begin{Bmatrix} \overline{N}_{x1} \\ \overline{V}_{y1} \\ \overline{M}_{z1} \\ \overline{N}_{x2} \\ \overline{V}_{y2} \\ \overline{M}_{z2} \end{Bmatrix}}_{\overline{\mathbf{f}}_e} = \overline{\mathbf{k}}_e^{tan} \underbrace{\begin{Bmatrix} \overline{u}_{x1} \\ \overline{v}_{y1} \\ \overline{\theta}_{z1} \\ \overline{u}_{x2} \\ \overline{v}_{y2} \\ \overline{\theta}_{z2} \end{Bmatrix}}_{\overline{\mathbf{d}}_e}$$

where, $\overline{\mathbf{k}}_e^{tan}$ is the element stiffness matrix in local reference.

$$\overline{\mathbf{k}}_e^{tan} = \begin{bmatrix} [EA]^{tan} & 0 & 0 & -[EA]^{tan} & 0 & 0 \\ 0 & [GA]^{tan} & 0 & 0 & -[GA]^{tan} & 0 \\ 0 & 0 & [EI_z]^{tan} & 0 & 0 & -[EI_z]^{tan} \\ -[EA]^{tan} & 0 & 0 & [EA]^{tan} & 0 & 0 \\ 0 & -[GA]^{tan} & 0 & 0 & [GA]^{tan} & 0 \\ 0 & 0 & -[EI_z]^{tan} & 0 & 0 & [EI_z]^{tan} \end{bmatrix}$$

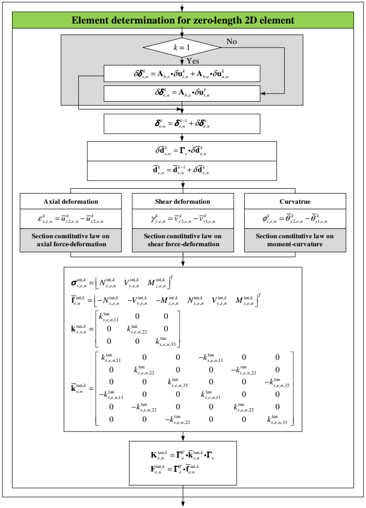
Note analogy with the simpler spring elements previously seen (Matrix analysis).

Coordinate system in zero-length 2D element is same as the one of the 2D stiffness element.



(a) Global reference in an element (b) Local reference in an element

$$\boldsymbol{\Gamma}_e = \left[\begin{array}{c|cccccc} & N_{X1} & V_{Y1} & M_{Z1} & N_{X2} & V_{Y2} & M_{Z2} \\ \hline \overline{N}_{x1} & \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ \overline{V}_{y1} & -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ \overline{M}_{z1} & 0 & 0 & 1 & 0 & 0 & 0 \\ \overline{N}_{x2} & 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ \overline{V}_{y2} & 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ \overline{M}_{z2} & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$


Element determination for zero-length 2D element
k = 1
No
Yes
Axial deformation
Shear deformation
Curvatrue
Section constitutive law on axial force-deformation
Section constitutive law on shear force-deformation
Section constitutive law on moment-curvature

1. Step 1: Determine the section deformation vector, axial deformation, shear deformation and curvature. For each deformation, we extract the associated components from $\overline{\mathbf{d}}^k_{e,n}$.

$$
\begin{aligned}
\overline{\mathbf{d}}^k_{e,n} &= \lfloor \overline{u}^k_{x1,e,n}\ \overline{v}^k_{y1,e,n}\ \overline{\theta}^k_{z1,e,n}\ \overline{u}^k_{x2,e,n}\ \overline{v}^k_{y2,e,n}\ \overline{\theta}^k_{z2,e,n} \rfloor^T \\
\boldsymbol{\varepsilon}^k_{s,e,n} &= \lfloor \varepsilon^k_{x,e,n},\ \gamma^k_{y,e,n},\ \phi^k_{z,e,n} \rfloor^T \\
\varepsilon^k_{x,e,n} &= \overline{u}^k_{x2,e,n} - \overline{u}^k_{x2,e,n} \\
\gamma^k_{y,e,n} &= \overline{v}^k_{y2,e,n} - \overline{v}^k_{y2,e,n} \\
\phi^k_{z,e,n} &= \overline{\theta}^k_{z2,e,n} - \overline{\theta}^k_{z2,e,n}
\end{aligned}
$$

which define axial section deformation, shear deformation, and curvature.

2. Step 2: Determine the section tangent stiffness associated with axial force-deformation, shear force-deformation, and moment-curvature in the section constitutive laws.

If we assume that the section constitutive law is explicitly known, $\mathbf{k}_{s,e,n}^{tan,k}$ and $\boldsymbol{\sigma}_{s,e,n}^{int,k}$ are determined from $\boldsymbol{\varepsilon}_{s,e,n}^{k}$.

In elastic section, we need not to compute $\mathbf{k}_{s,e,n}^{tan,k}$ again as it is identical to the initial section stiffness matrix $\mathbf{k}_{s,e}$. For an elastic section,

$$\begin{aligned}
\mathbf{k}_{s,e,n}^{tan} &= \mathbf{k}_{s,e} \\
\underbrace{\left\{ \begin{array}{c} N_{x,e,n}^{int,k} \\ V_{y,e,n}^{int,k} \\ M_{z,e,n}^{int,k} \end{array} \right\}}_{\boldsymbol{\sigma}_{s,e,n}^{int,k}} &= \mathbf{k}_{s,e,n}^{tan} \underbrace{\left\{ \begin{array}{c} \varepsilon_{x,e,n}^{k} \\ \gamma_{y,e,n}^{k} \\ \phi_{z,e,n}^{k} \end{array} \right\}}_{\boldsymbol{\varepsilon}_{s,e,n}^{k}}
\end{aligned}$$

where, $\mathbf{k}_{s,e,n}^{tan,k}$ is the section tangent stiffness matrix at k^{th} iteration.

3. Step 3: Determine the internal element nodal force vector and the element tangent stiffness matrix

$$\bar{\mathbf{f}}_{e,n}^{int,k} = \lfloor N_{x,e,n}^{int,k}, V_{y,e,n}^{int,k}, M_{z,e,n}^{int,k}, -N_{x,e,n}^{int,k}, -V_{y,e,n}^{int,k}, -M_{z,e,n}^{int,k} \rfloor^T \tag{10}$$

$$\bar{\mathbf{k}}_e^{tan,k} = \begin{bmatrix} EA_{e,n}^{tan,k} & 0 & 0 & -EA_{e,n}^{tan,k} & 0 & 0 \\ 0 & GA_{e,n}^{tan,k} & 0 & 0 & -GA_{e,n}^{tan,k} & 0 \\ 0 & 0 & EI_{z,e,n}^{tan,k} & 0 & 0 & -EI_{z,e,n}^{tan,k} \\ -EA_{e,n}^{tan,k} & 0 & 0 & EA_{e,n}^{tan,k} & 0 & 0 \\ 0 & -GA_{e,n}^{tan,k} & 0 & 0 & GA_{e,n}^{tan,k} & 0 \\ 0 & 0 & -EI_{z,e,n}^{tan,k} & 0 & 0 & EI_{z,e,n}^{tan,k} \end{bmatrix} \tag{11}$$

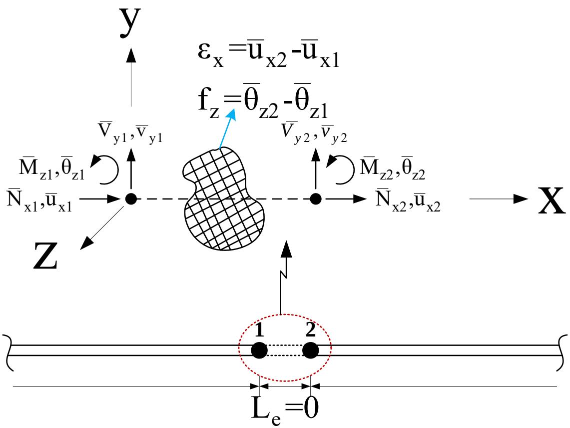
where, $\bar{\mathbf{k}}_{e,n}^{tan,k}$ is the element tangent stiffness matrix in local reference.

In global, we determine $\mathbf{F}_{e,n}^{int,k}$ and $\mathbf{K}_{e,n}^{tan,k}$.

$$\mathbf{F}_{e,n}^{int,k} = \boldsymbol{\Gamma}_e{}^T \cdot \bar{\mathbf{f}}_{e,n}^{int,k}$$

$$\mathbf{K}_{e,n}^{tan,k} = \boldsymbol{\Gamma}_e{}^T \cdot \bar{\mathbf{k}}_{e,n}^{tan,k} \cdot \boldsymbol{\Gamma}_e$$

Zero-length section element is analogous to the zero length element, however, it uses layer/fiber. This element enables us to model the shift in center of section rotation which may occur (in bar-slip for example). The element is formulated on the basis of coupled axial force and moment; No shear forces.

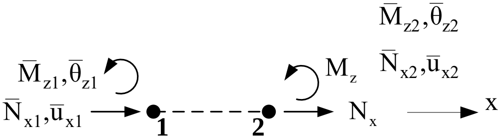


Constitutive law Section constitutive law is expressed as

$$\underbrace{\left\{ \begin{array}{c} N_x \\ M_z \end{array} \right\}}_{\boldsymbol{\sigma}_s} = \underbrace{\left[\begin{array}{cc} k^{tan}_{s,11} & k^{tan}_{s,12} \\ k^{tan}_{s,21} & k^{tan}_{s,22} \end{array} \right]}_{\mathbf{k}^{tan}_s} \cdot \underbrace{\left\{ \begin{array}{c} \overline{u}_{x2} - \overline{u}_{x1} \\ \overline{\theta}_{z2} - \overline{\theta}_{z1} \end{array} \right\}}_{\boldsymbol{\varepsilon}_s}$$

where, N and M are analogous to Eqs. 7, 8, and $\mathbf{k}^{tan}_s$ is the section tangent stiffness matrix obtained from layer/fiber state determination, analogous to Eq. 9.

Equilibrium Zero-length section element is based on Bernoulli beam theory.



Composing equilibrium equations

$$
\begin{aligned}
\overline{N}_{x1} &= k^{tan}_{s,11} \cdot (\overline{u}_{x1} - \overline{u}_{x2}) + k^{tan}_{s,12} \cdot (\overline{\theta}_{z1} - \overline{\theta}_{z2}) \\
\overline{M}_{z1} &= k^{tan}_{s,21} \cdot (\overline{u}_{x1} - \overline{u}_{x2}) + k^{tan}_{s,22} \cdot (\overline{\theta}_{z1} - \overline{\theta}_{z2})
\end{aligned}
\tag{12}
$$

Likewise

$$
\begin{aligned}
\overline{N}_{x2} &= k^{tan}_{s,11} \cdot (\overline{u}_{x2} - \overline{u}_{x1}) + k^{tan}_{s,12} \cdot (\overline{\theta}_{z2} - \overline{\theta}_{z1}) \\
\overline{M}_{z2} &= k^{tan}_{s,21} \cdot (\overline{u}_{x2} - \overline{u}_{x1}) + k^{tan}_{s,22} \cdot (\overline{\theta}_{z2} - \overline{\theta}_{z1})
\end{aligned}
\tag{13}
$$

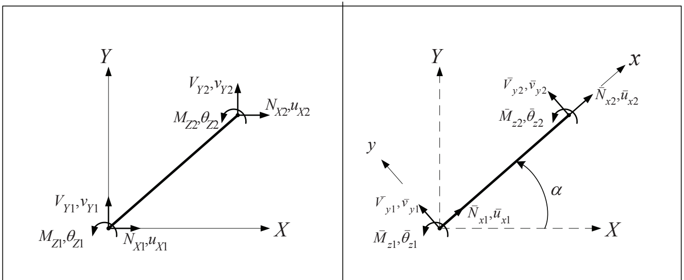
Rewriting Eq. 12 and 13 to matrix form, the relationship between element nodal force and displacement vector is given by

$$
\underbrace{\left\{ \begin{array}{c} \overline{N}_{x1} \\ 0 \\ \overline{M}_{z1} \\ \overline{N}_{x2} \\ 0 \\ \overline{M}_{z2} \end{array} \right\}}_{\overline{\mathbf{f}}_e} = \overline{\mathbf{k}}^{tan}_e \underbrace{\left\{ \begin{array}{c} \overline{u}_{x1} \\ 0 \\ \overline{\theta}_{z1} \\ \overline{u}_{x2} \\ 0 \\ \overline{\theta}_{z2} \end{array} \right\}}_{\overline{\mathbf{d}}_e}
$$

where, $\overline{\mathbf{k}}_e^{tan}$ is the element stiffness matrix in local reference.

$$\overline{\mathbf{k}}_e^{tan} = \begin{bmatrix} k_{s,11}^{tan} & 0 & k_{s,12}^{tan} & -k_{s,11}^{tan} & 0 & -k_{s,12}^{tan} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k_{s,21}^{tan} & 0 & k_{s,22}^{tan} & -k_{s,21}^{tan} & 0 & -k_{s,22}^{tan} \\ -k_{s,11}^{tan} & 0 & -k_{s,12}^{tan} & k_{s,11}^{tan} & 0 & k_{s,12}^{tan} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_{s,21}^{tan} & 0 & -k_{s,22}^{tan} & k_{s,21}^{tan} & 0 & k_{s,22}^{tan} \end{bmatrix} \tag{14}$$

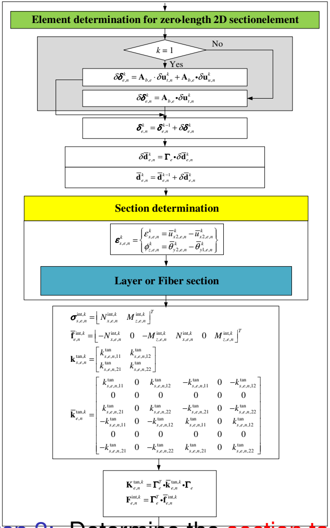
Coordinate system in zero-length 2D element is same as in 2D stiffness element.



(a) Global reference in an element

(b) Local reference in an element

$$
\boldsymbol{\Gamma}_e = \left[\begin{array}{c|cccccc}
 & N_{X1} & V_{Y1} & M_{Z1} & N_{X2} & V_{Y2} & M_{Z2} \\
\hline
\overline{N}_{x1} & \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\
\overline{V}_{y1} & -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\
\overline{M}_{z1} & 0 & 0 & 1 & 0 & 0 & 0 \\
\overline{N}_{x2} & 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\
\overline{V}_{y2} & 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\
\overline{M}_{z2} & 0 & 0 & 0 & 0 & 0 & 1
\end{array} \right]
$$



Step 1: Determine the section deformation vector, axial deformation and curvature. For each deformation, we extracts the associated components from $\overline{\mathbf{d}}^k_{e,n}$.

$$
\begin{aligned}
\overline{\mathbf{d}}^k_{e,n} &= \lfloor \overline{u}^k_{x1,e,n}\ 0\ \overline{\theta}^k_{z1,e,n}\ \overline{u}^k_{x2,e,n}\ 0\ \overline{\theta}^k_{z2,e,n} \rfloor^T \\
\boldsymbol{\varepsilon}^k_{s,e,n} &= \lfloor \varepsilon^k_{x,e,n},\ \phi^k_{z,e,n} \rfloor^T \\
\varepsilon^k_{x,e,n} &= \overline{u}^k_{x2,e,n} - \overline{u}^k_{x2,e,n} \\
\phi^k_{z,e,n} &= \overline{\theta}^k_{z2,e,n} - \overline{\theta}^k_{z2,e,n}
\end{aligned}
$$

Step 2: Determine the section tangent stiffness associated with axial force-deformation and moment-curvature using layer/fiber state determination as in for Layr/fiber. Determine next the internal section force vector. If we assume that the material constitutive law is explicitly known, $\mathbf{k}_{s,e,n}^{tan,k}$ and $\boldsymbol{\sigma}_{s,e,n}^{int,k}$ are determined from $\boldsymbol{\varepsilon}_{s,e,n}^{k}$. However, in the section with elastic material, we need not to compute $\mathbf{k}_{s,e,n}^{tan,k}$ again as it is identical to the initial section stiffness matrix $\mathbf{k}_{s,e}$. If we have a section with elastic material, then

$$\begin{aligned}
\mathbf{k}_{s,e,n}^{tan} &= \mathbf{k}_{s,e} \\
\underbrace{\left\{ \begin{array}{c} N_{x,e,n}^{int,k} \\ M_{z,e,n}^{int,k} \end{array} \right\}}_{\boldsymbol{\sigma}_{s,e,n}^{int,k}} &= \mathbf{k}_{s,e,n}^{tan} \underbrace{\left\{ \begin{array}{c} \varepsilon_{x,e,n}^{k} \\ \phi_{z,e,n}^{k} \end{array} \right\}}_{\boldsymbol{\varepsilon}_{s,e,n}^{k}}
\end{aligned}$$

where, $\mathbf{k}_{s,e,n}^{tan,k}$ is the section tangent stiffness matrix at k^{th} iteration.

Step 3: Determine the internal element nodal force vector and the element tangent stiffness matrix

$$\overline{\mathbf{f}}^{int,k}_{e,n} = \lfloor N^{int,k}_{x,e,n}, 0, M^{int,k}_{z,e,n}, -N^{int,k}_{x,e,n}, 0, -M^{int,k}_{z,e,n} \rfloor^T$$

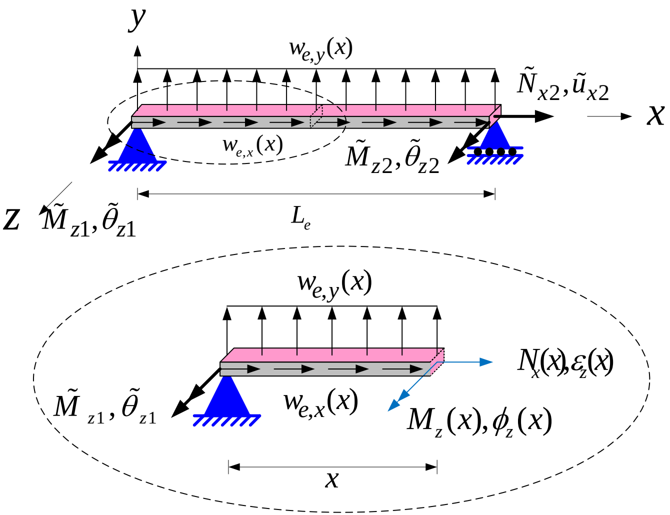
$$\overline{\mathbf{k}}^{tan,k}_{e,n} = \begin{bmatrix} \mathrm{k}^{tan,k}_{s,e,n,11} & 0 & \mathrm{k}^{tan,k}_{s,e,n,12} & -\mathrm{k}^{tan,k}_{s,e,n,11} & 0 & -\mathrm{k}^{tan,k}_{s,e,n,12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{k}^{tan,k}_{s,e,n,21} & 0 & \mathrm{k}^{tan,k}_{s,e,n,22} & -\mathrm{k}^{tan,k}_{s,e,n,21} & 0 & -\mathrm{k}^{tan,k}_{s,e,n,22} \\ -\mathrm{k}^{tan,k}_{s,e,n,11} & 0 & -\mathrm{k}^{tan,k}_{s,12e,n,} & \mathrm{k}^{tan,k}_{s,e,n,11} & 0 & \mathrm{k}^{tan,k}_{s,e,n,12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathrm{k}^{tan,k}_{s,e,n,21} & 0 & -\mathrm{k}^{tan,k}_{s,e,n,22} & \mathrm{k}^{tan,k}_{s,e,n,21} & 0 & \mathrm{k}^{tan,k}_{s,e,n,22} \end{bmatrix}$$

where, $\overline{\mathbf{k}}^{tan,k}_{e,n}$ is the element tangent stiffness matrix in local reference. We determine $\mathbf{F}^{int,k}_{e,n}$ and $\mathbf{K}^{tan,k}_{e,n}$.

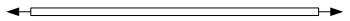
$$\begin{aligned} \mathbf{F}^{int,k}_{e,n} &= \mathbf{\Gamma}_e{}^T \cdot \overline{\mathbf{f}}^{int,k}_{e,n} \\ \mathbf{K}^{tan,k}_{e,n} &= \mathbf{\Gamma}_e{}^T \cdot \overline{\mathbf{k}}^{tan,k}_{e,n} \cdot \mathbf{\Gamma}_e \end{aligned}$$

- Flexibility based elements
  - Are nonconformist finite elements since they yield the element flexibility matrix rather than the classical stiffness matrix.
  - Are based on the equations of equilibrium rather than on assumed displacement field, while at the global level formulation is displacement based.
  - Offer some important advantages over stiffness based elements: fewer elements are needed (albeit at the cost of a more complex formulation); stiffness-based method formulations are approximate and flexibility-based method formulations are exact such as a section varying along the element and elements with material nonlinearity.
- We derive the element flexibility matrix $\tilde{\mathbf{c}}_e$ without rigid body modes and then invert it to obtain the corresponding element stiffness matrix $\tilde{\mathbf{k}}_e$ (again without rigid body modes). The retained degrees of freedom are the axial force at node 2, and the two end moments.
- There are two distinct formulations: a) with element iterations, and b) without element iterations. We will focus on the former.

- Whereas we have used the principle of virtual work (displacement) for the derivation of the stiffness based element, we shall now use the principle of complementary virtual work (force) through the usual three steps.



y
$w_{e,y}(x)$
$\tilde{N}_{x2}, \tilde{u}_{x2}$
X
$w_{e,x}(x)$
$\tilde{M}_{z2}, \tilde{\theta}_{z2}$
Z
$\tilde{M}_{z1}, \tilde{\theta}_{z1}$
L_e
$w_{e,y}(x)$
$N_x(x), \varepsilon_z(x)$
$\tilde{M}_{z1}, \tilde{\theta}_{z1}$
$w_{e,x}(x)$
$M_z(x), \phi_z(x)$
x



(a) Positive section axial force

(b) Positive section moment

- Equilibrium will now be strongly enforced (whereas it was satisfied in the weak sense previously) and we seek to derive the force shape functions:
  - For uniformly distributed axial forces, we have $dN_x(x) = w_x^{(e)} dx$ or $\frac{dN_x(x)}{dx} = w_x^{(e)}(x)$
  - For uniformly distributed transverse forces $\frac{dV_y(x)}{dx} = w_y^{(e)}(x))$ and $\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = w(x)$
  - Equilibrium can be expressed as

$$\underbrace{\mathbf{w}_e(x)}_{\text{External}} + \underbrace{\mathcal{L}_f \cdot \boldsymbol{\sigma}_s(x)}_{\text{Internal}} = \mathbf{0}; \quad \left\{ \begin{array}{c} w_x^{(e)}(x) \\ w_y^{(e)}(x) \end{array} \right\} + \left[\begin{array}{cc} \frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & \frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{array} \right] \left\{ \begin{array}{c} N_x(x) \\ M_z(x) \end{array} \right\} = \mathbf{0}$$

  $\mathbf{w}_e(x)$ is the external element traction vector, $\mathcal{L}_f$ is the force differential operator which enforces equilibrium. (Note in stiffness formulation, the compatibility was "strongly" enforced).

- We will write equilibrium of sectional stresses in terms of the nodal forces, and assume that there are no external element traction.
- Whereas we previously used displacement interpolation functions, we now need force interpolation functions, $\mathbf{N}_f(x)$ in order to exactly satisfy equilibrium along the element

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & \frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{bmatrix} \begin{Bmatrix} N_x(x) \\ M_z(x) \end{Bmatrix} = \mathbf{0}$$

- Integrating these equations, we obtain $N_x(x) = c_3$ and $M_z(x) = c_1 x + c_2$.
- We now seek to determine the shape functions that relate section internal forces at any point x to the nodal forces. We enforce natural boundary condition

$$\begin{array}{rclrclrcl} N_x(L) & = & \tilde{N}_{x2}; & M_z(0) & = & -\tilde{M}_{z1}; & M_z(L) & = & \tilde{M}_{z2}; \\ \Rightarrow c_1 & = & \frac{\tilde{M}_{z1}+\tilde{M}_{z2}}{L_e}; & c_2 & = & -\tilde{M}_{z1}; & c_3 & = & \tilde{N}_{x2}; \end{array}$$

- Substituting, we have the internal axial force and moment at any point (x) in terms of the nodal forces.

$$\underbrace{\left\{ \begin{array}{c} N_x(x) \\ M_z(x) \end{array} \right\}}_{\boldsymbol{\sigma}_s(x)} = \underbrace{\left[\begin{array}{ccc} 0 & 0 & 1 \\ \frac{x}{L_e} - 1 & \frac{x}{L_e} & 0 \end{array} \right]}_{\mathbf{N}_f(x)} \underbrace{\left\{ \begin{array}{c} \tilde{M}_{z1} \\ \tilde{M}_{z2} \\ \tilde{N}_{x2} \end{array} \right\}}_{\tilde{\mathbf{f}}_e}$$

  where, $\tilde{\mathbf{f}}_e$ is the element nodal force vector without rigid body modes.
- It should be noted that these shape functions enforce equilibrium at any section along the element

- Constitutive law: Previously expressed section forces in terms of section deformations, we now need to express section deformations in terms of section forces: $\boldsymbol{\varepsilon}_s(x) = \mathbf{c}_s(x) \cdot \boldsymbol{\sigma}_s(x)$ where, $\mathbf{c}_s(x)$ is the section flexibility matrix. If $\mathbf{c}_s(x)$ is not derived from fiber section, then for linear elastic analysis $\mathbf{c}_s(x)$ is simply.

$$\mathbf{c}_s(x) = \left[\begin{array}{cc} \frac{1}{E(x) \cdot A(x)} & 0 \\ 0 & \frac{1}{E(x) \cdot I_z(x)} \end{array} \right]$$

- Compatibility of displacements: enforced only in a weak form through the principle of complementary virtual work (as opposed to the principle of virtual work for the stiffness-based method).

$$\underbrace{\delta\tilde{\mathbf{f}}_e{}^T\tilde{\mathbf{d}}_e}_{\text{External}} = \underbrace{\int_0^{L_e}\delta\boldsymbol{\sigma}_s(x)^T\cdot\boldsymbol{\varepsilon}_s(x)\mathrm{d}x}_{\text{Internal}}$$

where $\tilde{\mathbf{d}}_e$ is the element nodal displacement vector without rigid body modes.

- Substituting

$$\begin{aligned}\delta\tilde{\mathbf{f}}_e{}^T\tilde{\mathbf{d}}_e &= \int_0^{L_e}\delta\tilde{\mathbf{f}}_e{}^T\cdot\mathbf{N}_f(x)^T\cdot\mathbf{c}_s(x)\cdot\boldsymbol{\sigma}_s(x)\,\mathrm{d}x\\ \tilde{\mathbf{d}}_e &= \int_0^{L_e}\mathbf{N}_f(x)^T\cdot\mathbf{c}_s(x)\cdot\boldsymbol{\sigma}_s(x)\,\mathrm{d}x = \underbrace{\int_0^{L_e}\mathbf{N}_f(x)^T\cdot\mathbf{c}_s(x)\cdot\mathbf{N}_f(x)\,\mathrm{d}x}_{\tilde{\mathbf{c}}_e}\cdot\tilde{\mathbf{f}}_e\end{aligned}$$

or

$$\tilde{\mathbf{d}}_e = \tilde{\mathbf{c}}_e\cdot\tilde{\mathbf{f}}_e$$

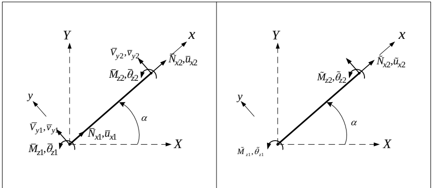
- The element flexibility matrix without rigid body modes in local reference is thus given by

$$\tilde{\mathbf{c}}_e = \int_0^{L_e} \mathbf{N}_f(x)^T \cdot \mathbf{c}_s(x) \cdot \mathbf{N}_f(x)\, \mathrm{d}x$$

- The corresponding element stiffness matrix without rigid body modes in local reference is simply

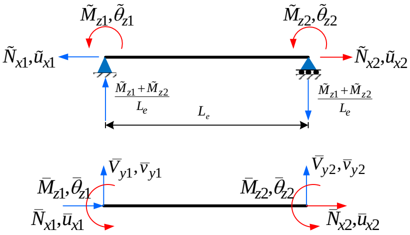
$$\tilde{\mathbf{k}}_e = [\tilde{\mathbf{c}}_e]^{-1}$$

  Note this is a 3x3 matrix, we still have to insert equilibrium relations and transform it into the usual 6x6 stiffness matrix



(a) Local reference in an element

(b) Local reference in an element without rigid body modes



- Contrarily to the reference system of the stiffness-based method, we need to consider forces and displacements in local reference with and without rigid body modes.
- Element nodal force vector without rigid body modes in local reference are (arbitrarily) selected as $\tilde{\mathbf{f}}_e = \lfloor \tilde{M}_{z1},\ \tilde{M}_{z2},\ \tilde{N}_{x2} \rfloor^T$, and the corresponding element nodal displacement vector without rigid body modes in local reference are given by $\tilde{\mathbf{d}}_e = \lfloor \tilde{\theta}_{z1},\ \tilde{\theta}_{z2},\ \tilde{u}_{x2} \rfloor^T$
- The relationship between rigid body modes and no rigid body modes is obtained through equilibrium

$$\underbrace{\left\{ \begin{array}{c} \overline{N}_{x1} \\ \overline{V}_{y1} \\ \overline{M}_{z1} \\ \overline{N}_{x2} \\ \overline{V}_{y2} \\ \overline{M}_{z2} \end{array} \right\}}_{\overline{\mathbf{f}}_e} = \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ \frac{1}{L_e} & \frac{1}{L_e} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{L_e} & -\frac{1}{L_e} & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\tilde{\mathbf{\Gamma}}_e^T} \underbrace{\left\{ \begin{array}{c} \tilde{M}_{z1} \\ \tilde{M}_{z2} \\ \tilde{N}_{x2} \end{array} \right\}}_{\tilde{\mathbf{f}}_e}$$

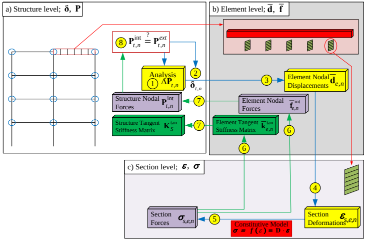
- Substituting, $\overline{\mathbf{f}}_e = \tilde{\mathbf{\Gamma}}_e^T \cdot \tilde{\mathbf{f}}_e$; $\overline{\mathbf{d}}_e = \tilde{\mathbf{\Gamma}}_e^T \cdot \tilde{\mathbf{d}}_e$; or

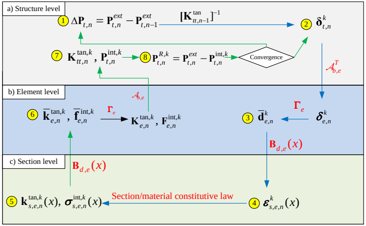
$$\mathbf{K}_e = \tilde{\mathbf{\Gamma}}_e^T \cdot \tilde{\mathbf{k}}_e \cdot \tilde{\mathbf{\Gamma}}_e$$

- Note that whereas previously $\mathbf{\Gamma}_e$ denoted a geometric transformation matrix (for stiffness based elements), it now corresponds to a statics matrix (also denoted as $\mathcal{B}$ previously).
- Derivation of the stiffness matrix from the flexibility one and the equations of equilibrium parallels the one earlier derived

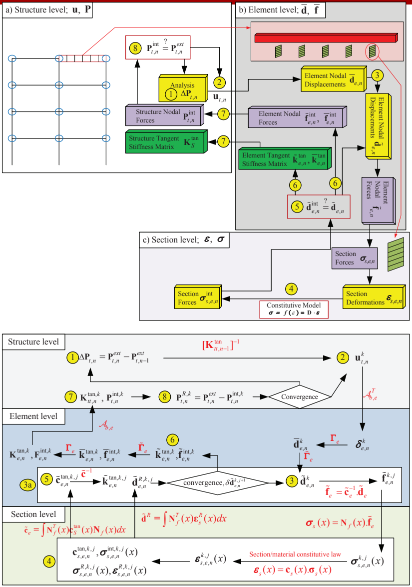
$$[\mathbf{K}] = \left[\begin{array}{c|c} [\mathbf{d}]^{-1} & [\mathbf{d}]^{-1}[\mathcal{B}]^T \\ \hline [\mathcal{B}][\mathbf{d}]^{-1} & [\mathcal{B}][\mathbf{d}]^{-1}[\mathcal{B}]^T \end{array}\right]$$

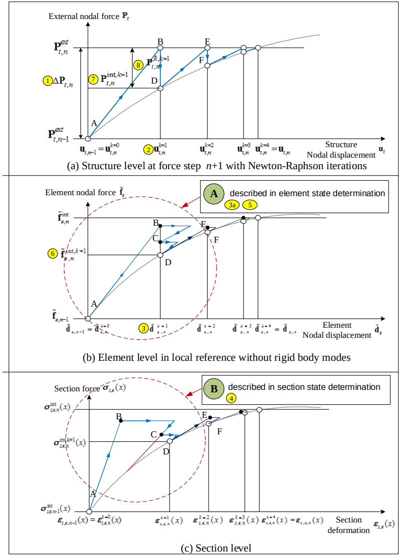
- The flexibility-based element (derived from the complementary principle of virtual work) does not have shape functions that relate deformation field inside the element with element nodal displacement vector, but shape functions which relate section forces to nodal forces.
- The global formulation is based on the stiffness (displacement) formulation, the element is based on a flexibility (force) formulation; the two will have to be reconciled (in the determination of the internal element force vectors).
- At the element level, the flexibility based element will provide nodal displacements which are not necessarily compatible with the ones coming from adjacent elements just as in the stiffness based formulation, forces were not compatible at the element level.
- We must ensure nodal displacement compatibility (in the same way as we ensured nodal equilibrium in the stiffness based formulation. Accomplished iteratively.
- Note that in the stiffness based method, there was a discontinuity in nodal forces.
- There are two algorithms for the mixed stiffness-based and flexibility-based methods: (a) with Newton-Raphson iteration in the element level to determine element state (Spacone), (b) without iteration in the element level to determine element state (Carol).

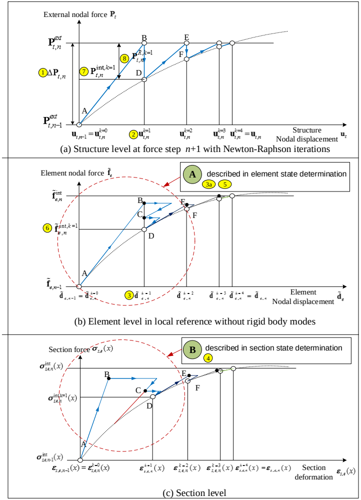




$\mathcal{A}_{b,e}$ and $\mathcal{A}_{b,e}^{T}$ are the displacement extracting operator and the force assembling operator.


a) Structure level; u, P
b) Element level; d, f
c) Section level; ε, σ
Analysis
Element Nodal Displacements
Structure Nodal Forces
Element Nodal Forces
Structure Tangent Stiffness Matrix
Element Tangent Stiffness Matrix
Element Nodal Displacements
Element Nodal Forces
Section Forces
Section Forces
Section Deformations
Constitutive Model
Structure level
Element level
Section level
Convergence
convergence
Section/material constitutive law

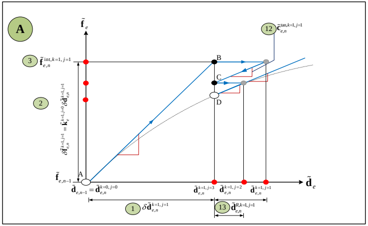

External nodal force
Structure Nodal displacement
(a) Structure level at force step n+1 with Newton-Raphson iterations
Element nodal force
described in element state determination
Element Nodal displacement
(b) Element level in local reference without rigid body modes
Section force
described in section state determination
Section deformation
(c) Section level



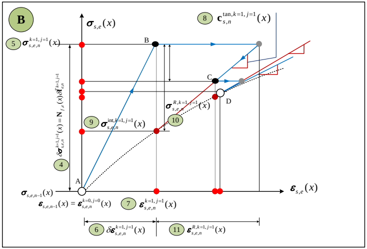
(a) Structure level at force step $n+1$ with Newton-Raphson iterations

(b) Element level in local reference without rigid body modes

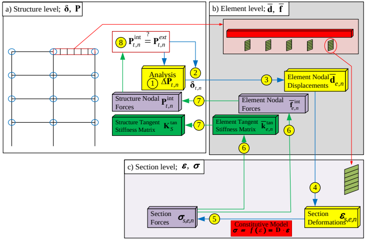
(c) Section level

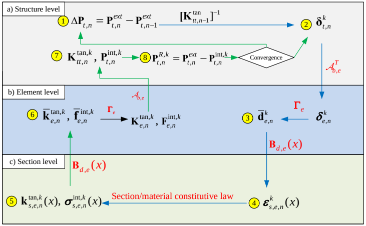


(a) Element state determination 3a 5

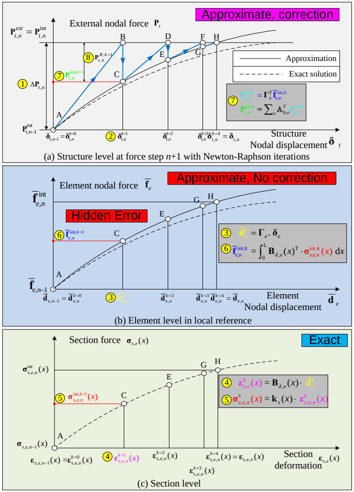


(b) Section state determination 4





$\mathcal{A}_{b,e}$ and $\mathcal{A}_{b,e}^{T}$ are the displacement extracting operator and the force assembling operator.



Compare with Stiffness based formulation

- In the flexibility based element we can not go directly from nodal displacements to section strains (as was the case in the stiffness based element), this is accomplished
  1. Determine the element nodal force vector $\tilde{\mathbf{f}}_{e,n}^{k,j}$ (⑥) from the current element nodal displacement vector using the element tangent stiffness matrix $\tilde{\mathbf{k}}_{e,n}^{tan,k,j-1}$ (③) of the previous iteration.
  2. Through the force interpolation functions $\mathbf{N}_{f,e}(x)$ determine the section force vectors $\boldsymbol{\sigma}_{s,e,n}^{k,j}(x)$ along the element.
  3. Determine the section strains by multiplying the constitutive model times the section forces.
- When we recompute the displacements corresponding to the strains.
- Compatibility of displacements at the structural level will not be satisfied.
- Thus we have an additional loop at the element level to reconcile structure based displacement and element based (through the flexibility matrix) ones, or compatibility of displacement.
- There are two complications in this procedure.

1. The determination of the section deformation vectors $\boldsymbol{\varepsilon}^{k,j}_{s,e,n}(x)$ from section force vectors since the nonlinear section force-deformation relation is commonly expressed as an explicit function of section deformation vector (④).
2. Changes in the section tangent stiffness matrices $\mathbf{k}^{tan}_{s,e,n}(x)$ produce a new element tangent stiffness matrix which, in turn, changes the element nodal force vector for the given element nodal displacement vector (⑥).

- The problem is solved through a nonlinear approach which first determines residual element nodal displacement vector $\tilde{\mathbf{d}}_{e,n}^{R,k,j}$ at each iteration. Then, compatibility of displacement at the structural level requires that this residual element nodal displacement vector be corrected.
- At the element level by applying corrective element nodal force vector based on the current element tangent stiffness matrix. The corresponding section force vectors are then determined from the force interpolation functions so that equilibrium will always be satisfied along the element. Section force vectors will not change during the section state determination in order to maintain equilibrium along the element.
- Linear approximation of section force-deformation relation about the present state results in residual section deformation vectors $\boldsymbol{\sigma}_{s,e,n}^{R,k,j}(x)$. These are then integrated along the element to obtain new residual element nodal displacement vector (⑤) and the whole process is repeated until convergence occurs.
- Compatibility of element nodal displacement vector and equilibrium along the element are always satisfied.

- The goal of the Newton-Raphson iteration loop in the element level is to determine the internal element nodal force vector (⑥) for the current element nodal displacement vector at the k^{th} Newton-Raphson iteration, hence $\tilde{\mathbf{d}}^{k}_{e,n} = \tilde{\mathbf{d}}^{k-1}_{e,n} + \delta\tilde{\mathbf{d}}^{k}_{e,n}$

1. The initial state of the element, represented by the point $\mathbf{A}$, and $j = 0$ and $k = 0$ corresponds to the state at the end of the last convergence in structural level. With the initial element tangent flexibility matrix given by $\tilde{\mathbf{c}}^{tan,k=1,j=0}_{e,n} = \tilde{\mathbf{c}}^{tan}_{e,n-1}$ and the given incremental element nodal displacement vector $\delta\tilde{\mathbf{d}}^{k=1,j=1}_{e,n} = \delta\tilde{\mathbf{d}}^{k=1}_{e,n}$ hence, the corresponding incremental element nodal force vector is $\delta\tilde{\mathbf{f}}^{k=1,j=1}_{e,n} = \left[\tilde{\mathbf{c}}^{tan,k=1,j=0}_{e,n}\right]^{-1} \cdot \delta\tilde{\mathbf{d}}^{k=1,j=1}_{e,n} = \tilde{\mathbf{k}}^{tan,k=1,j=0}_{e,n} \cdot \delta\tilde{\mathbf{d}}^{k=1,j=1}_{e,n}$
2. The incremental section force vectors can now be determined from the force interpolation functions $\delta\boldsymbol{\sigma}^{k=1,j=1}_{s,e,n}(x) = \mathbf{N}_{f,e}(x) \cdot \delta\tilde{\mathbf{f}}^{k=1,j=1}_{e,n}$ With the section tangent flexibility matrices at end of the last convergence in structural level given by $\mathbf{c}^{tan,k=1,j=0}_{s,e,n}(x) = \mathbf{c}^{tan}_{s,e,n-1}(x)$
3. The linearization of the section force-deformation relation yields the incremental section deformation vectors. $\delta\boldsymbol{\varepsilon}^{k=1,j=1}_{s,e,n}(x) = \mathbf{c}^{tan,k=1,j=0}_{s,e,n}(x) \cdot \delta\boldsymbol{\sigma}^{k=1,j=1}_{s,e,n}(x)$

4. The section deformation vectors are updated to the state that corresponds to point $\mathbf{B}$ and the updated section deformation vector (④) will be given by $\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=1}(x) = \boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=0}(x) + \delta\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=1}(x)$ For the sake of simplicity we will assume that the section force-deformation relation is explicitly known, then the section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=1}(x)$ will correspond to internal section force vectors $\boldsymbol{\sigma}_{s,e,n}^{int,k=1,j=1}(x)$ and updated section tangent flexibility matrices $\mathbf{c}_{s,e,n}^{tan,k=1,j=1}(x)$can be defined.

5. The residual section force vectors are then determined $\boldsymbol{\sigma}_{s,e,n}^{R,k=1,j=1}(x) = \boldsymbol{\sigma}_{s,e,n}^{k=1,j=1}(x) - \boldsymbol{\sigma}_{s,e,n}^{int,k=1,j=1}(x)$ and are transformed into residual section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{R,k=1,j=1}(x)$

$$\boldsymbol{\varepsilon}_{s,e,n}^{R,k=1,j=1}(x) = \mathbf{c}_{s,e,n}^{tan,k=1,j=1}(x) \cdot \boldsymbol{\sigma}_{s,e,n}^{R,k=1,j=1}(x)$$

6. The residual section deformation vectors are thus the linear approximation of the deformation error made in the linearization of the section force-deformation relation. While any suitable section flexibility matrix can be used to calculate the residual section deformation vector, the section tangent flexibility matrices offer the fastest convergence rate.

7. The residual section deformation vectors are integrated along the element using the complimentary principle of virtual work to obtain the residual element nodal displacement vector (⑤), $\tilde{\mathbf{d}}_{e,n}^{R,k=1,j=1} = \int_0^{L_e} \mathbf{N}_{f,e}(x)^T \cdot \boldsymbol{\varepsilon}_{s,e,n}^{R,k=1,j=1}(x)\mathrm{d}x$

8 At this stage the first iteration ($j = 1$) is completed. The final element and section states for $j = 1$ correspond to point $\mathbf{B}$. The residual section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{R,k=1,j=1}(x)$ and the residual element nodal displacement vector $\tilde{\mathbf{d}}_{e,n}^{R,k=1,j=1}$ were determined in the first iteration, but the corresponding element nodal displacement vector have not yet been updated. Instead, they constitute the starting point of the remaining steps within iteration loop j.

9 The presence of residual element nodal displacement vector $\tilde{\mathbf{d}}_{e,n}^{R,k=1,j=1}$ will violate compatibility, since elements sharing a common node would now have different element nodal displacement vector. In order to restore the inter-element compatibility, corrective force vector $\delta\tilde{\mathbf{f}}_{e,n}^{k=1,j=2}$ must be applied at the ends of the element as follows

$$\delta\tilde{\mathbf{f}}_{e,n}^{k=1,j=2} = -\left[\tilde{\mathbf{c}}_{e,n}^{k=1,j=1}\right]^{-1} \cdot \tilde{\mathbf{d}}_{e,n}^{R,k=1,j=1}; \;\; \tilde{\mathbf{c}}_{e,n}^{k=1,j=1} = \int_0^{L_e} \mathbf{N}_{f,e}(x)^T \cdot \mathbf{c}_{s,e,n}^{tan,k=1,j=1}(x) \cdot \mathbf{N}_{f,e}(x) \mathrm{d}$$

10. Thus, in the second iteration ($j = 2$), the element nodal force vector (⑥) is updated as $\tilde{\mathbf{f}}_{e,n}^{k=1,j=2} = \tilde{\mathbf{f}}_{e,n}^{k=1,j=1} + \delta\tilde{\mathbf{f}}_{e,n}^{k=1,j=2}$ and the section force and deformation vectors are also updated to

$$\begin{aligned}
\delta\boldsymbol{\sigma}_{s,e,n}^{k=1,j=2}(x) &= \mathbf{N}_{f,e}(x)\cdot\delta\tilde{\mathbf{f}}_{e,n}^{k=1,j=2}\\
\boldsymbol{\sigma}_{s,e,n}^{k=1,j=2}(x) &= \boldsymbol{\sigma}_{s,e,n}^{k=1,j=1}(x) + \delta\boldsymbol{\sigma}_{s,e,n}^{k=1,j=2}(x)\\
\delta\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=2}(x) &= \boldsymbol{\varepsilon}_{s,e,n}^{R,k=1,j=1}(x) + \mathbf{c}_{s,e,n}^{tan,k=1,j=1}(x)\cdot\delta\boldsymbol{\sigma}_{s,e,n}^{k=1,j=2}(x)\\
\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=2}(x) &= \boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=1}(x) + \delta\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=2}(x)
\end{aligned}$$

11. The state of the element and sections within the element at the end of the second iteration $j = 2$ corresponds to point $\mathbf{C}$.

- It should be noted that the updated tangent flexibility matrices $\mathbf{c}_{s,e,n}^{tan,k=1,j=2}(x)$ and residual section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{R,k=1,j=2}(x)$ are computed for all sections.
- Residual section deformation vectors are then integrated to obtain the residual element nodal deformation vector $\tilde{\mathbf{d}}_{e,n}^{R,k=1,j=2}$ and the new element tangent flexibility matrix $\tilde{\mathbf{c}}_{e,n}^{k=1,j=2}$ is determined by integration of the section flexibility matrices $\mathbf{c}_{s,e,n}^{tan,k=1,j=2}(x)$. This completes the second iteration within loop j.

- When incremental element nodal displacement vector $\delta\tilde{\mathbf{d}}_{e,n}^{k,j=1} = \delta\tilde{\mathbf{d}}_{e,n}^{k}$ is added to the element nodal displacement vector $\tilde{\mathbf{d}}_{e,n}^{k-1}$ at the end of the previous Newton-Raphson iteration, it is important to make sure that the element nodal displacement vector $\tilde{\mathbf{d}}_{e,n}^{k}$ do not change except in the first iteration $j = 1$ during iteration loop j
- Equilibrium along the element is always strictly satisfied since section force vectors (④) are derived from element nodal force vector by the force interpolation functions.

$$\boldsymbol{\sigma}_{s,e,n}^{k}(x) = \mathbf{N}_{f,e}(x) \cdot \tilde{\mathbf{f}}_{e,n}^{k} \quad \text{and} \quad \delta\boldsymbol{\sigma}_{s,e,n}^{k}(x) = \mathbf{N}_{f,e}(x) \cdot \delta\tilde{\mathbf{f}}_{e,n}^{k}$$

- Compatibility is also satisfied, not only at the element ends, but also along the element.

$$\begin{aligned}
\delta\tilde{\mathbf{f}}_{e,n}^{k,j} &= -\left[\tilde{\mathbf{c}}_{e,n}^{k,j-1}\right]^{-1} \cdot \tilde{\mathbf{d}}_{e,n}^{R,k,j-1} \\
\delta\boldsymbol{\sigma}_{s,e,n}^{k,j}(x) &= \mathbf{N}_{f,e}(x) \cdot \delta\tilde{\mathbf{f}}_{e,n}^{k,j} \\
\delta\boldsymbol{\varepsilon}_{s,e,n}^{k,j}(x) &= \boldsymbol{\varepsilon}_{s,e,n}^{R,k,j-1}(x) + \mathbf{c}_{s,e,n}^{tan,k,j-1}(x) \cdot \delta\boldsymbol{\sigma}_{s,e,n}^{k,j}(x)
\end{aligned}$$

- The second term expresses the relation between section deformation vectors and element nodal displacement vector. However, it should be noted that residual section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{R,k,j-1}(x)$ do not strictly satisfy this compatibility condition. This requirement can only be satisfied by integrating the residual section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{R,k,j-1}(x)$ to obtain $\tilde{\mathbf{d}}_{e,n}^{R,k,j-1}$. Since this is rather inefficient from a computational standpoint, the small compatibility error in the calculation of residual section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{R,k,j-1}(x)$ will be neglected.
- While equilibrium and compatibility are satisfied along the element during each iteration of loop j, the section force-deformation relation and the element force-deformation relation is only satisfied within a specified tolerance when convergence is achieved.

Non Linear Structural Analysis

Element Formulations; Condensed Version

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Table of Contents I

1. Compatibility of
   - Displacement Section displacements:

$$\mathbf{d}_s(x) = \left\{ \begin{array}{c} u(x) \\ v(x) \end{array} \right\} = \mathbf{N}_d(x) \cdot \underbrace{\lfloor \; \overline{u}_{x1}, \quad \overline{v}_{y1}, \quad \overline{\theta}_{z1}, \quad \overline{u}_{x2}, \quad \overline{v}_{y2}, \quad \overline{\theta}_{z2} \; \rfloor^T}_{\overline{\mathbf{d}}_e}$$

   - Deformation

$$\boldsymbol{\varepsilon}_s(x) = \left\{ \begin{array}{c} \varepsilon_x(x) \\ \phi_z(x) \end{array} \right\} = \mathbf{B}_d(x) \cdot \overline{\mathbf{d}}_e$$

2. Constitutive law axial strain and curvature to axial force and moment

$$\underbrace{\left\{ \begin{array}{c} N_x(x) \\ M_z(x) \end{array} \right\}}_{\boldsymbol{\sigma}_s(x)} = \mathbf{k}_s(x)\; \boldsymbol{\varepsilon}_s(x)$$

where $\boldsymbol{\sigma}_s(x)$ is the section force vector, and $\mathbf{k}_s(x)$ is the section stiffness matrix. If $\mathbf{k}_s(x)$ is not derived from layer/fiber discretization of the cross section, and for linear elastic case $\mathbf{k}_s(x)$ is simply equal to

$$\mathbf{k}_s(x) = \begin{bmatrix} E(x) \cdot A(x) & 0 \\ 0 & E(x) \cdot I_z(x) \end{bmatrix}$$

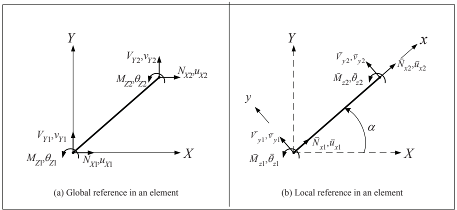
where, $E(x)$, $A(x)$, and $I_z(x)$ are elastic modulus, cross sectional area, and section moment of inertia.

3. Equilibrium will be satisfied only in the weak sense through the principle of virtual displacement expressed as

$$\underbrace{\delta\overline{\mathbf{d}}_e^T \cdot \overline{\mathbf{f}}_e}_{\text{External}} = \underbrace{\int_0^{L_e} \delta\boldsymbol{\varepsilon}_s(x)^T \cdot \boldsymbol{\sigma}_s(x)\, \mathrm{d}x}_{\text{Internal}}$$

Substituting:

$$\overline{\mathbf{k}}_e = \int_0^{L_e} \mathbf{B}_d(x)^T \cdot \mathbf{k}_s(x) \cdot \mathbf{B}_d(x)\, \mathrm{d}x$$



(a) Global reference in an element

(b) Local reference in an element

Element nodal forces and displacements are expressed with respect to the global reference

$$\mathbf{F}_e = \lfloor N_{X1},\ V_{Y1},\ M_{Z1},\ N_{X2},\ V_{Y2},\ M_{Z2} \rfloor^T$$
$$\boldsymbol{\delta}_e = \lfloor u_{X1},\ v_{Y1},\ \theta_{Z1},\ u_{X2},\ v_{Y2},\ \theta_{Z2} \rfloor^T$$

- Rotation matrix which transforms from global reference

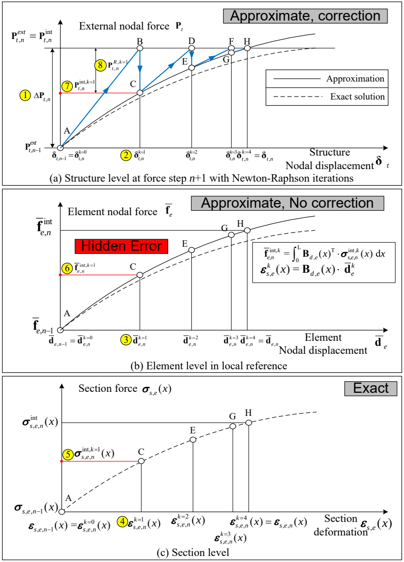
$$\overline{\mathbf{f}}_e = \boldsymbol{\Gamma}_e \cdot \mathbf{F}_e; \qquad \overline{\mathbf{d}}_e = \boldsymbol{\Gamma}_e \cdot \boldsymbol{\delta}_e; \qquad \mathbf{K}_e = \boldsymbol{\Gamma}_e^T \cdot \overline{\mathbf{k}}_e \cdot \boldsymbol{\Gamma}_e$$

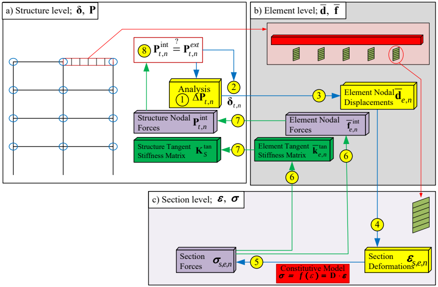
- three levels: a) structure, b) element, and c) section.
- State determination (internal forces and tangent stiffness matrix corresponding to element nodal displacements) for
  1. Section where internal section forces, are computed from section deformations which are in turn determined from element nodal displacements and the section stiffness matrix
  2. Element tangent stiffness matrices and internal element nodal forces of each element are determined from the internal section forces for each element which are in turn computed from section deformations.
  3. Structure: element tangent stiffness matrices and internal element force vector of all the elements are assembled to form the (augmented) tangent stiffness matrix $\mathbf{K}_S^{tan}$ and internal nodal force vector $\mathbf{P}_S^{int}$ ($\mathbf{P}_S^{int} = \mathbf{P}_t^{int} + \mathbf{P}_u^{int}$) of the structure. Subscript t and u refer to free and constrained degrees of freedom respectively (that is along the natural and essential boundaries).

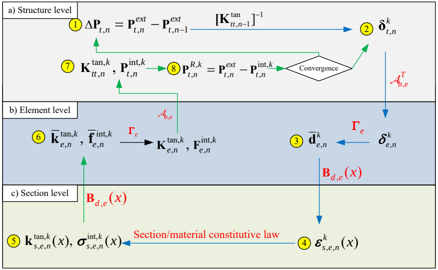
- Once the structure state determination is complete, the internal nodal force vector ($\mathbf{P}^{int}_{t,n}$) is compared with the total applied external nodal force vector ($\mathbf{P}^{ext}_{t,n}$)and the difference ($\mathbf{P}^{R}_{t,n}$), is the residual nodal force vector which is then reapplied to the structure in an iterative solution process until convergence (equilibrium) is satisfied.

Level	Internal Force	Tangent Stiffness matrix	"Displacement"
Section	$\underbrace{\left\{ \begin{array}{c} N_x(x) \\ M_z(x) \end{array} \right\}}_{\boldsymbol{\sigma}_s(x)} = \mathbf{k}_s(x)\boldsymbol{\varepsilon}_s(x)$	$\mathbf{k}_s(x) = \left[\begin{array}{cc} E(x)A(x) & 0 \\ 0 & E(x)I_z(x) \end{array} \right]$	$\boldsymbol{\varepsilon}_s(x) = \left\{ \begin{array}{c} \varepsilon_x(x) \\ \phi_z(x) \end{array} \right\} = \mathbf{B}_d(x)\overline{\mathbf{d}}_e$
Element Local Element Global	$\overline{\mathbf{f}}^{int}_e = \int_0^{Le} \mathbf{B}_{d,e}(x)^T \boldsymbol{\sigma}^{int}_{s,e}(x)dx$ $\mathbf{F}^{int,k}_{e,n} = \boldsymbol{\Gamma}^T_e \overline{\mathbf{f}}^{int,k}_{e,n}$	$\overline{\mathbf{k}}_e = \int_0^{Le} \mathbf{B}_d(x)^T \mathbf{k}_s(x)\mathbf{B}_d(x)dx$ $\mathbf{K}^{tan,k}_{e,n} = \boldsymbol{\Gamma}^T_e \overline{\mathbf{k}}^{tan,k}_{e,n} \boldsymbol{\Gamma}_e$	$\overline{\mathbf{d}}_e = \boldsymbol{\Gamma}_e \cdot \boldsymbol{\delta}_e$ $\boldsymbol{\delta}_e$
Structure	$\mathbf{P}^{int,k}_{t,n} = \sum_e \mathcal{A}^T_{b,e} \mathbf{F}^{int,k}_{e,n}$	$\mathbf{K}^{tan,k}_{S,n} = \sum_e \mathcal{A}^T_{b,e} \mathbf{K}^{tan,k}_{e,n} \mathcal{A}_{b,e}$	$\boldsymbol{\delta}_e$

$\mathcal{A}^T_{b,e}$ is a force assembling operator, and $\mathbf{K}^{tan,k}_{S,n}$ encompasses the four submatrices, $\mathbf{K}^{tan,k}_{tt,n}$, $\mathbf{K}^{tan,k}_{tu,n}$, $\mathbf{K}^{tan,k}_{ut,n}$, and $\mathbf{K}^{tan,k}_{uu,n}$

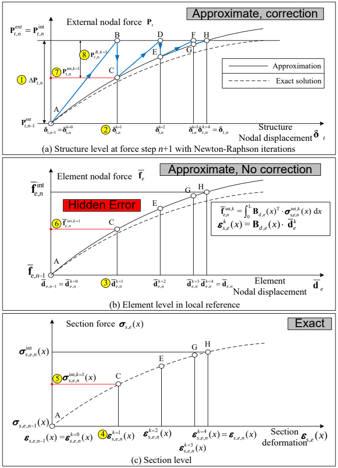






$\mathcal{A}_{b,e}$ and $\mathcal{A}_{b,e}^T$ are the displacement extracting operator and the force assembling operator.

"Hidden" Error



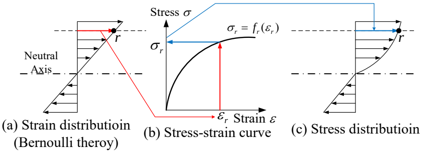
- Newton-Raphson iteration method operates in the global coordinate (structural level) system.
- At the k^{th} iteration, determine the section displacements $\mathbf{d}_{s,n}(x)$ from the element nodal displacements in local reference $\overline{\mathbf{d}}_{e,n}$ (③).
- Section deformation vectors $\boldsymbol{\varepsilon}^k_{s,e,n}(x)$ (④)for each section are computed. This is the first approximation of the element state determination, since $\mathbf{B}_{d,e}(x)$ is exact only in the linear elastic case.
- Assuming that the section constitutive law is explicitly known, the section tangent stiffness matrices $\mathbf{k}^{tan,k}_{s,e,n}(x)$ and the internal section force vectors $\boldsymbol{\sigma}^{int,k}_{s,e,n}(x)$ are readily determined from $\boldsymbol{\varepsilon}^k_{s,e,n}(x)$ (⑤).
- Element stiffness matrices $\overline{\mathbf{k}}^k_{e,n}$ in local reference and the internal element nodal force vectors in local reference $\overline{\mathbf{f}}^{int,k}_{e,n}$ are determined next (⑥).
- During assembly of the global stiffness matrix, the structure's tangent stiffness matrix and vector of nodal internal forces are determined (⑦), before the residual is computed (⑧) for convergence.
- Since $\mathbf{B}_{d,e}(x)$ is only approximate (since we are approximating the displacement field), the state variables: a) integrals for the element tangent stiffness matrix in local reference and b)internal element nodal force vector in local reference will also yield approximate results.
- The approximation of $\mathbf{B}_{d,e}(x)$ leads to stiffer solution. Note that the curve labeled "Exact solution" is only exact within the assumptions of the section constitutive law and the kinematic approximations that deformations are small and plane sections remain plane.
- Solutions: a) finer mesh discretization of the structure, especially, in frame regions that undergo highly nonlinear behaviors, such as the member ends.; b) use flexibility based elements.

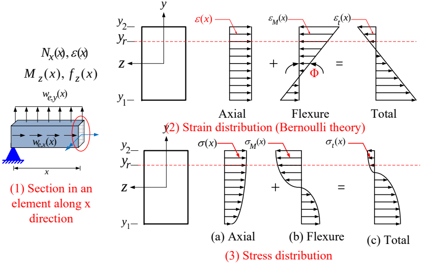
- So far, assumed that a section is characterized by a moment curvature relation, i.e when the moment reaches the plastic/yield moment, the whole section pastifies.
- This is only an approximation, as in reality there is a gradual plastification starting from the outer fibers, and this plastification zone gradually spreads inward until the whole section ultimately becomes plastic.
- To capture this gradual spread one can either resort to continuum 2D/3D solid (finite) elements, which is computationally expensive/inefficient, or use layered elements.
- Ultimately, our objective remains the derivation of $\mathbf{k}_{s,e}^{tan}(x)$ such that

$$\left\{ \begin{array}{c} N(x) \\ M_z(x) \end{array} \right\} = \mathbf{k}_{s,e}^{tan}(x) \left\{ \begin{array}{c} \varepsilon(x) \\ \phi_z(x) \end{array} \right\}$$

- Ignoring transverse shear deformation (accounted for in the so-called Timoshenko beam), and thus assuming a linear strain distribution (Euler-Bernouilli beam), but a non linear stress-stain behavior, the stress distribution is $\sigma_t(x) = \frac{N_x(x)}{A(x)} \pm \frac{M_z(x)}{I_z(x)} y$

- At this point, from the nodal displacement, we can determine the section deformations (axial strain, $\varepsilon(x)$ and curvature, $\phi(x)$) (and thus the linear strain distribution), and since we have a nonlinear material, the exact location of the neutral axis is not yet known, and at each fiber elevation we do have a different $E_r^{tan}(x)$. r is the fiber subscript.





- We know ε and Φ, must determine N and M

- Primary Terms are those due to pure axial and flexure:
  - Pure axial force due to $\sigma(x)$ is simply determined from
    $N_x(x) = \int_{-y_1}^{y_2} \sigma(x) dA = \int_{-y_1}^{y_2} \underbrace{E_r^{tan}(x) \cdot \varepsilon(x)}_{\sigma(x)} dA \simeq \sum_r E_r^{tan}(x) \cdot A_r(x) \cdot \varepsilon(x)$
  - Pure moment due to $\sigma_M(x)$ is considered next, and again we week an expression of ($M(x)$) in terms of the curvature and and $E_r^{tan}(x)$, and recalling that $I = \int y^2 dA$ and $\sigma_M(x)_{@y_r} = E_r^{tan}(x) \cdot \phi_z(x) \cdot y_r$

$$
\begin{aligned}
M_z(x) &= \int_{-y_1}^{y_2} \sigma_M(x) \cdot y dA = \int_{-y_1}^{y_2} \underbrace{E_r^{tan}(x) \cdot \underbrace{\phi_z(x) \cdot y}_{\varepsilon}}_{\sigma} \cdot y dA \\
&= \phi_z(x) \int_{-y_1}^{y_2} E_r^{tan}(x) \cdot y^2 dA \\
&\simeq \sum_r E_r^{tan}(x) \cdot A_r(x) \cdot y_r^2 \cdot \phi_z(x)
\end{aligned}
$$

- Secondary Terms are due to coupling and will result in non-zero off diagonal terms in the stiffness matrix. Note that this cancels out in linear elastic analysis.

- Second axial force due to curvature as there is no reason why the nonlinear flexural stress distribution will necessarily yield a summation of forces equal to zero.

$$\begin{aligned}
dN_x(x) &= -E_r^{tan}(x)\cdot\varepsilon_M(x)dA = -E_r^{tan}(x)\cdot\phi_z(x)\cdot ydA\\
N_x(x) &= -\int_{-y_1}^{y_2} E_r^{tan}(x)\cdot\phi_z(x)\cdot ydA\\
&\simeq -\sum_r E_r^{tan}(x)\cdot A_r(x)\cdot y_r\cdot\phi_z(x)
\end{aligned}$$

where the strain ($\varepsilon_M(x)$) is obtained from the curvature ($\phi_z(x)$).

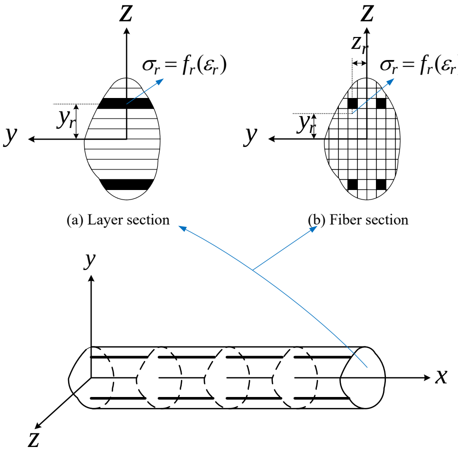
- Secondary moment due to axial strain as there is no reason why the location of the neutral axis is indeed correct resulting in a summation of moment equal to zero.

$$\begin{aligned}
dM_z(x) &= -E_r^{tan}(x)\cdot\varepsilon(x)\cdot y\cdot dA\\
M_z(x) &= -\int_{-y_1}^{y_2} E_r^{tan}(x)\cdot\varepsilon(x)\cdot ydA\\
&\simeq -\sum_r E_r^{tan}(x)\cdot A_r(x)\cdot y_r\cdot\varepsilon(x)
\end{aligned}$$

- Summing up within a matrix, $\mathbf{k}_{s,e}^{tan}(x)$ takes the form:

$$\underbrace{\left\{ \begin{array}{c} N \\ M \end{array} \right\}}_{\sigma_S} = \underbrace{\sum_r \left[\begin{array}{cc} E_r^{tan}(x) \cdot A_r(x) & -E_r^{tan}(x) \cdot A_r(x) \cdot y_r \\ -E_r^{tan}(x) \cdot A_r(x) \cdot y_r & E_r^{tan}(x) \cdot A_r(x) \cdot y_r^2 \end{array} \right]}_{\mathbf{k}_s^{tan,n}} \left\{ \begin{array}{c} \varepsilon(x) \\ \phi_z(x) \end{array} \right\}$$

- The implementation of this layer or fiber section will require an additional discretization of the cross section into layers or fibers


z
$\sigma_r = f_r(\varepsilon_r)$
y_r
y
(a) Layer section
z
z_r
$\sigma_r = f_r(\varepsilon_r)$
y_r
y
(b) Fiber section
y
x
z

- Noting that layer/fiber stress-strain relations are typically expressed as explicit functions of strain, state determination is given by

Layer or Fiber section

(Iteration loop: r = 1 to # of layers or fibers)

$$\varepsilon_{r,n} = \begin{bmatrix} 1 & -y_r \end{bmatrix} \cdot \boldsymbol{\varepsilon}^{k,j}_{s,e,n}$$

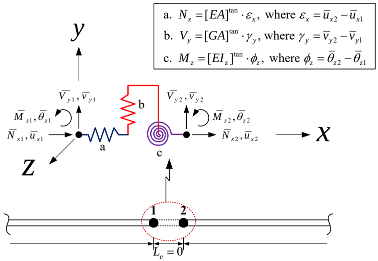
Each fiber material constitutive law

$\sigma_{r,n} = f_r(\varepsilon_{r,n})$, where f_r is function r^{th} layer/fiber with $\mathrm{E}^{\tan}_{r,n}$.

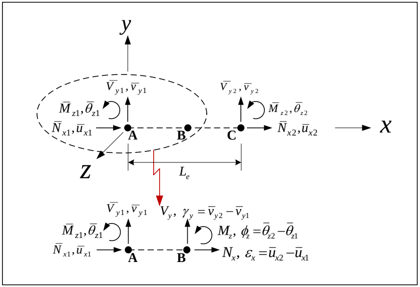
$$\sigma_{r,n}, \mathrm{E}^{\tan}_{r,n}$$

$$\boldsymbol{\sigma}^{\mathrm{int},k}_{s,e,n} = \sum_r \begin{bmatrix} 1 \\ -y_r \end{bmatrix} \cdot \sigma_{r,n}, \quad \mathbf{k}^{\tan,k}_{s,e,n}, \quad \mathbf{c}^{\tan,k}_{s,e,n} = \left[\mathbf{k}^{\tan,k}_{s,e,n} \right]^{-1}$$

- We note that this cross sectional definition allows us to easily specify longitudinal steel reinforcement. Shear reinforcement, on the other hand, can not be explicitly modeled, however, common practice is to assign modified properties to the confined concrete.
- Neutral Axis is implicitly determined. In the input data, we can assume the neutral axis to be in the bottom layer (for ease of determining layer elevation y_r), then at the global level equilibrium will not be satisfied, and then displacements will be adjusted, and indirectly strain distribution will be corrected by shifting the N.A. Faster convergence could be achieved if an *intelligent* guess is made for the location of the NA, and define all fibers with respect to that location. Alternatively, the program could immediately (first increment/iteration) determine the elastic neutral axis.



- Zero-length elements are needed for lumped plasticity models where plastic hinges form at the end of the element. They are more suitable for lateral loads than for vertical ones.
- Element end deformations in the reinforced concrete are composed of two types:
  - flexural deformation that causes inelastic strains
  - element end rotation which may be caused be the slip of longitudinal reinforcement in reinforced concrete or plastic hinges in steel members.



1. Constitutive law Section constitutive law is expressed as

$$\underbrace{\begin{Bmatrix} N_x \\ V_y \\ M_z \end{Bmatrix}}_{\boldsymbol{\sigma}_s} = \underbrace{\begin{bmatrix} [EA]^{tan} & 0 & 0 \\ 0 & [GA]^{tan} & 0 \\ 0 & 0 & [EI_z]^{tan} \end{bmatrix}}_{\mathbf{k}_s^{tan}} \underbrace{\begin{Bmatrix} \bar{u}_{x2} - \bar{u}_{x1} \\ \bar{v}_{y2} - \bar{v}_{y1} \\ \bar{\theta}_{z2} - \bar{\theta}_{z1} \end{Bmatrix}}_{\boldsymbol{\varepsilon}_s}$$

where, $[EA]^{tan}$, $[GA]^{tan}$ and $[EI_z]^{tan}$ are tangent stiffnesses associated with axial, shear and moment.

2. Equilibrium Composing equilibrium equations between point A and point B

$$\overline{N}_{x1} = [EA]^{tan} \cdot (\overline{u}_{x1} - \overline{u}_{x2}); \;\; \overline{V}_{y1} = [GA]^{tan} \cdot (\overline{v}_{y1} - \overline{v}_{y2}); \;\; \overline{M}_{z1} = [EI_z]^{tan} \cdot (\overline{\theta}_{z1} - \overline{\theta}_{z2})$$

Likewise between point B and point C,

$$\overline{N}_{x2} = [EA]^{tan} \cdot (\overline{u}_{x2} - \overline{u}_{x1}); \;\; \overline{V}_{y2} = [GA]^{tan} \cdot (\overline{v}_{y2} - \overline{v}_{y1}); \;\; \overline{M}_{z2} = [EI_z]^{tan} \cdot (\overline{\theta}_{z2} - \overline{\theta}_{z1})$$

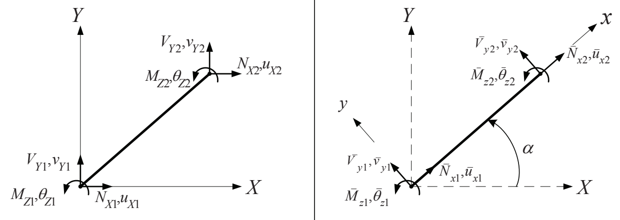
Rewriting in matrix form, the relationship between element nodal force and displacement vector is given by

$$\underbrace{\begin{Bmatrix} \overline{N}_{x1} \\ \overline{V}_{y1} \\ \overline{M}_{z1} \\ \overline{N}_{x2} \\ \overline{V}_{y2} \\ \overline{M}_{z2} \end{Bmatrix}}_{\overline{\mathbf{f}}_e} = \overline{\mathbf{k}}_e^{tan} \underbrace{\begin{Bmatrix} \overline{u}_{x1} \\ \overline{v}_{y1} \\ \overline{\theta}_{z1} \\ \overline{u}_{x2} \\ \overline{v}_{y2} \\ \overline{\theta}_{z2} \end{Bmatrix}}_{\overline{\mathbf{d}}_e}$$

where, $\overline{\mathbf{k}}_e^{tan}$ is the element stiffness matrix in local reference.

$$\overline{\mathbf{k}}_e^{tan} = \begin{bmatrix} [EA]^{tan} & 0 & 0 & -[EA]^{tan} & 0 & 0 \\ 0 & [GA]^{tan} & 0 & 0 & -[GA]^{tan} & 0 \\ 0 & 0 & [EI_z]^{tan} & 0 & 0 & -[EI_z]^{tan} \\ -[EA]^{tan} & 0 & 0 & [EA]^{tan} & 0 & 0 \\ 0 & -[GA]^{tan} & 0 & 0 & [GA]^{tan} & 0 \\ 0 & 0 & -[EI_z]^{tan} & 0 & 0 & [EI_z]^{tan} \end{bmatrix}$$

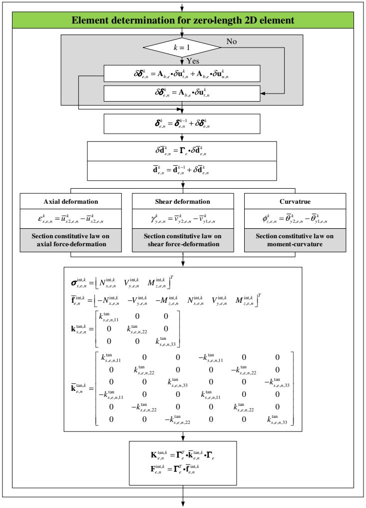
Coordinate system in zero-length 2D element is same as the one of the 2D stiffness element.



(a) Global reference in an element

(b) Local reference in an element

$$\Gamma_e = \begin{array}{c|cccccc} & N_{X1} & V_{Y1} & M_{Z1} & N_{X2} & V_{Y2} & M_{Z2} \\ \hline \overline{N}_{x1} & \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ \overline{V}_{y1} & -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ \overline{M}_{z1} & 0 & 0 & 1 & 0 & 0 & 0 \\ \overline{N}_{x2} & 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ \overline{V}_{y2} & 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ \overline{M}_{z2} & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$


Element determination for zero-length 2D element
$k = 1$
No
Yes
$\delta\boldsymbol{\delta}_{e,n}^{k} = \mathbf{A}_{b,e} \bullet \delta\mathbf{u}_{t,n}^{k} + \mathbf{A}_{b,e} \bullet \delta\mathbf{u}_{u,n}^{k}$
$\delta\boldsymbol{\delta}_{e,n}^{k} = \mathbf{A}_{b,e} \bullet \delta\mathbf{u}_{t,n}^{k}$
$\boldsymbol{\delta}_{e,n}^{k} = \boldsymbol{\delta}_{e,n}^{k-1} + \delta\boldsymbol{\delta}_{e,n}^{k}$
$\delta\overline{\mathbf{d}}_{e,n}^{k} = \boldsymbol{\Gamma}_{e} \bullet \delta\mathbf{d}_{e,n}^{k}$
$\overline{\mathbf{d}}_{e,n}^{k} = \overline{\mathbf{d}}_{e,n}^{k-1} + \delta\overline{\mathbf{d}}_{e,n}^{k}$
Axial deformation
$\varepsilon_{x,e,n}^{k} = \overline{u}_{x2,e,n}^{k} - \overline{u}_{x2,e,n}^{k}$
Section constitutive law on axial force-deformation
Shear deformation
$\gamma_{y,e,n}^{k} = \overline{v}_{y2,e,n}^{k} - \overline{v}_{y1,e,n}^{k}$
Section constitutive law on shear force-deformation
Curvatrue
$\phi_{z,e,n}^{k} = \overline{\theta}_{y2,e,n}^{k} - \overline{\theta}_{y1,e,n}^{k}$
Section constitutive law on moment-curvature
$\boldsymbol{\sigma}_{s,e,n}^{\mathrm{int},k} = \lfloor N_{x,e,n}^{\mathrm{int},k} \quad V_{y,e,n}^{\mathrm{int},k} \quad M_{z,e,n}^{\mathrm{int},k} \rfloor^{T}$
$\overline{\mathbf{f}}_{e,n}^{\mathrm{int},k} = \lfloor -N_{x,e,n}^{\mathrm{int},k} \quad -V_{y,e,n}^{\mathrm{int},k} \quad -M_{z,e,n}^{\mathrm{int},k} \quad N_{x,e,n}^{\mathrm{int},k} \quad V_{y,e,n}^{\mathrm{int},k} \quad M_{z,e,n}^{\mathrm{int},k} \rfloor^{T}$
$\mathbf{k}_{s,e,n}^{\tan,k} = \begin{bmatrix} k_{s,e,n,11}^{\tan} & 0 & 0 \\ 0 & k_{s,e,n,22}^{\tan} & 0 \\ 0 & 0 & k_{s,e,n,33}^{\tan} \end{bmatrix}$
$\overline{\mathbf{k}}_{e,n}^{\tan,k} = \begin{bmatrix} k_{s,e,n,11}^{\tan} & 0 & 0 & -k_{s,e,n,11}^{\tan} & 0 & 0 \\ 0 & k_{s,e,n,22}^{\tan} & 0 & 0 & -k_{s,e,n,22}^{\tan} & 0 \\ 0 & 0 & k_{s,e,n,33}^{\tan} & 0 & 0 & -k_{s,e,n,33}^{\tan} \\ -k_{s,e,n,11}^{\tan} & 0 & 0 & k_{s,e,n,11}^{\tan} & 0 & 0 \\ 0 & -k_{s,e,n,22}^{\tan} & 0 & 0 & k_{s,e,n,22}^{\tan} & 0 \\ 0 & 0 & -k_{s,e,n,22}^{\tan} & 0 & 0 & k_{s,e,n,33}^{\tan} \end{bmatrix}$
$\mathbf{K}_{e,n}^{\tan,k} = \boldsymbol{\Gamma}_{e}^{T} \bullet \overline{\mathbf{k}}_{e,n}^{\tan,k} \bullet \boldsymbol{\Gamma}_{e}$
$\mathbf{F}_{e,n}^{\mathrm{int},k} = \boldsymbol{\Gamma}_{e}^{T} \bullet \overline{\mathbf{f}}_{e,n}^{\mathrm{int},k}$

Step 1: Determine the section deformation vector, axial deformation, shear deformation and curvature. For each deformation, we extract the associated components from $\overline{\mathbf{d}}_{e,n}^{k}$.

$$\begin{aligned}
\overline{\mathbf{d}}_{e,n}^{k} &= \lfloor \overline{u}_{x1,e,n}^{k}\ \overline{v}_{y1,e,n}^{k}\ \overline{\theta}_{z1,e,n}^{k}\ \overline{u}_{x2,e,n}^{k}\ \overline{v}_{y2,e,n}^{k}\ \overline{\theta}_{z2,e,n}^{k} \rfloor^{T} \\
\boldsymbol{\varepsilon}_{s,e,n}^{k} &= \lfloor \varepsilon_{x,e,n}^{k},\ \gamma_{y,e,n}^{k},\ \phi_{z,e,n}^{k} \rfloor^{T} \\
\varepsilon_{x,e,n}^{k} &= \overline{u}_{x2,e,n}^{k} - \overline{u}_{x2,e,n}^{k} \\
\gamma_{y,e,n}^{k} &= \overline{v}_{y2,e,n}^{k} - \overline{v}_{y2,e,n}^{k} \\
\phi_{z,e,n}^{k} &= \overline{\theta}_{z2,e,n}^{k} - \overline{\theta}_{z2,e,n}^{k}
\end{aligned}$$

which define axial section deformation, shear deformation, and curvature.

Step 2: Determine the section tangent stiffness associated with axial force-deformation, shear force-deformation, and moment-curvature in the section constitutive laws. Section constitutive laws modified with several variables in function of material constitutive law associated with uniaxial stress-strain relationship can be used. The internal section force vector is determined next. If we assume that the section constitutive law is explicitly known, $\mathbf{k}^{tan,k}_{s,e,n}$ and $\boldsymbol{\sigma}^{int,k}_{s,e,n}$ are determined from $\boldsymbol{\varepsilon}^{k}_{s,e,n}$. However, in elastic section, we need not to compute $\mathbf{k}^{tan,k}_{s,e,n}$ again as it is identical to the initial section stiffness matrix $\mathbf{k}_{s,e}$. For an elastic section,

$$
\begin{aligned}
\mathbf{k}^{tan}_{s,e,n} &= \mathbf{k}_{s,e} \\
\underbrace{\left\{ \begin{array}{c} N^{int,k}_{x,e,n} \\ V^{int,k}_{y,e,n} \\ M^{int,k}_{z,e,n} \end{array} \right\}}_{\boldsymbol{\sigma}^{int,k}_{s,e,n}} &= \mathbf{k}^{tan}_{s,e,n} \underbrace{\left\{ \begin{array}{c} \varepsilon^{k}_{x,e,n} \\ \gamma^{k}_{y,e,n} \\ \phi^{k}_{z,e,n} \end{array} \right\}}_{\boldsymbol{\varepsilon}^{k}_{s,e,n}}
\end{aligned}
$$

where, $\mathbf{k}^{tan,k}_{s,e,n}$ is the section tangent stiffness matrix at k^{th} iteration.

Step 3: Determine the internal element nodal force vector and the element tangent stiffness matrix

$$\bar{\mathbf{f}}_{e,n}^{int,k} = \lfloor N_{x,e,n}^{int,k}, V_{y,e,n}^{int,k}, M_{z,e,n}^{int,k}, -N_{x,e,n}^{int,k}, -V_{y,e,n}^{int,k}, -M_{z,e,n}^{int,k} \rfloor^T$$

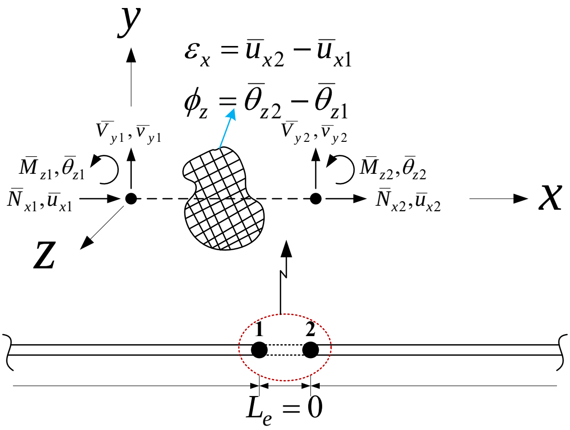
$$\bar{\mathbf{k}}_e^{tan,k} = \begin{bmatrix} EA_{e,n}^{tan,k} & 0 & 0 & -EA_{e,n}^{tan,k} & 0 & 0 \\ 0 & GA_{e,n}^{tan,k} & 0 & 0 & -GA_{e,n}^{tan,k} & 0 \\ 0 & 0 & EI_{z,e,n}^{tan,k} & 0 & 0 & -EI_{z,e,n}^{tan,k} \\ -EA_{e,n}^{tan,k} & 0 & 0 & EA_{e,n}^{tan,k} & 0 & 0 \\ 0 & -GA_{e,n}^{tan,k} & 0 & 0 & GA_{e,n}^{tan,k} & 0 \\ 0 & 0 & -EI_{z,e,n}^{tan,k} & 0 & 0 & EI_{z,e,n}^{tan,k} \end{bmatrix}$$

where, $\bar{\mathbf{k}}_{e,n}^{tan,k}$ is the element tangent stiffness matrix in local reference.

We determine $\mathbf{F}_{e,n}^{int,k}$ and $\mathbf{K}_{e,n}^{tan,k}$.

$$\begin{aligned} \mathbf{F}_{e,n}^{int,k} &= \boldsymbol{\Gamma}_e{}^T \cdot \bar{\mathbf{f}}_{e,n}^{int,k} \\ \mathbf{K}_{e,n}^{tan,k} &= \boldsymbol{\Gamma}_e{}^T \cdot \bar{\mathbf{k}}_{e,n}^{tan,k} \cdot \boldsymbol{\Gamma}_e \end{aligned}$$

Zero-length section element is analogous to the zero length element, however, it uses layer/fiber. This element enables us to model the shift in center of section rotation which may occur (in bar-slip for example). The element is formulated on the basis of coupled axial force and moment.

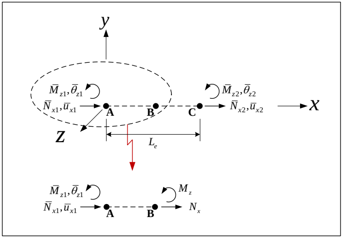


Constitutive law Section constitutive law is expressed as

$$\underbrace{\left\{\begin{array}{c} N_x \\ M_z \end{array}\right\}}_{\boldsymbol{\sigma}_s} = \underbrace{\left[\begin{array}{cc} \mathrm{k}_{s,11}^{tan} & \mathrm{k}_{s,12}^{tan} \\ \mathrm{k}_{s,21}^{tan} & \mathrm{k}_{s,22}^{tan} \end{array}\right]}_{\mathbf{k}_s^{tan}} \cdot \underbrace{\left\{\begin{array}{c} \overline{u}_{x2} - \overline{u}_{x1} \\ \overline{\theta}_{z2} - \overline{\theta}_{z1} \end{array}\right\}}_{\boldsymbol{\varepsilon}_s}$$

where, $\mathbf{k}_s^{tan}$ is the section tangent stiffness matrix obtained from layer/fiber state determination.

Equilibrium Zero-length section element is based on Bernoulli beam theory.



Composing equilibrium equations between point A and point B.

$$\begin{aligned}\overline{N}_{x1} &= k^{tan}_{s,11}\cdot(\overline{u}_{x1}-\overline{u}_{x2}) + k^{tan}_{s,12}\cdot(\overline{\theta}_{z1}-\overline{\theta}_{z2})\\ \overline{M}_{z1} &= k^{tan}_{s,21}\cdot(\overline{u}_{x1}-\overline{u}_{x2}) + k^{tan}_{s,22}\cdot(\overline{\theta}_{z1}-\overline{\theta}_{z2})\end{aligned} \tag{1}$$

Likewise between point B and point C,

$$\begin{aligned}\overline{N}_{x2} &= k^{tan}_{s,11}\cdot(\overline{u}_{x2}-\overline{u}_{x1}) + k^{tan}_{s,12}\cdot(\overline{\theta}_{z2}-\overline{\theta}_{z1})\\ \overline{M}_{z2} &= k^{tan}_{s,21}\cdot(\overline{u}_{x2}-\overline{u}_{x1}) + k^{tan}_{s,22}\cdot(\overline{\theta}_{z2}-\overline{\theta}_{z1})\end{aligned} \tag{2}$$

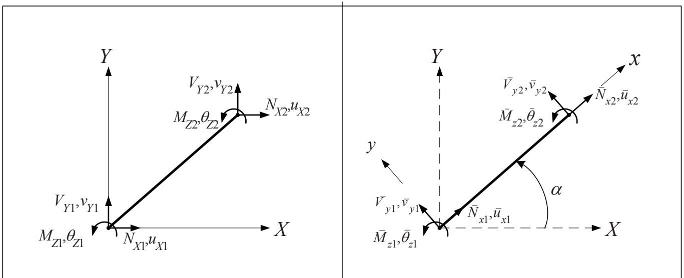
Rewriting Eq. 1 and 2 to matrix form, the relationship between element nodal force and displacement vector is given by

$$\underbrace{\begin{Bmatrix}\overline{N}_{x1}\\ 0\\ \overline{M}_{z1}\\ \overline{N}_{x2}\\ 0\\ \overline{M}_{z2}\end{Bmatrix}}_{\overline{\mathbf{f}}_e} = \overline{\mathbf{k}}^{tan}_e \underbrace{\begin{Bmatrix}\overline{u}_{x1}\\ 0\\ \overline{\theta}_{z1}\\ \overline{u}_{x2}\\ 0\\ \overline{\theta}_{z2}\end{Bmatrix}}_{\overline{\mathbf{d}}_e}$$

where, $\overline{\mathbf{k}}_e^{tan}$ is the element stiffness matrix in local reference.

$$\overline{\mathbf{k}}_e^{tan} = \begin{bmatrix} \mathrm{k}_{s,11}^{tan} & 0 & \mathrm{k}_{s,12}^{tan} & -\mathrm{k}_{s,11}^{tan} & 0 & -\mathrm{k}_{s,12}^{tan} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{k}_{s,21}^{tan} & 0 & \mathrm{k}_{s,22}^{tan} & -\mathrm{k}_{s,21}^{tan} & 0 & -\mathrm{k}_{s,22}^{tan} \\ -\mathrm{k}_{s,11}^{tan} & 0 & -\mathrm{k}_{s,12}^{tan} & \mathrm{k}_{s,11}^{tan} & 0 & \mathrm{k}_{s,12}^{tan} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathrm{k}_{s,21}^{tan} & 0 & -\mathrm{k}_{s,22}^{tan} & \mathrm{k}_{s,21}^{tan} & 0 & \mathrm{k}_{s,22}^{tan} \end{bmatrix} \tag{3}$$

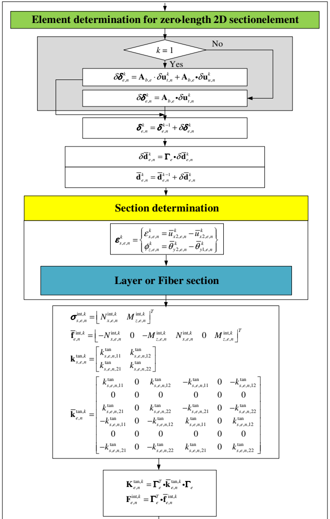
Coordinate system in zero-length 2D element is same as in 2D stiffness element.



(a) Global reference in an element

(b) Local reference in an element

$$\boldsymbol{\Gamma}_e = \left[\begin{array}{c|cccccc} & N_{X1} & V_{Y1} & M_{Z1} & N_{X2} & V_{Y2} & M_{Z2} \\ \hline \overline{N}_{x1} & \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ \overline{V}_{y1} & -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ \overline{M}_{z1} & 0 & 0 & 1 & 0 & 0 & 0 \\ \overline{N}_{x2} & 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ \overline{V}_{y2} & 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ \overline{M}_{z2} & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$



Step 1: Determine the section deformation vector, axial deformation and curvature. For each deformation, we extracts the associated components from $\overline{\mathbf{d}}^k_{e,n}$.

$$\begin{aligned}
\overline{\mathbf{d}}^k_{e,n} &= \lfloor \overline{u}^k_{x1,e,n} \; 0 \; \overline{\theta}^k_{z1,e,n} \; \overline{u}^k_{x2,e,n} \; 0 \; \overline{\theta}^k_{z2,e,n} \rfloor^T \\
\boldsymbol{\varepsilon}^k_{s,e,n} &= \lfloor \varepsilon^k_{x,e,n}, \; \phi^k_{z,e,n} \rfloor^T \\
\varepsilon^k_{x,e,n} &= \overline{u}^k_{x2,e,n} - \overline{u}^k_{x2,e,n} \\
\phi^k_{z,e,n} &= \overline{\theta}^k_{z2,e,n} - \overline{\theta}^k_{z2,e,n}
\end{aligned}$$

Step 2: Determine the section tangent stiffness associated with axial force-deformation and moment-curvature using layer/fiber state determination as in for Layr/fiber. Determine next the internal section force vector. If we assume that the material constitutive law is explicitly known, $\mathbf{k}_{s,e,n}^{tan,k}$ and $\boldsymbol{\sigma}_{s,e,n}^{int,k}$ are determined from $\boldsymbol{\varepsilon}_{s,e,n}^{k}$. However, in the section with elastic material, we need not to compute $\mathbf{k}_{s,e,n}^{tan,k}$ again as it is identical to the initial section stiffness matrix $\mathbf{k}_{s,e}$. If we have a section with elastic material, then

$$\begin{aligned}
\mathbf{k}_{s,e,n}^{tan} &= \mathbf{k}_{s,e} \\
\underbrace{\left\{\begin{array}{c} N_{x,e,n}^{int,k} \\ M_{z,e,n}^{int,k} \end{array}\right\}}_{\boldsymbol{\sigma}_{s,e,n}^{int,k}} &= \mathbf{k}_{s,e,n}^{tan} \underbrace{\left\{\begin{array}{c} \varepsilon_{x,e,n}^{k} \\ \phi_{z,e,n}^{k} \end{array}\right\}}_{\boldsymbol{\varepsilon}_{s,e,n}^{k}}
\end{aligned}$$

where, $\mathbf{k}_{s,e,n}^{tan,k}$ is the section tangent stiffness matrix at k^{th} iteration.

Step 3: Determine the internal element nodal force vector and the element tangent stiffness matrix

$$\overline{\mathbf{f}}^{int,k}_{e,n} = \lfloor N^{int,k}_{x,e,n}, 0, M^{int,k}_{z,e,n}, -N^{int,k}_{x,e,n}, 0, -M^{int,k}_{z,e,n} \rfloor^T$$

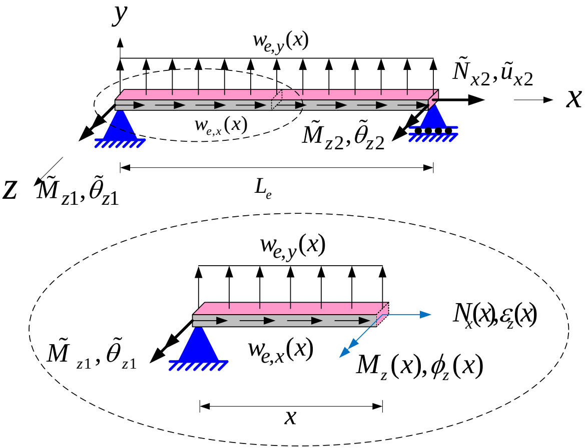
$$\overline{\mathbf{k}}^{tan,k}_{e,n} = \begin{bmatrix} \mathsf{k}^{tan,k}_{s,e,n,11} & 0 & \mathsf{k}^{tan,k}_{s,e,n,12} & -\mathsf{k}^{tan,k}_{s,e,n,11} & 0 & -\mathsf{k}^{tan,k}_{s,e,n,12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathsf{k}^{tan,k}_{s,e,n,21} & 0 & \mathsf{k}^{tan,k}_{s,e,n,22} & -\mathsf{k}^{tan,k}_{s,e,n,21} & 0 & -\mathsf{k}^{tan,k}_{s,e,n,22} \\ -\mathsf{k}^{tan,k}_{s,e,n,11} & 0 & -\mathsf{k}^{tan,k}_{s,12e,n,} & \mathsf{k}^{tan,k}_{s,e,n,11} & 0 & \mathsf{k}^{tan,k}_{s,e,n,12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathsf{k}^{tan,k}_{s,e,n,21} & 0 & -\mathsf{k}^{tan,k}_{s,e,n,22} & \mathsf{k}^{tan,k}_{s,e,n,21} & 0 & \mathsf{k}^{tan,k}_{s,e,n,22} \end{bmatrix}$$

where, $\overline{\mathbf{k}}^{tan,k}_{e,n}$ is the element tangent stiffness matrix in local reference. We determine $\mathbf{F}^{int,k}_{e,n}$ and $\mathbf{K}^{tan,k}_{e,n}$.

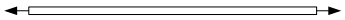
$$\begin{aligned} \mathbf{F}^{int,k}_{e,n} &= \boldsymbol{\Gamma}_e{}^T \cdot \overline{\mathbf{f}}^{int,k}_{e,n} \\ \mathbf{K}^{tan,k}_{e,n} &= \boldsymbol{\Gamma}_e{}^T \cdot \overline{\mathbf{k}}^{tan,k}_{e,n} \cdot \boldsymbol{\Gamma}_e \end{aligned}$$

- Flexibility based elements
  - Are nonconformist finite elements since they yield the element flexibility matrix rather than the classical stiffness matrix.
  - Are based on the equations of equilibrium rather than on assumed displacement field, while at the global level formulation is displacement based.
  - Offer some important advantages over stiffness based elements: fewer elements are needed (albeit at the cost of a more complex formulation); stiffness-based method formulations are approximate and flexibility-based method formulations are exact such as a section varying along the element and elements with material nonlinearity.
- We derive the element flexibility matrix $\tilde{\mathbf{c}}_e$ without rigid body modes and then invert it to obtain the corresponding element stiffness matrix $\tilde{\mathbf{k}}_e$ (again without rigid body modes). The retained degrees of freedom are the axial force at node 2, and the two end moments.
- There are two distinct formulations: a) with element iterations, and b) without element iterations. We will focus on the former.

- Whereas we have used the principle of virtual work (displacement) for the derivation of the stiffness based element, we shall now use the principle of complementary virtual work (force) through the usual three steps.



y
$w_{e,y}(x)$
$\tilde{N}_{x2}, \tilde{u}_{x2}$
x
$w_{e,x}(x)$
$\tilde{M}_{z2}, \tilde{\theta}_{z2}$
Z
$\tilde{M}_{z1}, \tilde{\theta}_{z1}$
L_e
$w_{e,y}(x)$
$N_x(x), \varepsilon_z(x)$
$\tilde{M}_{z1}, \tilde{\theta}_{z1}$
$w_{e,x}(x)$
$M_z(x), \phi_z(x)$
x



(a) Positive section axial force

(b) Positive section moment

- Equilibrium will now be strongly enforced (whereas it was satisfied in the weak sense previously) and we seek to derive the force shape functions:
  - For uniformly distributed axial forces, we have $dN_x(x) = w_x^{(e)} dx$ or $\frac{dN_x(x)}{dx} = w_x^{(e)}(x)$
  - For uniformly distributed transverse forces $\frac{dV_y(x)}{dx} = w_y^{(e)}(x))$ and $\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = w(x)$
  - Equilibrium can be expressed as

$$\underbrace{\mathbf{w}_e(x)}_{\text{External}} + \underbrace{\mathcal{L}_f \cdot \boldsymbol{\sigma}_s(x)}_{\text{Internal}} = \mathbf{0}; \quad \left\{ \begin{array}{c} w_x^{(e)}(x) \\ w_y^{(e)}(x) \end{array} \right\} + \left[\begin{array}{cc} \frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & \frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{array} \right] \left\{ \begin{array}{c} N_x(x) \\ M_z(x) \end{array} \right\} = \mathbf{0}$$

  $\mathbf{w}_e(x)$ is the external element traction vector, $\mathcal{L}_f$ is the force differential operator which enforces equilibrium. (Note in stiffness formulation, the compatibility was “strongly” enforced).

- We will write equilibrium of sectional stresses in terms of the nodal forces, and assume that there are no external element traction.
- Whereas we previously used displacement interpolation functions, we now need force interpolation functions, $\mathbf{N}_f(x)$ in order to exactly satisfy equilibrium along the element

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & \frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{bmatrix} \begin{Bmatrix} N_x(x) \\ M_z(x) \end{Bmatrix} = \mathbf{0}$$

- Integrating these equations, we obtain $N_x(x) = c_3$ and $M_z(x) = c_1 x + c_2$.
- We now seek to determine the shape functions that relate section internal forces at any point x to the nodal forces. We enforce natural boundary condition

$$\begin{aligned} N_x(L) &= \tilde{N}_{x2}; & M_z(0) &= -\tilde{M}_{z1}; & M_z(L) &= \tilde{M}_{z2}; \\ \Rightarrow c_1 &= \frac{\tilde{M}_{z1}+\tilde{M}_{z2}}{L_e}; & c_2 &= -\tilde{M}_{z1}; & c_3 &= \tilde{N}_{x2}; \end{aligned}$$

- Substituting, we have the internal axial force and moment at any point (x) in terms of the nodal forces.

$$\underbrace{\left\{ \begin{array}{c} N_x(x) \\ M_z(x) \end{array} \right\}}_{\boldsymbol{\sigma}_s(x)} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ \frac{x}{L_e} - 1 & \frac{x}{L_e} & 0 \end{bmatrix}}_{\mathbf{N}_f(x)} \underbrace{\left\{ \begin{array}{c} \tilde{M}_{z1} \\ \tilde{M}_{z2} \\ \tilde{N}_{x2} \end{array} \right\}}_{\tilde{\mathbf{f}}_e}$$

  where, $\tilde{\mathbf{f}}_e$ is the element nodal force vector without rigid body modes.
- It should be noted that these shape functions enforce equilibrium at any section along the element

- Constitutive law: Previously expressed section forces in terms of section deformations, we now need to express section deformations in terms of section forces: $\boldsymbol{\varepsilon}_s(x) = \mathbf{c}_s(x) \cdot \boldsymbol{\sigma}_s(x)$ where, $\mathbf{c}_s(x)$ is the section flexibility matrix. If $\mathbf{c}_s(x)$ is not derived from fiber section, then for linear elastic analysis $\mathbf{c}_s(x)$ is simply.

$$\mathbf{c}_s(x) = \begin{bmatrix} \frac{1}{E(x)\cdot A(x)} & 0 \\ 0 & \frac{1}{E(x)\cdot I_z(x)} \end{bmatrix}$$

- Compatibility of displacements: enforced only in a weak form through the principle of complementary virtual work (as opposed to the principle of virtual work for the stiffness-based method).

$$\underbrace{\delta\tilde{\mathbf{f}}_e{}^T\tilde{\mathbf{d}}_e}_{\text{External}} = \underbrace{\int_0^{L_e} \delta\boldsymbol{\sigma}_s(x)^T \cdot \boldsymbol{\varepsilon}_s(x)\mathrm{d}x}_{\text{Internal}}$$

where $\tilde{\mathbf{d}}_e$ is the element nodal displacement vector without rigid body modes.

- Substituting

$$\begin{aligned} \delta\tilde{\mathbf{f}}_e{}^T\tilde{\mathbf{d}}_e &= \int_0^{L_e} \delta\tilde{\mathbf{f}}_e{}^T \cdot \mathbf{N}_f(x)^T \cdot \mathbf{c}_s(x) \cdot \boldsymbol{\sigma}_s(x)\,\mathrm{d}x \\ \tilde{\mathbf{d}}_e &= \int_0^{L_e} \mathbf{N}_f(x)^T \cdot \mathbf{c}_s(x) \cdot \boldsymbol{\sigma}_s(x)\,\mathrm{d}x = \underbrace{\int_0^{L_e} \mathbf{N}_f(x)^T \cdot \mathbf{c}_s(x) \cdot \mathbf{N}_f(x)\,\mathrm{d}x}_{\tilde{\mathbf{c}}_e} \cdot \tilde{\mathbf{f}}_e \end{aligned}$$

or

$$\tilde{\mathbf{d}}_e = \tilde{\mathbf{c}}_e \cdot \tilde{\mathbf{f}}_e$$

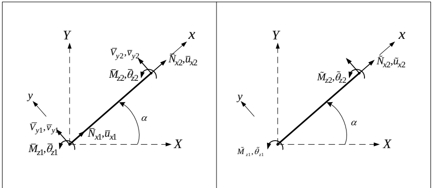
- The element flexibility matrix without rigid body modes in local reference is thus given by

$$\tilde{\mathbf{c}}_e = \int_0^{L_e} \mathbf{N}_f(x)^T \cdot \mathbf{c}_s(x) \cdot \mathbf{N}_f(x)\,\mathrm{d}x$$

- The corresponding element stiffness matrix without rigid body modes in local reference is simply

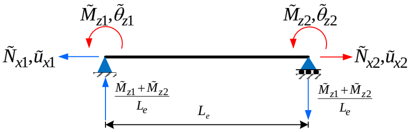
$$\tilde{\mathbf{k}}_e = [\tilde{\mathbf{c}}_e]^{-1}$$

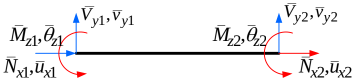
  Note this is a 3x3 matrix, we still have to insert equilibrium relations and transform it into the usual 6x6 stiffness matrix



(a) Local reference in an element

(b) Local reference in an element without rigid body modes





- Contrarily to the reference system of the stiffness-based method, we need to consider forces and displacements in local reference with and without rigid body modes.
- Element nodal force vector without rigid body modes in local reference are (arbitrarily) selected as $\tilde{\mathbf{f}}_e = \lfloor \tilde{M}_{z1},\ \tilde{M}_{z2},\ \tilde{N}_{x2} \rfloor^T$, and the corresponding element nodal displacement vector without rigid body modes in local reference are given by $\tilde{\mathbf{d}}_e = \lfloor \tilde{\theta}_{z1},\ \tilde{\theta}_{z2},\ \tilde{u}_{x2} \rfloor^T$
- The relationship between rigid body modes and no rigid body modes is obtained through equilibrium

$$\underbrace{\left\{\begin{array}{c} \overline{N}_{x1} \\ \overline{V}_{y1} \\ \overline{M}_{z1} \\ \overline{N}_{x2} \\ \overline{V}_{y2} \\ \overline{M}_{z2} \end{array}\right\}}_{\overline{\mathbf{f}}_e} = \underbrace{\left[\begin{array}{ccc} 0 & 0 & -1 \\ \frac{1}{L_e} & \frac{1}{L_e} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{L_e} & -\frac{1}{L_e} & 0 \\ 0 & 1 & 0 \end{array}\right]}_{\tilde{\mathbf{r}}_e^T} \underbrace{\left\{\begin{array}{c} \tilde{M}_{z1} \\ \tilde{M}_{z2} \\ \tilde{N}_{x2} \end{array}\right\}}_{\tilde{\mathbf{f}}_e}$$

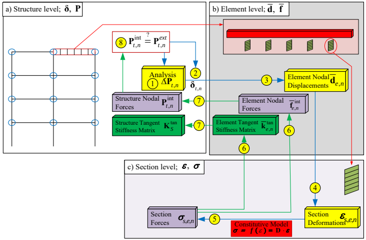
- Substituting, $\overline{\mathbf{f}}_e = \tilde{\mathbf{\Gamma}}_e^T \cdot \tilde{\mathbf{f}}_e$; $\overline{\mathbf{d}}_e = \tilde{\mathbf{\Gamma}}_e^T \cdot \tilde{\mathbf{d}}_e$; or

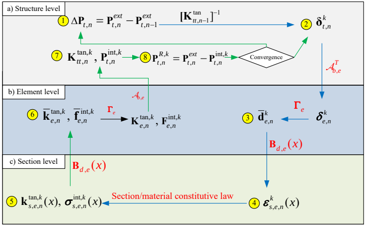
$$\mathbf{K}_e = \tilde{\mathbf{\Gamma}}_e^T \cdot \tilde{\mathbf{k}}_e \cdot \tilde{\mathbf{\Gamma}}_e$$

- Note that whereas previously $\mathbf{\Gamma}_e$ denoted a geometric transformation matrix (for stiffness based elements), it now corresponds to a statics matrix (also denoted as $\mathcal{B}$ previously).
- Derivation of the stiffness matrix from the flexibility one and the equations of equilibrium parallels the one earlier derived

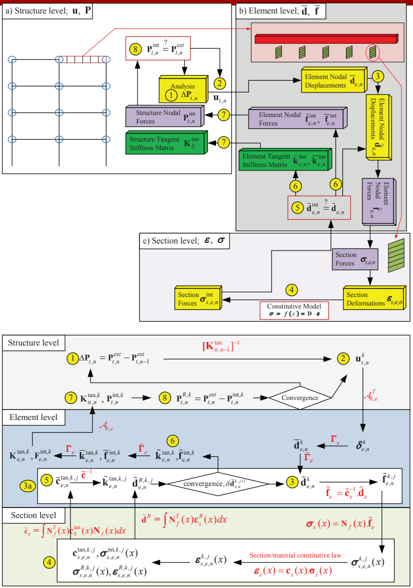
$$[\mathbf{K}] = \left[\begin{array}{c|c} [\mathbf{d}]^{-1} & [\mathbf{d}]^{-1}[\mathcal{B}]^T \\ \hline [\mathcal{B}][\mathbf{d}]^{-1} & [\mathcal{B}][\mathbf{d}]^{-1}[\mathcal{B}]^T \end{array}\right]$$

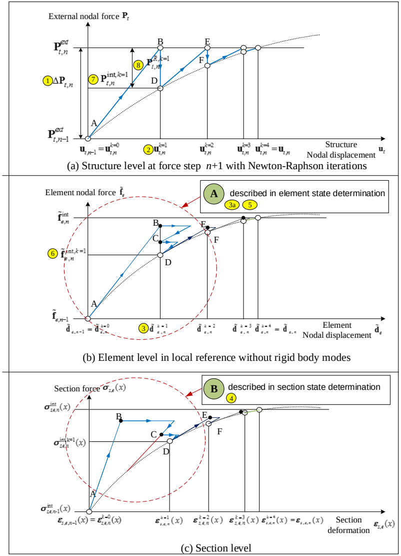
- The flexibility-based element (derived from the complementary principle of virtual work) does not have shape functions that relate deformation field inside the element with element nodal displacement vector, but shape functions which relate section forces to nodal forces.
- The global formulation is based on the stiffness (displacement) formulation, the element is based on a flexibility (force) formulation; the two will have to be reconciled (in the determination of the internal element force vectors).
- At the element level, the flexibility based element will provide nodal displacements which are not necessarily compatible with the ones coming from adjacent elements just as in the stiffness based formulation, forces were not compatible at the element level.
- We must ensure nodal displacement compatibility (in the same way as we ensured nodal equilibrium in the stiffness based formulation. Accomplished iteratively.
- Note that in the stiffness based method, there was a discontinuity in nodal forces.
- There are two algorithms for the mixed stiffness-based and flexibility-based methods: (a) with Newton-Raphson iteration in the element level to determine element state (Spacone), (b) without iteration in the element level to determine element state (Carol).

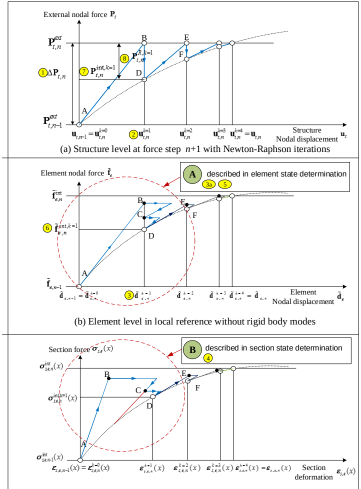




$\mathcal{A}_{b,e}$ and $\mathcal{A}_{b,e}^{T}$ are the displacement extracting operator and the force assembling operator.


a) Structure level; u, P
b) Element level; d, f
c) Section level; ε, σ
Analysis
Structure Nodal Forces
Structure Tangent Stiffness Matrix
Element Nodal Displacements
Element Nodal Forces
Element Tangent Stiffness Matrix
Element Nodal Displacements
Element Nodal Forces
Section Forces
Section Forces
Section Deformations
Constitutive Model
Structure level
Convergence
Element level
convergence
Section level
Section/material constitutive law

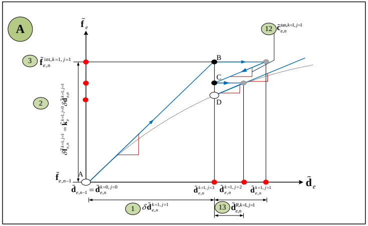

External nodal force
Structure Nodal displacement
(a) Structure level at force step n+1 with Newton-Raphson iterations
Element nodal force
described in element state determination
Element Nodal displacement
(b) Element level in local reference without rigid body modes
Section force
described in section state determination
Section deformation
(c) Section level



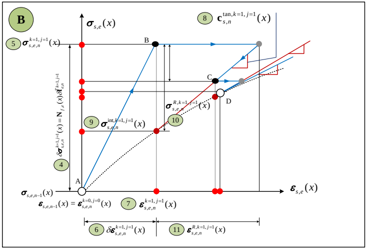
(a) Structure level at force step $n+1$ with Newton-Raphson iterations

(b) Element level in local reference without rigid body modes

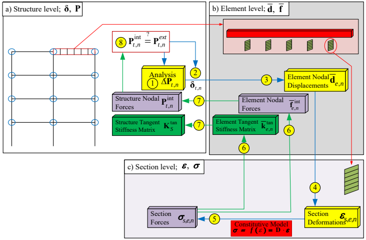
(c) Section level

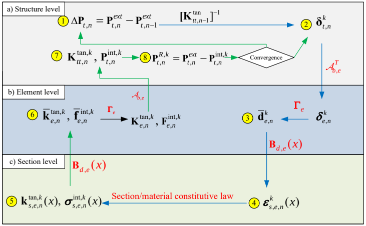


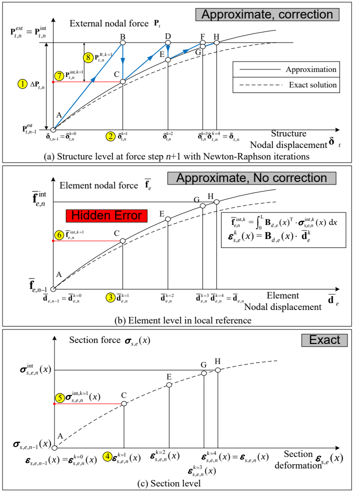
(a) Element state determination 3a 5



(b) Section state determination 4







$\mathcal{A}_{b,e}$ and $\mathcal{A}_{b,e}^T$ are the displacement extracting operator and the force assembling operator.

Compare with Stiffness based formulation

- In the flexibility based element we can not go directly from nodal displacements to section strains (as was the case in the stiffness based element), this is accomplished
  1. Determine the element nodal force vector $\tilde{\mathbf{f}}_{e,n}^{k,j}$ (⑥) from the current element nodal displacement vector using the element tangent stiffness matrix $\tilde{\mathbf{k}}_{e,n}^{tan,k,j-1}$ (③) of the previous iteration.
  2. Through the force interpolation functions $\mathbf{N}_{f,e}(x)$ determine the section force vectors $\boldsymbol{\sigma}_{s,e,n}^{k,j}(x)$ along the element.
  3. Determine the section strains by multiplying the constitutive model times the section forces.
- When we recompute the displacements corresponding to the strains.
- Compatibility of displacements at the structural level will not be satisfied.
- Thus we have an additional loop at the element level to reconcile structure based displacement and element based (through the flexibility matrix) ones, or compatibility of displacement.
- There are two complications in this procedure.

1. The determination of the section deformation vectors $\boldsymbol{\varepsilon}^{k,j}_{s,e,n}(x)$ from section force vectors since the nonlinear section force-deformation relation is commonly expressed as an explicit function of section deformation vector (④).
2. Changes in the section tangent stiffness matrices $\mathbf{k}^{tan}_{s,e,n}(x)$ produce a new element tangent stiffness matrix which, in turn, changes the element nodal force vector for the given element nodal displacement vector (⑥).

- The problem is solved through a nonlinear approach which first determines residual element nodal displacement vector $\tilde{\mathbf{d}}_{e,n}^{R,k,j}$ at each iteration. Then, compatibility of displacement at the structural level requires that this residual element nodal displacement vector be corrected.
- At the element level by applying corrective element nodal force vector based on the current element tangent stiffness matrix. The corresponding section force vectors are then determined from the force interpolation functions so that equilibrium will always be satisfied along the element. Section force vectors will not change during the section state determination in order to maintain equilibrium along the element.
- Linear approximation of section force-deformation relation about the present state results in residual section deformation vectors $\boldsymbol{\sigma}_{s,e,n}^{R,k,j}(x)$. These are then integrated along the element to obtain new residual element nodal displacement vector (⑤) and the whole process is repeated until convergence occurs.
- Compatibility of element nodal displacement vector and equilibrium along the element are always satisfied.

- The goal of the Newton-Raphson iteration loop in the element level is to determine the internal element nodal force vector (⑥) for the current element nodal displacement vector at the k^{th} Newton-Raphson iteration, hence $\tilde{\mathbf{d}}_{e,n}^{k} = \tilde{\mathbf{d}}_{e,n}^{k-1} + \delta\tilde{\mathbf{d}}_{e,n}^{k}$

1. The initial state of the element, represented by the point $\mathbf{A}$, and $j = 0$ and $k = 0$ corresponds to the state at the end of the last convergence in structural level. With the initial element tangent flexibility matrix given by $\tilde{\mathbf{c}}_{e,n}^{tan,k=1,j=0} = \tilde{\mathbf{c}}_{e,n-1}^{tan}$ and the given incremental element nodal displacement vector $\delta\tilde{\mathbf{d}}_{e,n}^{k=1,j=1} = \delta\tilde{\mathbf{d}}_{e,n}^{k=1}$ hence, the corresponding incremental element nodal force vector is $\delta\tilde{\mathbf{f}}_{e,n}^{k=1,j=1} = \left[\tilde{\mathbf{c}}_{e,n}^{tan,k=1,j=0}\right]^{-1} \cdot \delta\tilde{\mathbf{d}}_{e,n}^{k=1,j=1} = \tilde{\mathbf{k}}_{e,n}^{tan,k=1,j=0} \cdot \delta\tilde{\mathbf{d}}_{e,n}^{k=1,j=1}$
2. The incremental section force vectors can now be determined from the force interpolation functions $\delta\boldsymbol{\sigma}_{s,e,n}^{k=1,j=1}(x) = \mathbf{N}_{f,e}(x) \cdot \delta\tilde{\mathbf{f}}_{e,n}^{k=1,j=1}$ With the section tangent flexibility matrices at end of the last convergence in structural level given by $\mathbf{c}_{s,e,n}^{tan,k=1,j=0}(x) = \mathbf{c}_{s,e,n-1}^{tan}(x)$
3. The linearization of the section force-deformation relation yields the incremental section deformation vectors. $\delta\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=1}(x) = \mathbf{c}_{s,e,n}^{tan,k=1,j=0}(x) \cdot \delta\boldsymbol{\sigma}_{s,e,n}^{k=1,j=1}(x)$

4. The section deformation vectors are updated to the state that corresponds to point $\mathbf{B}$ and the updated section deformation vector (④) will be given by $\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=1}(x) = \boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=0}(x) + \delta\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=1}(x)$ For the sake of simplicity we will assume that the section force-deformation relation is explicitly known, then the section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=1}(x)$ will correspond to internal section force vectors $\boldsymbol{\sigma}_{s,e,n}^{int,k=1,j=1}(x)$ and updated section tangent flexibility matrices $\mathbf{c}_{s,e,n}^{tan,k=1,j=1}(x)$can be defined.

5. The residual section force vectors are then determined $\boldsymbol{\sigma}_{s,e,n}^{R,k=1,j=1}(x) = \boldsymbol{\sigma}_{s,e,n}^{k=1,j=1}(x) - \boldsymbol{\sigma}_{s,e,n}^{int,k=1,j=1}(x)$ and are transformed into residual section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{R,k=1,j=1}(x)$

$$\boldsymbol{\varepsilon}_{s,e,n}^{R,k=1,j=1}(x) = \mathbf{c}_{s,e,n}^{tan,k=1,j=1}(x) \cdot \boldsymbol{\sigma}_{s,e,n}^{R,k=1,j=1}(x)$$

6. The residual section deformation vectors are thus the linear approximation of the deformation error made in the linearization of the section force-deformation relation. While any suitable section flexibility matrix can be used to calculate the residual section deformation vector, the section tangent flexibility matrices offer the fastest convergence rate.

7. The residual section deformation vectors are integrated along the element using the complimentary principle of virtual work to obtain the residual element nodal displacement vector (⑤), $\tilde{\mathbf{d}}_{e,n}^{R,k=1,j=1} = \int_0^{L_e} \mathbf{N}_{f,e}(x)^T \cdot \boldsymbol{\varepsilon}_{s,e,n}^{R,k=1,j=1}(x)\mathrm{d}x$

8. At this stage the first iteration ($j = 1$) is completed. The final element and section states for $j = 1$ correspond to point $\mathbf{B}$. The residual section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{R,k=1,j=1}(x)$ and the residual element nodal displacement vector $\tilde{\mathbf{d}}_{e,n}^{R,k=1,j=1}$ were determined in the first iteration, but the corresponding element nodal displacement vector have not yet been updated. Instead, they constitute the starting point of the remaining steps within iteration loop j.

9. The presence of residual element nodal displacement vector $\tilde{\mathbf{d}}_{e,n}^{R,k=1,j=1}$ will violate compatibility, since elements sharing a common node would now have different element nodal displacement vector. In order to restore the inter-element compatibility, corrective force vector $\delta\tilde{\mathbf{f}}_{e,n}^{k=1,j=2}$ must be applied at the ends of the element as follows

$$\delta\tilde{\mathbf{f}}_{e,n}^{k=1,j=2} = -\left[\tilde{\mathbf{c}}_{e,n}^{k=1,j=1}\right]^{-1} \cdot \tilde{\mathbf{d}}_{e,n}^{R,k=1,j=1}; \;\; \tilde{\mathbf{c}}_{e,n}^{k=1,j=1} = \int_0^{L_e} \mathbf{N}_{f,e}(x)^T \cdot \mathbf{c}_{s,e,n}^{tan,k=1,j=1}(x) \cdot \mathbf{N}_{f,e}(x) \mathrm{d}$$

10. Thus, in the second iteration ($j = 2$), the element nodal force vector (⑥) is updated as $\tilde{\mathbf{f}}_{e,n}^{k=1,j=2} = \tilde{\mathbf{f}}_{e,n}^{k=1,j=1} + \delta\tilde{\mathbf{f}}_{e,n}^{k=1,j=2}$ and the section force and deformation vectors are also updated to

$$\begin{aligned}
\delta\boldsymbol{\sigma}_{s,e,n}^{k=1,j=2}(x) &= \mathbf{N}_{f,e}(x)\cdot\delta\tilde{\mathbf{f}}_{e,n}^{k=1,j=2}\\
\boldsymbol{\sigma}_{s,e,n}^{k=1,j=2}(x) &= \boldsymbol{\sigma}_{s,e,n}^{k=1,j=1}(x) + \delta\boldsymbol{\sigma}_{s,e,n}^{k=1,j=2}(x)\\
\delta\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=2}(x) &= \boldsymbol{\varepsilon}_{s,e,n}^{R,k=1,j=1}(x) + \mathbf{c}_{s,e,n}^{tan,k=1,j=1}(x)\cdot\delta\boldsymbol{\sigma}_{s,e,n}^{k=1,j=2}(x)\\
\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=2}(x) &= \boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=1}(x) + \delta\boldsymbol{\varepsilon}_{s,e,n}^{k=1,j=2}(x)
\end{aligned}$$

11. The state of the element and sections within the element at the end of the second iteration $j = 2$ corresponds to point $\mathbf{C}$.

- It should be noted that the updated tangent flexibility matrices $\mathbf{c}_{s,e,n}^{tan,k=1,j=2}(x)$ and residual section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{R,k=1,j=2}(x)$ are computed for all sections.
- Residual section deformation vectors are then integrated to obtain the residual element nodal deformation vector $\tilde{\mathbf{d}}_{e,n}^{R,k=1,j=2}$ and the new element tangent flexibility matrix $\tilde{\mathbf{c}}_{e,n}^{k=1,j=2}$ is determined by integration of the section flexibility matrices $\mathbf{c}_{s,e,n}^{tan,k=1,j=2}(x)$. This completes the second iteration within loop j.

- When incremental element nodal displacement vector $\delta\tilde{\mathbf{d}}_{e,n}^{k,j=1} = \delta\tilde{\mathbf{d}}_{e,n}^{k}$ is added to the element nodal displacement vector $\tilde{\mathbf{d}}_{e,n}^{k-1}$ at the end of the previous Newton-Raphson iteration, it is important to make sure that the element nodal displacement vector $\tilde{\mathbf{d}}_{e,n}^{k}$ do not change except in the first iteration $j = 1$ during iteration loop j
- Equilibrium along the element is always strictly satisfied since section force vectors (④) are derived from element nodal force vector by the force interpolation functions.

$$\boldsymbol{\sigma}_{s,e,n}^{k}(x) = \mathbf{N}_{f,e}(x) \cdot \tilde{\mathbf{f}}_{e,n}^{k} \quad \text{and} \quad \delta\boldsymbol{\sigma}_{s,e,n}^{k}(x) = \mathbf{N}_{f,e}(x) \cdot \delta\tilde{\mathbf{f}}_{e,n}^{k}$$

- Compatibility is also satisfied, not only at the element ends, but also along the element.

$$\begin{aligned}
\delta\tilde{\mathbf{f}}_{e,n}^{k,j} &= -\left[\tilde{\mathbf{c}}_{e,n}^{k,j-1}\right]^{-1} \cdot \tilde{\mathbf{d}}_{e,n}^{R,k,j-1} \\
\delta\boldsymbol{\sigma}_{s,e,n}^{k,j}(x) &= \mathbf{N}_{f,e}(x) \cdot \delta\tilde{\mathbf{f}}_{e,n}^{k,j} \\
\delta\boldsymbol{\varepsilon}_{s,e,n}^{k,j}(x) &= \boldsymbol{\varepsilon}_{s,e,n}^{R,k,j-1}(x) + \mathbf{c}_{s,e,n}^{tan,k,j-1}(x) \cdot \delta\boldsymbol{\sigma}_{s,e,n}^{k,j}(x)
\end{aligned}$$

- The second term expresses the relation between section deformation vectors and element nodal displacement vector. However, it should be noted that residual section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{R,k,j-1}(x)$ do not strictly satisfy this compatibility condition. This requirement can only be satisfied by integrating the residual section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{R,k,j-1}(x)$ to obtain $\tilde{\mathbf{d}}_{e,n}^{R,k,j-1}$. Since this is rather inefficient from a computational standpoint, the small compatibility error in the calculation of residual section deformation vectors $\boldsymbol{\varepsilon}_{s,e,n}^{R,k,j-1}(x)$ will be neglected.
- While equilibrium and compatibility are satisfied along the element during each iteration of loop j, the section force-deformation relation and the element force-deformation relation is only satisfied within a specified tolerance when convergence is achieved.

Non Linear Structural Analysis

Modelling

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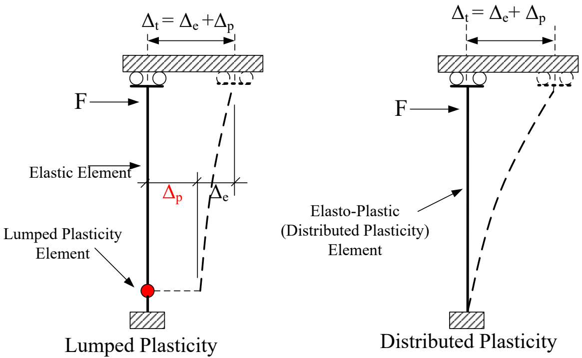
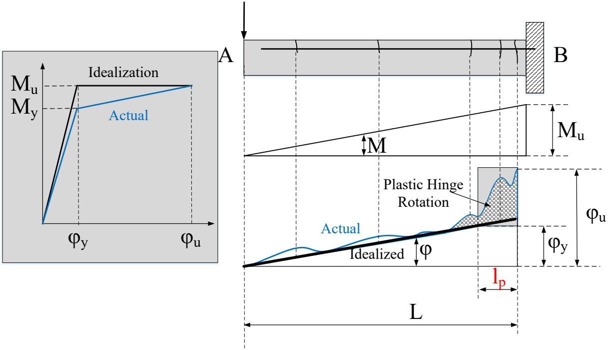
Fall 2020

Table of Contents I

- So far we have covered:
  - Classical plasticity
  - Computational Methods
    - Elements (including flexibility based, zero length, layers, limit states).
    - Constitutive models
  - methods of analysis
    - Nonlinear (static) Push over analysis
    - Nonlinear transient (dynamic) analysis
  - Basis for Performance Based Structural Design
- We now can finally talk about modeling
- Modeling is the science and art of putting together a mathematical model, i.e. mesh, material properties, load.
- Modeling in the context of nonlinear frame analysis is not as simple as “meshing” a 2D or 3D solid for stress analysis.
- There is never a single, unique, correct way of putting a mesh.
- Before we start, we should ask ourseleves:
  - 2D or 3D?
  - Lumped or distributed plasticity?
  - Layered section or Sectional forces?
  - Bond slip?
  - Limit state?
  - Pushover or transient analysis?
  - Damping: Rayleigh and/or hysteretic?
  - How much non-linearity to expect? up to peak?, post-peak?
  - Rigorous single analysis or approximate multiple analysis (Monte-Carlo)?
- It is always a compromise between:
  - Needs, time constraint
  - Our understanding of the problem and of nonlinear analysis,
  - Tools available
  - Quality of results expected.


$\Delta_t = \Delta_e + \Delta_p$
F
Elastic Element
Δ_p
Δ_e
Lumped Plasticity
Element
Lumped Plasticity
$\Delta_t = \Delta_e + \Delta_p$
F
Elasto-Plastic
(Distributed Plasticity)
Element
Distributed Plasticity



- *M* diagram, Linear
- Curvature $\phi = \frac{M}{EI}$ diagram depends on the corresponding moment of inertia, whether gross or cracked.
- At crack location, there is an increase in the curvature.

- Moment curvature has two distinct points corresponding to ϕ_y and ϕ_u.
- Considering a cantilevered beam cracking occur at the support, and inelastic rotation $(\phi_y < \phi < \phi_u)$ will occur at the "plastic hinge" close to the critical section.
- We define an equivalent plastic hinge the length over which the plastic curvature is considered constant.
- Rotation θ is given by

$$\theta_{AB} = \int_A^B \phi dx = \begin{cases} \theta_e = \int_A^B \frac{M}{EI} dx & \text{Elastic rotation} \\ \theta_p = (\phi_u - \phi_y) l_p & \text{Inelastic rotation} \end{cases} \quad (1)$$

- For the cantilevered beam, θ_{AB} is the area of the curvature diagram.

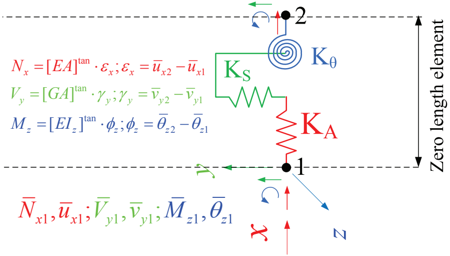
$$\begin{aligned} \theta_{AB} &= \theta_e + \theta_p \\ &= \phi_y \frac{L}{2} + (\phi_u - \phi_y) l_p \end{aligned} \quad (2)$$

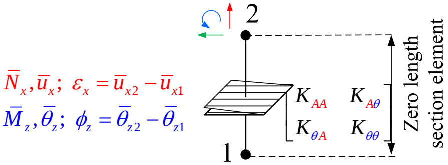
- The displacement between A and B is given by the second Moment curvature theorem

$$\Delta_{AB} = \int_A^B x\phi_{AB}dx \quad \text{where } x \text{ is the distance of } dx \text{ from } A \tag{3}$$

$$= \left(\frac{\phi_y L}{2}\frac{2L}{3}\right) + (\phi_u - \phi_y)l_p\left(L - \frac{l_p}{2}\right) \tag{4}$$

- Not addressed her is the importance of using the gross or cracked elastic moduli

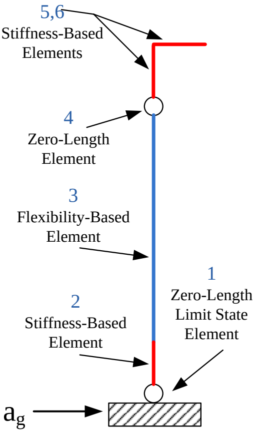




- We have three springs, can not specify only one spring.
- Can constraint appropriate dof
- if a -negative material tag is given to a spring, it is assumed rigid

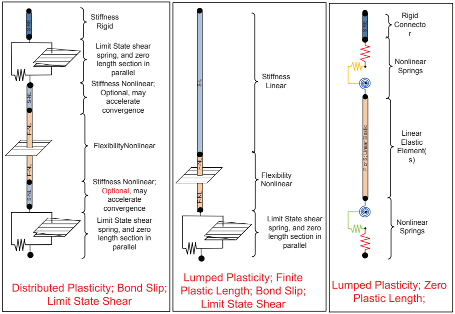
- Used to model bond slip.
- Ideally Fiber section must match the one of adjacent elements
- Careful, consult Ghannoum's model for correct properties (tricky).

- All of these model require proper calibration to determine spring stiffnesses.
- Reference Haselton *et al.*, *Beam-Column Element Model Calibrated for Predicting Flexural Response Leading to Global Collapse of RC Frame Buildings*, PEER Report 2007/03

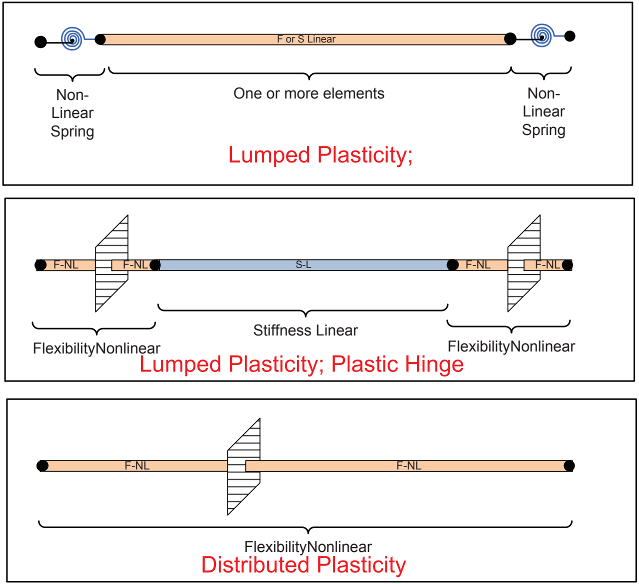


How do we model lateral deformation of column using LS model?

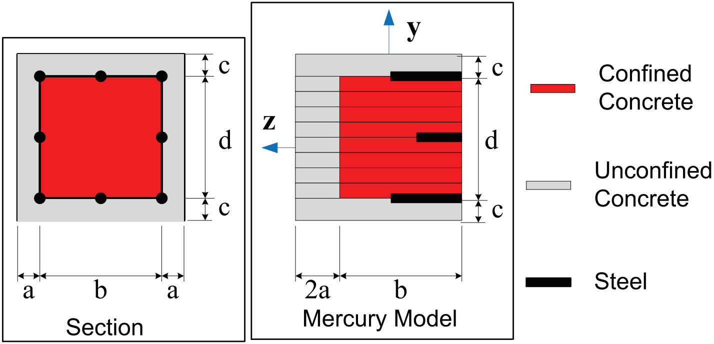
- Flexibility-based element: Elastic response
- Zero-length elements
  - Limit state shear spring with stiff axial and rotational springs
  - Shear spring to model column shear plastic response
- Stiffness-based elements; Rigid element connectors



- Shown are the rigid elements for the connection.
- If you anticipate excessive nonlinear deformation in the distributed plasticity model, insert the nonlinear stiffness based element at each end.
- The flexibility based element is almost invariably used with layered elements (it could also be defined in terms of section forces (N_x and M_z)).
- If bond slip is to be modeled, can place zero-length section element in parallel with the zero-length element.
- In case of mild nonlinearity, one flexibility element should suffice, but at least 3-4 stiffness based elements would be necessary.
- Stiffness based element: 3 IP; Flexibility based element: 5 IP.
- Choice between stiffness or flexibility based element is not obvious.
- Determination of the non-zero length of the plastic hinge, L_p can be "tricky", consult the work of Spacone.



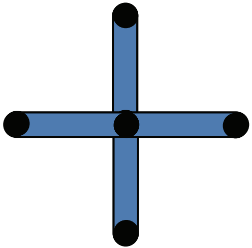
- Usually, we do not include bond slip for beams.
- In the lumped plasticity, the rotational spring can be either nonlinear, or elasto perfectly plastic (EPP)
- Determination of the non-zero length of the plastic hinge, L_p can be “tricky”, consult the work of Spacone.



- In 2D analysis, we refer to layers as opposed to fibers (3D).
- The z position of the layer is irrelevant (and need not be specified).
- Distinguish between
  - Unconfined concrete

- Confined concrete
- Reinforcement

- Can place multiple layers at same y elevations.
- Mercury will provide stress and strain for each layer.



4 Rigid elastic stiffness based elements

- Can not connect zero length elements amongst themselves.
- Adjacent model does not allow independent joint shear and rotational deformation; It is a rigid connection
- For non-rigid connectors, consult work of Lowes

Non Linear Structural Analysis

Constitutive Models

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Table of Contents II

- Constitutive models are at the heart of the finite element (material) non-linear analysis.
- In finite element of solids this would require $\mathbf{D}_t$, however in the context we deal exclusively with one dimensional formulation, hence we will be seeking E^{tan}.
- "classical" (1D) plasticity based models for steel were covered in the first plasticity lecture, all other models are "heuristically" based as they best capture the nonlinear cyclic response we are primarily interested in.
- Two parts:
    - Major focus on fiber/layered elements (thus distributed plasticity), where we seek the non-linear stress-strain relationship[1].
    - Lumped plasticity to be characterized by nonlinear moment-curvature relations.

[1] Zero length elements can also be characterized by non-linear stress-strain relations.

- This is how it ought to be.
- From continuum mechanics, we select for a convex thermodynamic potential a positive definite quadratic function in the components of the strain tensor

$$\Psi = \frac{1}{2\rho}\mathbf{a}:\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}$$

  and by definition $\boldsymbol{\sigma} = \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}} = \mathbf{a}:\boldsymbol{\varepsilon}$ which is Hooke's law.
- Isotropy and linearity require that the potential Ψ be a quadratic invariant of the strain tensor, i.e. a linear combination of the square of the first invariant $\varepsilon_I^2 = [\text{tr }(\varepsilon)]^2$, and the second invariant $\varepsilon_{II}^2 = \frac{1}{2}\text{tr }(\varepsilon^2)$

$$\Psi = \frac{1}{\rho}\left[\frac{1}{2}\left(\lambda\varepsilon_I^2 + 4\mu\varepsilon_{II}\right) - (3\lambda + 2\mu)\alpha\theta\varepsilon_I\right] - \frac{C_\varepsilon}{2T_0}\theta^2$$

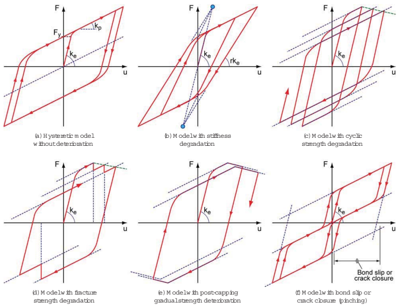
  where λ and μ are Lame's parameters.

- Differentiating

$$
\begin{aligned}
\boldsymbol{\sigma} &= \rho\frac{\partial\Psi}{\partial\varepsilon} = \lambda \text{tr}\,(\varepsilon)\mathbf{I} + 2\mu\varepsilon - (3\lambda+2\mu)\alpha\theta\mathbf{I} \\
\sigma_{ij} &= \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} - (3\lambda+2\mu)\alpha\theta\delta_{ij}
\end{aligned}
$$

- However too complex to develop a potential that can capture complex cyclic load and accompanying deterioration.

- Ultimately, we seek to capture the complex nonlinear response of steel and reinforced concretes structures subjected to reverse cyclic loading (earthquakes).
- There are very few thermodynamically rooted models which can achieve that (except those based on damage mechanics).
- As an alternative, models can be heuristically developed on the basis of laboratory observations resulting in empirical relations. All the models subsequently presented will fall in that category and are thus phenomenological models.

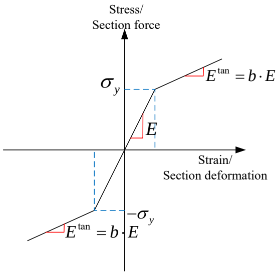


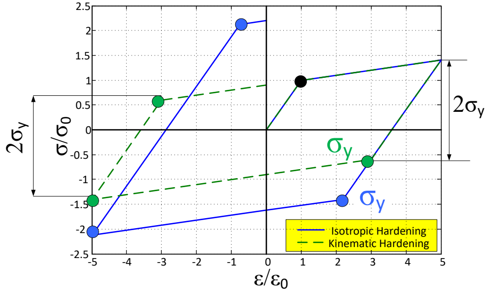
- Cyclic load
- Degradation
  - Strength
    - Tension
    - Tension and Compression
  - Stiffness
- Crack closure

- Steel model for random cyclic excitations present only minor difficulties.
- Most models are heuristic, analytically defined, and the most successful ones are those with variable parameters.
- Within this group, we distinguish three different formulations:
  1. An explicit algebraic equation of the stress: $\sigma = f(\varepsilon)$.
  2. An implicit algebraic equation of the stress: $f(\sigma, \varepsilon) = 0$.
  3. A first order differential equation: $\frac{d\sigma}{d\varepsilon^p} = E^p = f(\sigma)$
- A commonly used explicit model is:

$$\frac{\sigma}{\sigma_0} = b\frac{\varepsilon}{\varepsilon_0} + \frac{(1-b)\frac{\varepsilon}{\varepsilon_0}}{\left[1+\left(\frac{\varepsilon}{\varepsilon_0}\right)^R\right]1/R}$$

where σ_0, ε_0, b, and R are the yield stress and strain, strain hardening parameter, and a coefficient which account for the Baushinger effect and varies depending on the magnitude of the excursion ε_{max} into the inelastic range.





- Recall that

  Isotropic Hardening yield surface expands isotropically and keeps growing. Ultimately most of the response is linear.

  Kinematic hardening yield surface remains constant, translates with respect to the original position.

- Model originally developed by Filippou (1983).

- Rather than determining E^{tan} through H ($E^{tan} = \frac{E.H}{E+H}$) the simplified bilinear model computes it through a strain-hardening coefficient b which is the ratio of the post-yield tangent modulus E^{tan} and the initial elastic modulus E, and considers only isotropic hardening $E^{tan} = b \cdot E$
- To account for the evolution of elastic domain in isotropic harding, a stress shift σ_Δ is determined:
- If the incremental strain $\Delta\varepsilon$ changes a positive value into a negative one:

$$\Delta^N = 1 + a_1 \cdot \left(\frac{\varepsilon^{max} - \varepsilon^{min}}{2 \cdot a_2 \cdot \varepsilon_y} \right)^{0.8} ; \quad \sigma_\Delta = \Delta^N \cdot \sigma_y \cdot (1 - b)$$

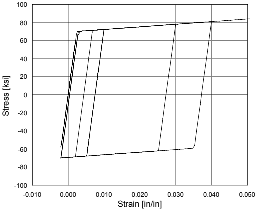
- If the incremental strain $\Delta\varepsilon$ changes a negative value into a positive one:

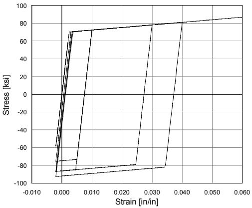
$$\Delta^P = 1 + a_3 \cdot \left(\frac{\varepsilon^{max} - \varepsilon^{min}}{2 \cdot a_4 \cdot \varepsilon_y} \right)^{0.8} ; \quad \sigma_\Delta = \Delta^P \cdot \sigma_y \cdot (1 - b)$$

- a_1 and a_3 are isotropic hardening parameter which reflect an increase of the compression yield envelope through a fraction of the yield strength after a plastic strain $a_2 \cdot \frac{\sigma_y}{E}$, and tension yield envelope as a fraction of the yield strength after a plastic strain of $a_4 \cdot \frac{\sigma_y}{E}$.
- a_2 and a_4 are isotropic hardening parameter with respect to a_1 and a_3, and ε_{max} and ε_{min} are the strain at the maximum and minimum strain reversal point.
- Limiting factor of this model is that a_1, a_2, a_3 and a_4 must be determined through curve fitting of the model with experimental results.
- Default values are $a_1 = 0$, $a_2 = 55$, $a_3 = 0$, and $a_4 = 55$ in OpenSees.

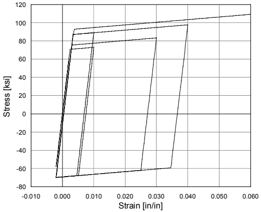
Requires input data in Mercury:

- `mattag`: Material tag
- `modulus`: Young's modulus of a material with `mattag`
- `sigmaY0`: Initial yield stress of a material with `mattag`
- `b`: Strain-hardening ratio between post-yield tangent and Young's modulus of a material with `mattag`
- `a1`: Isotropic hardening coefficient 1 of a material with `mattag` - increase of compression yield envelope as proportion of initial yield stress after a plastic strain of `a2` × (`SigmaY0/modulus`); (optional)
- `a2`: Isotropic hardening coefficient 2 of a material with `mattag`
- `a3`: Isotropic hardening coefficient 3 of a material with `mattag` - increase of tension yield envelope as proportion of initial yield stress after a plastic strain of `a4` × (`SigmaY0/modulus`)
- `a4`: Isotropic hardening coefficient 4 of a material with `mattag` (optional)
- `density`: Density of a material with `mattag`





b)

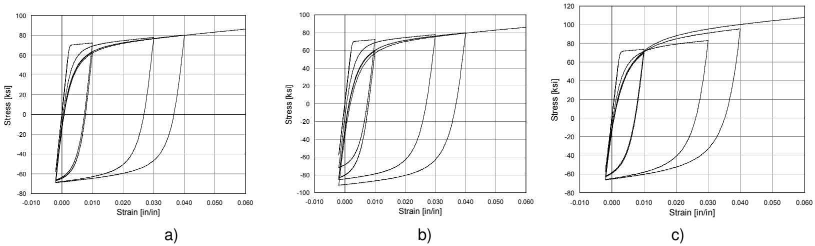


c)

a)

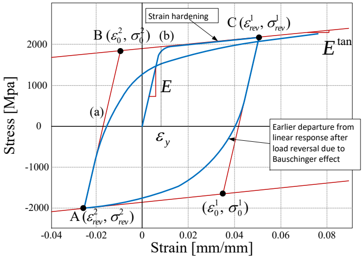
- a) Hysteretic Behavior of Model w/o Isotropic Hardening
- b) Hysteretic Behavior of Model with Isotropic Hardening in Compression
- c) Hysteretic Behavior of Model with Isotropic Hardening in Tension

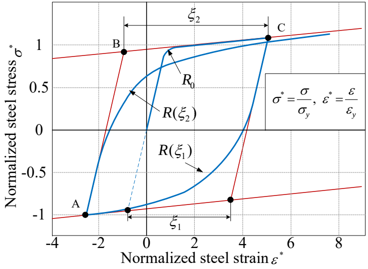
- Model was originally developed by Giuffre, Menegotto and Pinto. It was then modified by Filippou to include strain hardening.
- Main characteristic is the smooth curve which describes a behavior similar to the experimental one.



  - a) Hysteretic Behavior of Model w/o Isotropic Hardening
  - b) Hysteretic Behavior of Model with Isotropic Hardening in Compression
  - c) Hysteretic Behavior of Model with Isotropic Hardening in Tension
- Note that for cyclic load (load/reload)

- Isotropic hardening is not desirable as the yield stress keeps on increasing and at some point we only have an elastic response.
- Kinematic hardening is desirable as it accounts for Bauschinger effect under cyclic load.





- Starts with empirical stress-strain relation

$$\sigma^* = b \cdot \varepsilon^* + \frac{(1-b)\cdot\varepsilon^*}{(1+\varepsilon^{*R})^{1/R}}$$

where,

$$\varepsilon^* = \frac{\varepsilon - \varepsilon_{rev}}{\varepsilon_0 - \varepsilon_{rev}}; \quad \sigma^* = \frac{\sigma - \sigma_{rev}}{\sigma_0 - \sigma_{rev}}$$

- σ_0 and ε_0 are the stress and strain at the point where the two asymptotes of the branch under consideration meet (B); σ_{rev} and ε_{rev} are the stress and strain at the point where the last strain reversal took place (A).
- The tangent modulus E^{tan} is obtained by differentiating

$$E^{tan} = \frac{d\sigma}{d\varepsilon} = \frac{\sigma_0 - \sigma_{rev}}{\varepsilon_0 - \varepsilon_{rev}} \cdot \frac{d\sigma^*}{d\varepsilon^*}; \quad \frac{d\sigma^*}{d\varepsilon^*} = b + \left[\frac{1-b}{(1+\varepsilon^{*R})^{1/R}}\right] \cdot \left[1 - \frac{\varepsilon^{*R}}{1+\varepsilon^{*R}}\right]$$

- There is a curved transition from a straight line asymptote with slope E (a) to another asymptote with slope E^{tan} (b).
- σ_{rev}, ε_{rev} are the stress and strain at the point of strain reversal (point A), which also forms the origin of the asymptote with slope E (a).

- σ_0 and ε_0 are the stress and strain at the point of intersection of the two asymptotes (point B).
- b is the strain hardening ratio between slope E^{tan} and E, and R is a parameter that influences the curvature of the transition curve between the two asymptotes and permits a good representation of the Bauschinger effect.
- σ_0, ε_0, σ_{rev} and ε_{rev} are updated after each strain reversal.
- R depends on the absolute strain difference between the current asymptote intersection point (point B) and the previous maximum or minimum strain reversal point (point C) depending on whether the current strain is increasing or decreasing, respectively.
- There are two reported expression for $R(\xi)$:
  - Menegotto-Pinto original model $R(\xi) = R_0 - \frac{cR_1 \cdot \xi}{cR_2 + \xi}$
  - The one reported in OpenSees: $R(\xi) = R_0 \left(1 - \frac{cR_1 \cdot \xi}{cR_2 + \xi}\right)$

where, R_0 is the value of the parameter R during first loading, and cR_1 and cR_2 are experimentally determined parameters to be defined together with R_0. ξ can be expressed as

$$\xi = \left| \frac{\varepsilon^m - \varepsilon_0}{\varepsilon_y} \right| \tag{1}$$

where, ε^m is the strain at the previous maximum or minimum strain reversal point depending on whether the current strain is increasing or decreasing, respectively. ε_0 is the strain at the current intersection point of the two asymptotes.

- both ε^m and ε_0 lie along the same asymptote and ε_y is the initial yield strain.
- So far, model does not account for isotropic hardening in reverse cyclic load. Filippou proposed a shift of σ_0 and ε_0 in the linearly yield asymptote as follows:
  - If the incremental strain $\Delta\varepsilon$ changes a positive value to a negative value:

$$\begin{aligned}
\Delta^N &= 1 + a_1 \cdot \left(\frac{\varepsilon^{max} - \varepsilon^{min}}{2 \cdot a_2 \cdot \varepsilon_y} \right)^{0.8} \\
\varepsilon_0 &= \frac{-\sigma_y \cdot \Delta^N + E^{tan} \cdot \varepsilon_y \cdot \Delta^N - \sigma_{rev} + E \cdot \varepsilon_{rev}}{E - E^{tan}} \\
\sigma_0 &= -\sigma_y \cdot \Delta^N + E^{tan} \cdot (\varepsilon_o + \varepsilon_y \cdot \Delta^N)
\end{aligned}$$

- If the incremental strain $\Delta\varepsilon$ changes a negative value to a positive value,

$$
\begin{aligned}
\Delta^P &= 1 + a_3 \cdot \left(\frac{\varepsilon^{max} - \varepsilon^{min}}{2 \cdot a_4 \cdot \varepsilon_y} \right)^{0.8} \\
\varepsilon_0 &= \frac{\sigma_y \cdot \Delta^P - E^{tan} \cdot \varepsilon_y \cdot \Delta^P - \sigma_{rev} + E \cdot \varepsilon_{rev}}{E - E^{tan}} \\
\sigma_0 &= \sigma_y \cdot \Delta^P + E^{tan} \cdot (\varepsilon_o - \varepsilon_y \cdot \Delta^P)
\end{aligned}
$$

where, a_1 and a_3 are isotropic hardening parameter which reflect an increase of the compression yield envelope through a fraction of the yield strength after a plastic strain $a_2 \cdot \frac{\sigma_y}{E}$, and tension yield envelope as a fraction of the yield strength after a plastic strain of $a_4 \cdot \frac{\sigma_y}{E}$. a_2 and a_4 are isotropic hardening parameter with respect to a_1 and a_3, and ε_{max} and ε_{min} are the strain at the maximum and minimum strain reversal point.

- The problem is that a_1, a_2, a_3 and a_4 must be determined through curve fitting of the model with experimental results. Default values are $a_1 = 0$, $a_2 = 55$, $a_3 = 0$, and $a_4 = 55$. Note similarity with previous model.

Required Input data in Mercury:

- `mattag`: Material tag
- `modulus`: Young's modulus of a material with `mattag`
- `sigmaY0`: Initial yield stress of a material with `mattag`
- `b`: Strain-hardening ratio between post-yield tangent and Young's modulus of a material with `mattag`
- `R0`: Coefficient 0 of a material with `mattag` to control the transition from elastic to plastic branches - value between 10 and 20 is recommended
- `cR1`: Coefficient 1 of a material with `mattag` to control the transition from elastic to plastic branches - 0.925 is recommended
- `cR2`: Coefficient 1 of a material with `mattag` to control the transition from elastic to plastic branches - 0.15 is recommended
- `a1`: Isotropic hardening coefficient 1 of a material with `mattag` - increase of compression yield envelope as proportion of initial yield stress after a plastic strain of `a2` $\times$ (`SigmaY0/modulus`)
- `a2`: Isotropic hardening coefficient 2 of a material with `mattag`
- `a3`: Isotropic hardening coefficient 3 of a material with `mattag` - increase of tension yield envelope as proportion of initial yield stress after a plastic strain of `a4` $\times$ (`SigmaY0/modulus`)

- `a4`: Isotropic hardening coefficient 4 of a material with `mattag`
- `density`: Density of a material with `mattag`

Giuffre-Menegotto-Pinto Model Modified by Filippou et al. determination

$$\varepsilon_{r,n}$$

$$E_r^{sh} = b_r \cdot E_r, \varepsilon_{y,r} = \frac{\sigma_{y,r}}{E_r}, \text{if } (\sigma_{ini,r} \neq 0), \text{then } \varepsilon_{ini,r} = \frac{\sigma_{ini,r}}{E_r} \text{ and } \varepsilon_{r,n} = \varepsilon_{r,n} + \varepsilon_{ini,r}$$

$$\varepsilon_{rev,r,n} = \varepsilon_{rev,r,n-1}, \text{ where } \varepsilon_{rev,r,n=0} = 0, \ \sigma_{rev,r,n} = \sigma_{rev,r,n-1}, \text{ where } \sigma_{rev,r,n=0} = 0$$

$$\varepsilon_{0,r,n} = \varepsilon_{0,r,n-1}, \text{ where } \varepsilon_{0,r,n=0} = 0, \ \sigma_{0,r,n} = \sigma_{0,r,n-1}, \text{ where } \sigma_{0,r,n=0} = 0$$

$$\varepsilon_{r,n}^{\max} = \varepsilon_{r,n-1}^{\max}, \text{ where } \varepsilon_{r,n=0}^{\max} = \frac{\sigma_{y,r}}{E_r}, \ \varepsilon_{r,n}^{\min} = \varepsilon_{r,n-1}^{\min}, \text{ where } \varepsilon_{r,n=0}^{\min} = -\frac{\sigma_{y,r}}{E_r}$$

$$\varepsilon_{pe,n} = \varepsilon_{pe,n-1}, \text{ where } \varepsilon_{pe,n=0} = 0$$

$$\text{Forceflag}_{r,n} = \text{Forceflag}_{r,n-1}, \text{ where Forceflag}_{r,n} = 0$$

$$\Delta\varepsilon = \varepsilon_{r,n} - \varepsilon_{r,n-1}$$

if $\text{Foceflag}_{r,n} = 0$	else if $\text{Foceflag}_{r,n} = 1$	else if $\text{Foceflag}_{r,n} = -1$

Determination (1) for Giuffre-Menegotto-Pinto Model Modified by Filippou et al.

if $\text{Foceflag}_{r,n} = 0$

if $\Delta\varepsilon = 0$	else if $\Delta\varepsilon < 0$	else if $\Delta\varepsilon > 0$
$\varepsilon_{rev,r,n} = 0$ $\sigma_{rev,r,n} = 0$ $\varepsilon_{0,r,n} = 0$ $\sigma_{0,r,n} = 0$ $\varepsilon_{r,n}^{\max} = \sigma_{y,r} / E_r$ $\varepsilon_{r,n}^{\min} = -\sigma_{y,r} / E_r$ $\text{Forceflag}_{r,n} = 0$ $\sigma_{r,n} = \sigma_{ini,r}$ $E_{r,n}^{\tan} = E_r$	$\varepsilon_{rev,r,n} = 0$ $\sigma_{rev,r,n} = 0$ $\varepsilon_{0,r,n} = \sigma_{y,r} / E_r$ $\sigma_{0,r,n} = \sigma_{y,r}$ $\varepsilon_{r,n}^{\max} = \sigma_{y,r} / E_r$ $\varepsilon_{r,n}^{\min} = -\sigma_{y,r} / E_r$ $\varepsilon_{pe,n} = \varepsilon_{r,n}^{\min}$ $\text{Forceflag}_{r,n} = 1$ $\xi = \left\| \frac{\varepsilon_{pe,n} - \varepsilon_{0,r,n}}{\varepsilon_{y,r}} \right\|$ $R = R_0 - R_0^{nratio} \cdot \frac{cR_1 \cdot \xi}{cR_2 + \xi}$ $\dot{\varepsilon} = \frac{\varepsilon_{r,n} - \varepsilon_{rev,r,n}}{\varepsilon_{0,r,n} - \varepsilon_{rev,r,n}}$ $c_1 = 1 + \|\dot{\varepsilon}\|^R$ $c_2 = c_1^{1/R}$ $\sigma_{r,n} = b_r \cdot \dot{\varepsilon} + (1 - b_r) \cdot \frac{\dot{\varepsilon}}{c_2}$ $\sigma_{r,n} = \sigma_{r,n} \cdot (\sigma_{0,r,n} - \sigma_{rev,r,n}) + \sigma_{rev,r,n}$ $E_{r,n}^{\tan} = b_r + \frac{(1 - b_r)}{c_1 \cdot c_2}$ $E_{r,n}^{\tan} = E_{r,n}^{\tan} \cdot \frac{\sigma_{0,r,n} - \sigma_{rev,r,n}}{\varepsilon_{0,r,n} - \varepsilon_{rev,r,n}}$	$\varepsilon_{rev,r,n} = 0$ $\sigma_{rev,r,n} = 0$ $\varepsilon_{0,r,n} = -\sigma_{y,r} / E_r$ $\sigma_{0,r,n} = -\sigma_{y,r}$ $\varepsilon_{r,n}^{\max} = \sigma_{y,r} / E_r$ $\varepsilon_{r,n}^{\min} = -\sigma_{y,r} / E_r$ $\varepsilon_{pe,n} = \varepsilon_{r,n}^{\max}$ $\text{Forceflag}_{r,n} = -1$ $\xi = \left\| \frac{\varepsilon_{pe,n} - \varepsilon_{0,r,n}}{\varepsilon_{y,r}} \right\|$ $R = R_0 - R_0^{nratio} \cdot \frac{cR_1 \cdot \xi}{cR_2 + \xi}$ $\dot{\varepsilon} = \frac{\varepsilon_{r,n} - \varepsilon_{rev,r,n}}{\varepsilon_{0,r,n} - \varepsilon_{rev,r,n}}$ $c_1 = 1 + \|\dot{\varepsilon}\|^R$ $c_2 = c_1^{1/R}$ $\sigma_{r,n} = b_r \cdot \dot{\varepsilon} + (1 - b_r) \cdot \frac{\dot{\varepsilon}}{c_2}$ $\sigma_{r,n} = \sigma_{r,n} \cdot (\sigma_{0,r,n} - \sigma_{rev,r,n}) + \sigma_{rev,r,n}$ $E_{r,n}^{\tan} = b_r + \frac{(1 - b_r)}{c_1 \cdot c_2}$ $E_{r,n}^{\tan} = E_{r,n}^{\tan} \cdot \frac{\sigma_{0,r,n} - \sigma_{rev,r,n}}{\varepsilon_{0,r,n} - \varepsilon_{rev,r,n}}$

Determination (2) for Giuffre-Menegotto-Pinto Model Modified by Filippou et al.

else if $\text{Foceflag}_{r,n} = 1$

if $\Delta\varepsilon < 0$	else if $\Delta\varepsilon > 0$
$\varepsilon_{rev,r,n},\quad \sigma_{rev,r,n}$ $\Delta^N = 1 + a_1 \cdot \left(\dfrac{\varepsilon^{\max}_{r,n} - \varepsilon^{\min}_{r,n}}{2 \cdot a_2 \cdot \varepsilon_y} \right)^{0.8}$ $\varepsilon_{0,r,n} = \dfrac{-\sigma_{y,r} \cdot \Delta^N + E_r^{sh} \cdot \varepsilon_{y,r} \cdot \Delta^N - \sigma_{rev,r,n} + E_r \cdot \varepsilon_{rev,r,n}}{E_r - E_r^{sh}}$ $\sigma_{0,r,n} = -\sigma_{y,r} \cdot \Delta^N + E_r^{sh} \cdot (\varepsilon_{0,r,n} + \varepsilon_{y,r} \cdot \Delta^N)$ $\varepsilon^{\max}_{r,n} = \text{Max}\left[\varepsilon^{\max}_{r,n}, \varepsilon_{r,n-1}\right]$ $\varepsilon^{\min}_{r,n} = \text{Min}\left[\varepsilon^{\min}_{r,n}, \varepsilon_{r,n-1}\right]$ $\varepsilon_{pe,n} = \varepsilon^{\min}_{r,n}$ $\text{Forceflag}_{r,n} = -1$ $\xi = \left\| \dfrac{\varepsilon_{pe,n} - \varepsilon_{0,r,n}}{\varepsilon_{y,r}} \right\|$ $R = R_0 - R_0^{nratio} \cdot \dfrac{cR_1 \cdot \xi}{cR_2 + \xi}$ $\dot{\varepsilon} = \dfrac{\varepsilon_{r,n} - \varepsilon_{rev,r,n}}{\varepsilon_{0,r,n} - \varepsilon_{rev,r,n}}$ $c_1 = 1 + \|\dot{\varepsilon}\|^R$ $c_2 = c_1^{1/R}$ $\sigma_{r,n} = b_r \cdot \dot{\varepsilon} + (1 - b_r) \cdot \dfrac{\dot{\varepsilon}}{c_2}$ $\sigma_{r,n} = \sigma_{r,n} \cdot (\sigma_{0,r,n} - \sigma_{rev,r,n}) + \sigma_{rev,r,n}$ $E^{\tan}_{r,n} = b_r + \dfrac{(1 - b_r)}{c_1 \cdot c_2}$ $E^{\tan}_{r,n} = E^{\tan}_{r,n} \cdot \dfrac{\sigma_{0,r,n} - \sigma_{rev,r,n}}{\varepsilon_{0,r,n} - \varepsilon_{rev,r,n}}$	$\varepsilon_{rev,r,n},\quad \sigma_{rev,r,n},\quad \varepsilon_{0,r,n},\quad \sigma_{0,r,n}$ $\varepsilon^{\max}_{r,n} = \text{Max}\left[\varepsilon^{\max}_{r,n}, \varepsilon_{r,n-1}\right]$ $\varepsilon^{\min}_{r,n} = \text{Min}\left[\varepsilon^{\min}_{r,n}, \varepsilon_{r,n-1}\right]$ $\varepsilon_{pe,n} = \varepsilon^{\max}_{r,n}$ $\text{Forceflag}_{r,n} = 1$ $\xi = \left\| \dfrac{\varepsilon_{pe,n} - \varepsilon_{0,r,n}}{\varepsilon_{y,r}} \right\|$ $R = R_0 - R_0^{nratio} \cdot \dfrac{cR_1 \cdot \xi}{cR_2 + \xi}$ $\dot{\varepsilon} = \dfrac{\varepsilon_{r,n} - \varepsilon_{rev,r,n}}{\varepsilon_{0,r,n} - \varepsilon_{rev,r,n}}$ $c_1 = 1 + \|\dot{\varepsilon}\|^R$ $c_2 = c_1^{1/R}$ $\sigma_{r,n} = b_r \cdot \dot{\varepsilon} + (1 - b_r) \cdot \dfrac{\dot{\varepsilon}}{c_2}$ $\sigma_{r,n} = \sigma_{r,n} \cdot (\sigma_{0,r,n} - \sigma_{rev,r,n}) + \sigma_{rev,r,n}$ $E^{\tan}_{r,n} = b_r + \dfrac{(1 - b_r)}{c_1 \cdot c_2}$ $E^{\tan}_{r,n} = E^{\tan}_{r,n} \cdot \dfrac{\sigma_{0,r,n} - \sigma_{rev,r,n}}{\varepsilon_{0,r,n} - \varepsilon_{rev,r,n}}$

Determination (3) for Giuffre-Menegotto-Pinto Model Modified by Filippou et al.

else if $\text{Foceflag}_{r,n} = -1$

if $\Delta\varepsilon > 0$

$$\varepsilon_{rev,r,n},\quad \sigma_{rev,r,n}$$

$$\Delta^{P} = 1 + a_3 \cdot \left(\frac{\varepsilon^{\max}_{r,n} - \varepsilon^{\min}_{r,n}}{2 \cdot a_4 \cdot \varepsilon_y} \right)^{0.8}$$

$$\varepsilon_{0,r,n} = \frac{\sigma_{y,r} \cdot \Delta^{P} - E^{sh}_{r} \cdot \varepsilon_{y,r} \cdot \Delta^{P} - \sigma_{rev,r,n} + E_r \cdot \varepsilon_{rev,r,n}}{E_r - E^{sh}_{r}}$$

$$\sigma_{0,r,n} = \sigma_{y,r} \cdot \Delta^{P} + E^{sh}_{r} \cdot (\varepsilon_{0,r,n} - \varepsilon_{y,r} \cdot \Delta^{P})$$

$$\varepsilon^{\max}_{r,n} = \text{Max}\left[\varepsilon^{\max}_{r,n}, \varepsilon_{r,n-1}\right]$$

$$\varepsilon^{\min}_{r,n} = \text{Min}\left[\varepsilon^{\min}_{r,n}, \varepsilon_{r,n-1}\right]$$

$$\varepsilon_{pe,n} = \varepsilon^{\max}_{r,n}$$

$$\text{Forceflag}_{r,n} = 1$$

$$\xi = \left| \frac{\varepsilon_{pe,n} - \varepsilon_{0,r,n}}{\varepsilon_{y,r}} \right|$$

$$R = R_0 - R_0^{nratio} \cdot \frac{cR_1 \cdot \xi}{cR_2 + \xi}$$

$$\dot{\varepsilon} = \frac{\varepsilon_{r,n} - \varepsilon_{rev,r,n}}{\varepsilon_{0,r,n} - \varepsilon_{rev,r,n}}$$

$$c_1 = 1 + |\dot{\varepsilon}|^{R}$$

$$c_2 = c_1^{1/R}$$

$$\sigma_{r,n} = b_r \cdot \dot{\varepsilon} + (1 - b_r) \cdot \frac{\dot{\varepsilon}}{c_2}$$

$$\sigma_{r,n} = \sigma_{r,n} \cdot (\sigma_{0,r,n} - \sigma_{rev,r,n}) + \sigma_{rev,r,n}$$

$$E^{\tan}_{r,n} = b_r + \frac{(1 - b_r)}{c_1 \cdot c_2}$$

$$E^{\tan}_{r,n} = E^{\tan}_{r,n} \cdot \frac{\sigma_{0,r,n} - \sigma_{rev,r,n}}{\varepsilon_{0,r,n} - \varepsilon_{rev,r,n}}$$

else if $\Delta\varepsilon < 0$

$$\varepsilon_{rev,r,n},\quad \sigma_{rev,r,n},\quad \varepsilon_{0,r,n},\quad \sigma_{0,r,n}$$

$$\varepsilon^{\max}_{r,n} = \text{Max}\left[\varepsilon^{\max}_{r,n}, \varepsilon_{r,n-1}\right]$$

$$\varepsilon^{\min}_{r,n} = \text{Min}\left[\varepsilon^{\min}_{r,n}, \varepsilon_{r,n-1}\right]$$

$$\varepsilon_{pe,n} = \varepsilon^{\min}_{r,n}$$

$$\text{Forceflag}_{r,n} = -1$$

$$\xi = \left| \frac{\varepsilon_{pe,n} - \varepsilon_{0,r,n}}{\varepsilon_{y,r}} \right|$$

$$R = R_0 - R_0^{nratio} \cdot \frac{cR_1 \cdot \xi}{cR_2 + \xi}$$

$$\dot{\varepsilon} = \frac{\varepsilon_{r,n} - \varepsilon_{rev,r,n}}{\varepsilon_{0,r,n} - \varepsilon_{rev,r,n}}$$

$$c_1 = 1 + |\dot{\varepsilon}|^{R}$$

$$c_2 = c_1^{1/R}$$

$$\sigma_{r,n} = b_r \cdot \dot{\varepsilon} + (1 - b_r) \cdot \frac{\dot{\varepsilon}}{c_2}$$

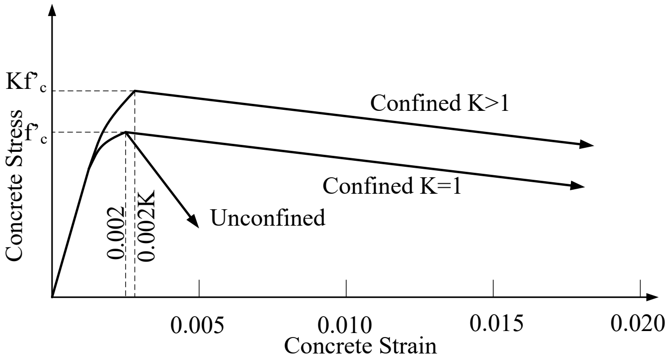
$$\sigma_{r,n} = \sigma_{r,n} \cdot (\sigma_{0,r,n} - \sigma_{rev,r,n}) + \sigma_{rev,r,n}$$

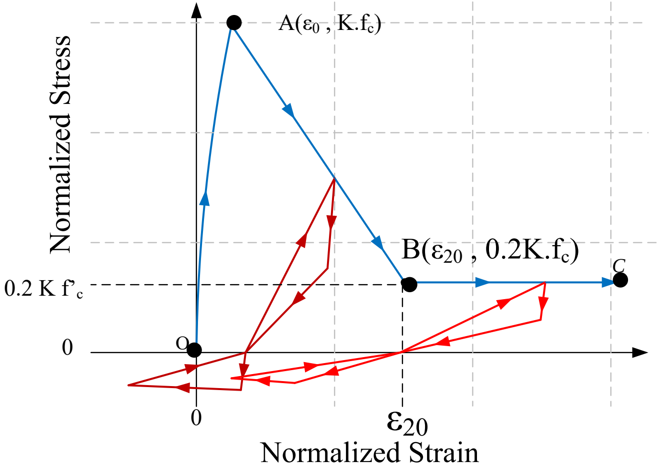
$$E^{\tan}_{r,n} = b_r + \frac{(1 - b_r)}{c_1 \cdot c_2}$$

$$E^{\tan}_{r,n} = E^{\tan}_{r,n} \cdot \frac{\sigma_{0,r,n} - \sigma_{rev,r,n}}{\varepsilon_{0,r,n} - \varepsilon_{rev,r,n}}$$

Determination (4) for Giuffre-Menegotto-Pinto Model Modified by Filippou et al.

- Concrete is much more difficult to model than steel.
- We need to address nonlinearity in compression, tension stiffening, and softening following tensile strength.
- Most popular model: Modified Kent and Park.


Kf'c
f'c
Concrete Stress
0.002
0.002K
Confined K>1
Confined K=1
Unconfined
0.005
0.010
0.015
0.020
Concrete Strain



A(ε0 , K.fc)
Normalized Stress
B(ε20 , 0.2K.fc)
C
0.2 K f'c
O
0
0
ε20
Normalized Strain

- A "good" concrete model must account for
  - Effect of concrete confinement (by shear reinforcement) on the monotonic envelope curve in compression
  - Successive degradation of stiffness of both the unloading and reloading curves, for increasing values of compressive strain
  - Effect of tension stiffening: ability of concrete between cracks to resist tensile stress and contribute to the flexural stiffness of the member. As the magnitude of load increases, additional cracks form at closer intervals, hence reducing the tensile stress that can be developed in the concrete. Therefore tension stiffening is gradually reduced as load is increased in the post-cracking stage.
- Hysteretic response under cyclic loading in compression
- In compression stress-strain relation is empirically defined by three regions (compression positive)

$$OA : \varepsilon_c \leq \varepsilon_0 \Rightarrow \sigma_c = K \cdot f_c \cdot \left[2 \cdot \frac{\varepsilon_c}{\varepsilon_0} - \left(\frac{\varepsilon_c}{\varepsilon_0}\right)^2\right] \text{ and } E^{tan} = \frac{2 \cdot K \cdot f_c}{\varepsilon_0} \cdot \left(1 - \frac{\varepsilon_c}{\varepsilon_0}\right)$$

From this equation, we can determine the maximum compressive strength of confined concrete (by simply setting $\varepsilon_c = \varepsilon_0$) $f_{c,confined} = Kf_c$

$$AB : \varepsilon_0 < \varepsilon_c \leq \varepsilon_{20} \Rightarrow \sigma_c = K \cdot f_c \cdot [1 - Z(\varepsilon_c - \varepsilon_0)] \text{ and } E^{tan} = -Z \cdot K \cdot f_c$$
$$BC : \varepsilon_c > \varepsilon_{20} \Rightarrow \sigma_c = 0.2 \cdot K \cdot f_c \text{ and } E^{tan} = 0$$

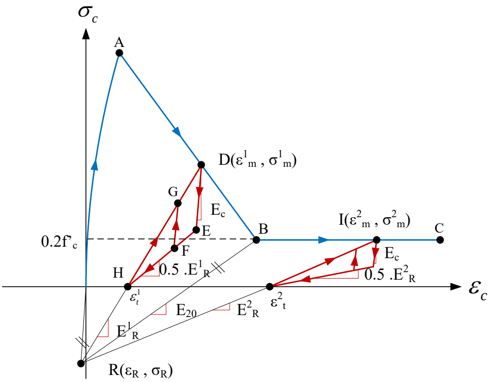
where

$\varepsilon_0^{unconfined}$	0.003
$\varepsilon_0^{confined}$	$0.002 \cdot K$
K	$1 + \frac{\rho_s \cdot f_{ys}}{f_c}$
Z	$\frac{0.5}{\frac{3+0.29\cdot f_c}{145\cdot f_c - 1,000} + 0.75\cdot\rho_s\cdot\sqrt{\frac{h}{s_h}} - 0.002\cdot K}$
$\varepsilon_0^{unconfined}$	Concrete strain corresponding maximum stress usually set to 0.003
ε_{20}	Concrete strain at 20 percent of maximum stress
K	factor which accounts for the strength increase due to confinement
Z	Strain softening slope
f_c	Concrete compressive cylinder strength in MPa (1 MPa = 145 psi)
f_{ys}	Yield strength of stirrups in MPa
ρ_s	Ratio of the volume of hoop reinforcement to the volume of concrete core measured to outside of stirrups
h	Width of concrete core measured to outside of stirrups
s_h	Center to center spacing of stirrups or hoop sets

- The cyclic unloading and reloading behavior is represented by a set of straight lines. Hysteretic behavior occurs under, both, tensile and compressive stress.

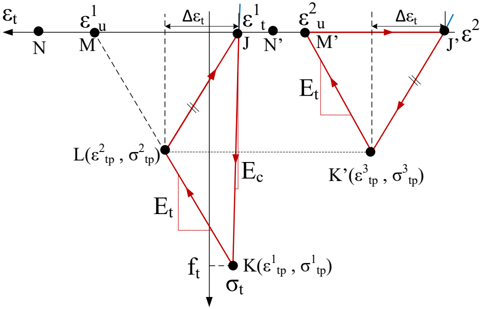
	ε_0	f_c	f_t	E_0	E_t	$f_{c,20}$	ε_{20}
Unconfined	0.003	Test	$7.5\sqrt{f_c^{unc}}$	$2\frac{f_c^{unc}}{\varepsilon_0^{unc}}$	$\frac{E_0^{unc}}{5}$	$f_c^{unc}/3$	$3\varepsilon_0^{unc}$
Confined	$0.002K$	Kf_c^{unc}	$7.5\sqrt{f_c^{unc}}$	$2\frac{f_c^{conf}}{\varepsilon_0^{conf}}$	$\frac{E_0^{conf}}{5}$	$0.2f_c^{conf}$	$\varepsilon_0^{conf}+\frac{Kf_c^{unc}-f_{c,20}^{confined}}{ZKf_c^{unc}}$

f_c in psi for f_t; An alternative to the Kent and Park residual stress/strain is to use $f_{c,res}^{conf}=0.9f_c$ and ε_{res}^{conf} is assigned a "large" value to ensure gradual descent. This combination ensures stable analysis (Ghannoum, 2011).



- On the compressive side of the model, there is a successive degradation of stiffness of both the unloading and reloading lines for increasing values of maximum strain.

- The degradation of stiffness is such that the projections of all reloading lines intersect at a common point R
- R is determined by the intersection of the tangent to the monotonic envelope curve at the origin and the projection of the unloading line from point B that corresponds to concrete strength of $0.2 \cdot f_c$



- The tensile behavior of the model takes into account tension stiffening and the degradation of the unloading and reloading stiffness for increasing values of maximum tensile strain after initial cracking. The maximum tensile strength of the concrete (modulus of rupture) is assumed equal to be $f_t = 0.6228\sqrt{f_c}$ where f_t and f_c are expressed in MPa.

- Tensile stress-strain relation is defined by three points with coordinates $(\varepsilon_t, 0)$, $(\varepsilon_{tp}, \sigma_{tp})$ and $(\varepsilon_u, 0)$, as represented by points J, K and M. ε_t is the strain at the point where the unloading line from the compressive stress region crosses the strain axis. ε_t changes with maximum compressive strain. ε_{tp} and σ_{tp} are the strain and stress at the peak of the tensile stress-strain relation.
- Given these three control points, the tensile stress-strain relation and tangent moduli are defined by the following equations (tension is positive),

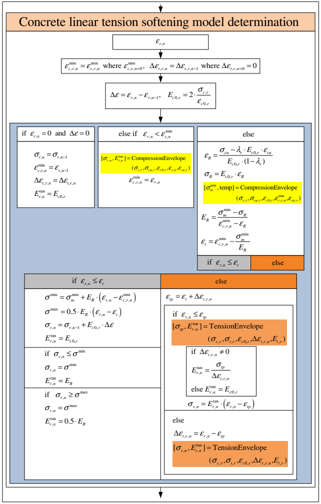
$$\begin{array}{rl} JK & : \varepsilon_t < \varepsilon_c \leq \varepsilon_{tp},\ \sigma_c = E^{tan} \cdot (\varepsilon_c - \varepsilon_t), \quad E^{tan} = \frac{\sigma_{tp}}{\varepsilon_{tp} - \varepsilon_t} \\ KM & : \varepsilon_{tp} < \varepsilon_c \leq \varepsilon_u,\ \sigma_c = \sigma_{tp} + E^{tan} \cdot (\varepsilon_c - \varepsilon_{tp}), \quad E^{tan} = -E_t \\ MN & : \varepsilon_c > \varepsilon_u,\ \sigma_c = 0, \quad E^{tan} = 0 \end{array}$$

- Model can be better understood by following the example load paths.
- As the model unloads from compression, it crosses the strain axis at the point J.
- It then loads in tension until initial cracking occurs at point K.
- Beyond point K softening commences until the strain reversal point L.
- The unloading path follows a straight line from point L to point J where the model reloads in compression.

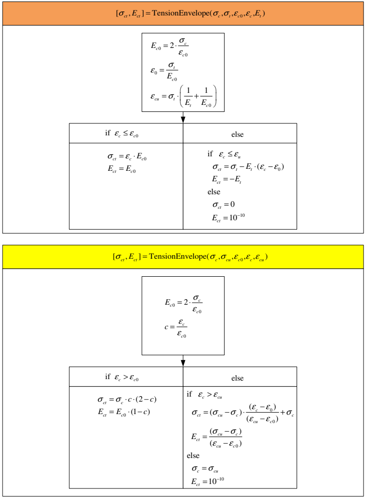
- The second time the model goes into tension is at point J'. The reloading path J'K' is exactly the duplication of the previous unloading path LJ that has been shifted a distance JJ' along the strain axis.
- At point K' the model rejoins the softening branch which continues until the tensile stress is reduced to zero at point M'. The stress remains zero through the strain reversal point N' until the model reloads in compression at point J'. Henceforth, the tensile stress capacity of the model is reduced to zero.
- This concrete model is relatively economical in terms of the amount of memory required of the past stress-strain history. The parameters that are used as memory can be listed as follows:
  - the stress and strain at the point corresponding to the last model state
  - the strain at the last unloading point on the compressive monotonic envelope, ε_m
  - The differential $\Delta\varepsilon_t$ between maximum previous tensile strain and ε_t

Required Input data in Mercury:

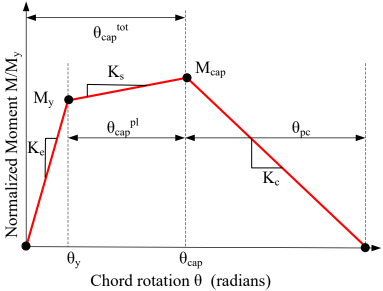
- `mattag`: Material tag
- σ_c: Compressive yield stress of a material with `mattag` - -ve
- ε_c: Compressive yield strain of a material with `mattag` - -ve
- σ_{cu}: Compressive crushing stress of a material with `mattag` - -ve
- ε_{cu}: Compressive crushing strain of a material with `mattag` - -ve
- λ: Ratio between unloading slope at ε_c and slope Young's modulus of a material with `mattag`
- σ_t: Tensile yield stress of a material with `mattag`
- `modulus`: Tension softening stiffness(absolute value) - slope of the linear tension softening branch of a material with `mattag`
- `density`: Density of a material with `mattag`



Determination (1) for modified Kent and Park model

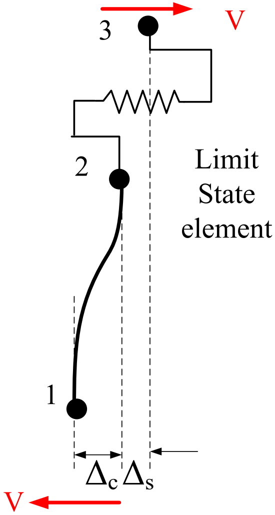


Determination (2) for modified Kent and Park model

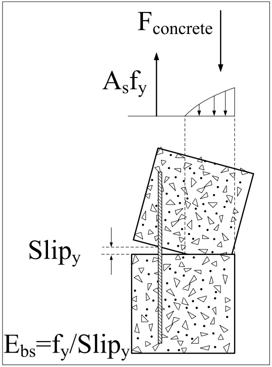


M_y	"Yield" Moment
θ_y	Chord rotation at "Yield"
θ_{cap}	Chord rotation (monotonic) at onset of strength loss (capping)
K_S	Hardening
K_c	Post-capping stiffness

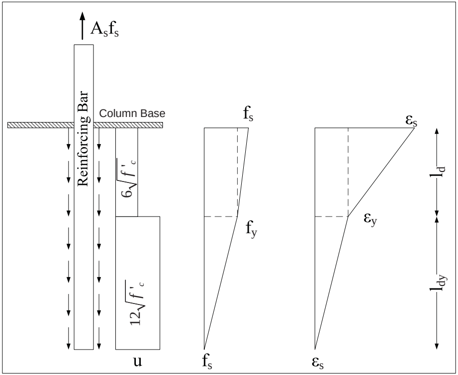
- Parameters obtained through calibration with experimental data.
- Caution with unload (not addressed here)



- Zero-length shear spring in series with beam-column constitutive model
- Upon reaching failure surface, shear spring stiffness degraded to user defined value (K_{deg})
- Member total lateral response (Δ) is sum of shear spring displacement (Δ_s) and beam-column displacement (Δ_f)
- Results in
  - Increased deformation/drift
  - Shear strength loss
  - Flexural yielding
  - Loss of axial load carrying capacity leading to collapse



- Slip due to longitudinal reinforcing bar near the column and from the anchorage can be easily determined if we assume a uniform bond stress u_b along the bars within the development length inside the footing or the beam-column joint.
- From equilibrium $u_b(\pi d_b)l_d = \frac{\pi d_b^2}{4} f_s$ where d_b is the bar diameter, l_d is the development length over which the slip occurs, solving for l_d, $l_d = \frac{d_b f_s}{4u_b}$
- Assuming that the maximum strain occurs at the end of the column, and a linear variation of strain along the development length, the integral of the strain curve will give the total bar slip at the footing-column interface or beam-column interface is the slip given by $S = \frac{\varepsilon_s l_d}{2} = \frac{f_s l_d}{2E_s}$
- Substituting $S = \frac{\varepsilon_s d_b f_s}{8u_b}$
- Assuming that the cross section rotates about its neutral axis when slip occurs ($\phi_y = \varepsilon_y/(d-c)$), the displacement related to the bar slip at a point at a distance L from the column base will be $\Delta_{slip} = \frac{\phi_y d_b f_y L}{8u_b}$



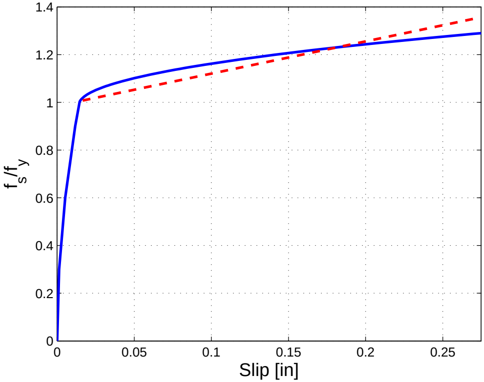
- A simplified bond model for bond stress in terms of the actual steel stress assumes a constant bond stress of $u_e = 12\sqrt{f'_c}$ prior to steel yielding, and another constant bond stress of $u_p = 6\sqrt{f'_c}$ past steel yielding
- Based on this assumption, the total bar slip S at the edge of the anchorage is obtained by integrating the steel strains over the embedded length.
- This model was used to obtain a monotonic relation for bar slip versus bar stress at the column base. Assuming sufficient anchorage:

$$S_1 = \frac{\varepsilon_s f_s}{8u_e} d_b; \;\; \varepsilon_s \leq \varepsilon_y; \;\; S_2 = \frac{\varepsilon_y f_y}{8u_e} d_b + \frac{(\varepsilon_s + \varepsilon_y)(f_s - f_y)}{8u_p} d_b; \;\; \varepsilon_s > \varepsilon_y$$

where d_b is the bar diameter, u_e is the elastic bond stress $= 12\sqrt{f'_c}$ (psi), and u_p is the plastic bond stress $= 6\sqrt{f'_c}$.

```
clear
'+++++++'
%% Input parameters
d_b=3/8; % bar diameter in inches
f_c=4000; % compressive strength (psi)
E_s=27300000; % psi
b=0.01; %
fy=64000;% original yield stress in psi
f_y=1.25*fy; % yield stress increaed by 25% for rate effect
%f_u=100000; % ksi
%
epsilon_y=f_y/E_s;
%% Bond
u_e=12*sqrt(f_c);
u_p=6*sqrt(f_c);
%
k=0;
epsilon_final=30*epsilon_y;
delta_epsilon=epsilon_final/100;
for epsilon_s=0:delta_epsilon:epsilon_final
    k=k+1;
    if epsilon_s<=epsilon_y
        f_s=epsilon_s*E_s;
        slip_y=epsilon_s*f_s*d_b/(8*u_e);
        slip(k)=slip_y;
    else
        f_s=epsilon_y*E_s+b*(epsilon_s epsilon_y)*E_s;
        slip_u=epsilon_y*f_y*d_b/(8*u_e)+(epsilon_s+epsilon_y)*(f_s f_y)*d_b/(8*u_p);
        slip(k)=slip_u;
    end
```

```
    normalized_stress(k)=f_s/f_y;
end
plot(slip,normalized_stress,'linewidth',2);grid;xlim([0,max(slip)]);
xlabel('Slip [in]','fontsize',14);ylabel('f_s/f_y','fontsize',14);
print deps 'bond slip curve.pdf'
```



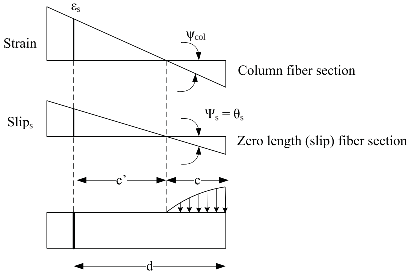
This section is an adaptation from Ghannoum's model

- Zero length section element should be used only when fiber elements are used if we want to capture the bond slip between concrete and rebar.
- We have a nonlinear post-peak response of bond stress vs bond slip, and we need to linearize it, and then solve for u_p (which will be different than the previously given value of $6\sqrt{f_c'}$ suitable for the nonlinear hardening segment.
- We seek to have the linearized segment intersect the nonlinear one at $1.25 f_y$, hence $\varepsilon_u = \varepsilon_y + 0.25\frac{f_y}{E_s/h} = \frac{f_y}{E_s} + 0.25h\frac{f_y}{E_s} = 0.26h\varepsilon_y$ where h is the hardening parameter set to 100. Substituting with $S_2 = \frac{\varepsilon_y f_y}{8u_e}d_b + \frac{(\varepsilon_s+\varepsilon_y)(f_s-f_y)}{8u_p}d_b$; we obtain$S_u = S_y + \frac{6.75\varepsilon_y f_y}{8u_p}d_b$
- We can reasonably assume that $S_u = \frac{\varepsilon_u}{\varepsilon_y}S_y = 26S_y$, upon substitution, we get:

$$u_p = 3.24\sqrt{f_c'}$$

u_p may be used in so-called limit state elements to assess bond slip induced failure.

- Irrespective of which steel model is used in the beam-column, it is recommended to use the bilinear one for this element. Using a bilinear model, with $h = 100$ will be equivalent to having a bar slip curve such that the second segment intersect the exact one at $f_s = 1.25 f_y$ with $u_p = 3.24\sqrt{f'_c}$.
- In the steel bilinear model, Young's modulus should be adjusted to reflect bond slip, by replacing E_s by E_{bs}; $E_{bs} = \frac{f_y}{S_y}$
- It should be noted that inherent in this assumption is a unit length of the zero length element.
- Finally, to maintain the same material stiffness ratio between bar-slip steel in the zero length section element, and the one in the frame element (longitudinal steel), we multiply the bar slip concrete material strains by E_s/E_{bs}.
- The concrete properties for the zero length section element are such that the location of the neutral axis in the beam-column element and the zero length fiber section is the same.



Thus

$$\left.\begin{array}{rcl}\theta_s & = & \frac{S_s}{c'} \\ \Psi_{col} & = & \frac{\varepsilon_s}{c'}\end{array}\right\} \theta_s = \Psi_{col}\frac{S_s}{\varepsilon_s} \tag{2}$$

- Hence all fiber strains (corresponding to steel and concrete) in the zero length section must be scaled by $\frac{S_s}{\varepsilon_s}$
- This can be easily achieved in altering the material input data such that
  1. All stress values remain unchanged
  2. Strains are scaled by $\frac{S_y}{\varepsilon_y}$

Non Linear Structural Analysis

Engineering Seismic Risk Analysis

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Fall 2014

Table of Contents I

Introduction

- By now, we have a good basic understandings of the tools to undertake a nonlinear analysis.
- We still have to review fundamental issues associated with time-history analysis.
- Ultimately, those tools will be used to undertake a modern Performance Based Earthquake Engineering investigation.
- The methodology of PBEE hinges on some basic definitions which must be understood.
- This lecture will present those ingredients necessary (but not yet how to combine them into a meal)

Performance Levels and Acceptance Criteria I

- PBE seeks first to identify discrete performance levels for the major structural components which significantly affect the building function and safety.
- ASCE 41 (ASCE 2007) (and other codes) generally provide guidance three performance levels
  - Immediate Occupancy where an essentially elastic behavior is sought by limiting structural damage (e.g., yielding of steel, significant cracking of concrete, and nonstructural damage.)
  - Life Safety Limit damage of structural and nonstructural components so as to minimize the risk of injury or casualties and to keep essential circulation routes accessible.
  - Collapse Prevention Ensure a small risk of partial or complete building collapse by limiting structural deformations and forces to the onset of significant strength and stiffness degradation.
- The engineer decides which performance levels

Performance Levels and Acceptance Criteria II

- Performance Based Engineering 1 Most recent code, FEMA 750-p developed by the Building Seismic Safety Council for FEMA. It builds on previous pre-Standards.

New Design	FEMA 310 (ASCE 1998)	ASCE/SEI 31 (2003)
Existing Buildings	FEMA 356 (ASCE 2000)	ASCE/SEI 41 (2006)

Performance Levels and Acceptance Criteria III

NEHRP Recommended Seismic Provisions

for New Buildings and Other Structures

FEMA P-750 / 2009 Edition

FEMA

nehrp

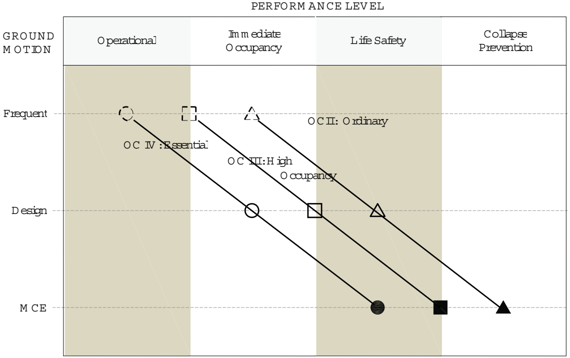


Figure C11.5-1 Expected performance as related to occupancy category (OC) and level of ground motion.

Capacity and Demand

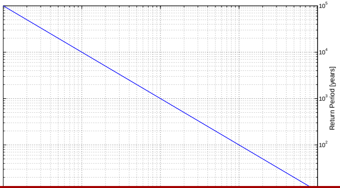
- We will need to identify specific engineering demand parameters (EDP) and appropriate acceptance criteria to quantitatively evaluate the performance levels.
- The demand parameters typically include peak (shear) forces and deformations, inter-story drifts, and floor accelerations in structural and nonstructural components.
- Performance is checked by comparing computed demands with acceptance criteria (capacity) for the desired performance level.
- Depending on the structural configuration, the results of nonlinear analyses can be sensitive to assumed input parameters and the types of models used.
- One must have clear expectations about those portions of the structure that are expected to undergo inelastic deformations and then use the analyses to
    1. Confirm the locations of inelastic deformations
    2. Characterize
        - Deformation demands of yielding elements
        - Force demands in non-yielding elements.
- Capacity design concepts can provide reliable performance.

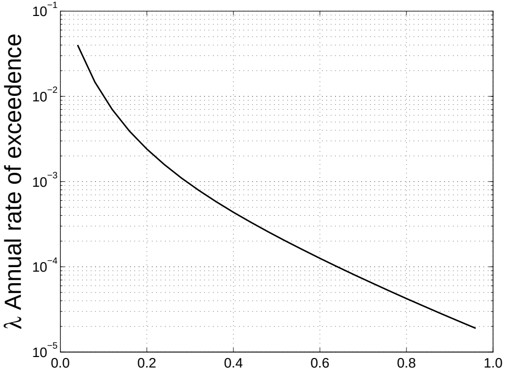
Capacity Design

- Capacity Design is indeed the approach where the engineer decides *a priori* which elements will yield (and thus need to be ductile) and those which will not yield (and will need to be stiff and with sufficient strength).
- Advantages
  - Safeguard against brittle failure of elements which can not be designed as ductile.
  - Limiting the location of the structure where expensive ductile detailing is required (they act as fuses).
  - Reliable energy dissipation by enforcing deformation modes where inelastic deformations are routed to ductile elements.
- Very similar to the structural design of a car.
- Example: strong column/weak beam.

Seismic Hazard Analysis

- In the context of PBEE, one must first conduct a seismic hazard analysis (SHA) which includes location identification (with respect to a fault), geotechnical conditions (shear wave velocity), magnitude of previously recorded earthquakes, size of the rupture area, type of fault, crustal rock damping characteristics, rock properties.
- From the corresponding analysis one can determine annual rate of exceedance λ vs intensity measure (IM) a measure of the ground motion characteristic, typically the (peak or spectral) ground acceleration.





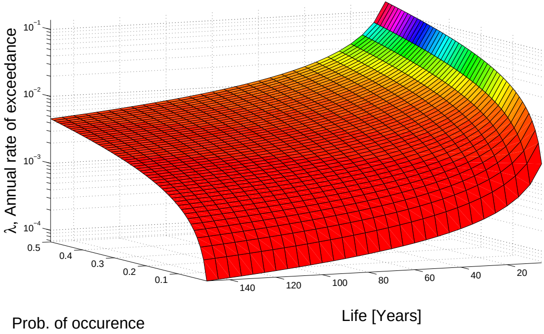
Engineering Seismic Risk Analysis I

- The annual rate of exceedance of the ground motion amplitude, λ, (inverse of return period T_R) for Design Base Level (DBL) and Maximum Design Level (MDL) are determined from a Poisson probability model

  $$\lambda = -\frac{\text{Ln}\left(1 - P_E\right)}{t}$$

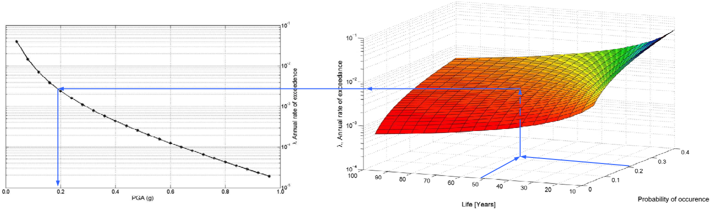
  where P_E is the probability of occurrence of at least one event (i.e. an earthquake) during the life time t.
- t is usually taken as 50 years for buildings, and 100 years for dams.
- P_E for ground motion is usually assumed to be in the ranges [20%   64%] for DBL and [10%   20%] for MDL.
- Assuming a lifetime of 100 years, the corresponding $T_r = 1/\lambda$ is determined for 450 and 1,000 years for DBL and MDL, respectively from.

Engineering Seismic Risk Analysis II



PSHA=SHA+ESRA I

- Probability Seismic Hazard Analysis or PSHA=SHA+ESRA.
- Engineering Seismic Risk Analysis yielded annual rate of exceedance λ in terms of probability of occurrence of at least one event and life time t.
- Seismic hazard analysis yielded annual rate of exceedance λ vs intensity measure.
- Select λ from the first curve, and PGA from the second.



- with the PGA known, one selects (or generate) a set of n ground motion acceleration time histories to perform multiple analyses.

PSHA=SHA+ESRA II

- From the corresponding analysis one plots

  Intensity Measure (IM) a measure of the ground motion characteristic, typically the (peak or spectral) ground acceleration.

  Engineering demand parameter (EDP) which corresponds to any outcome of the analysis of relevance to the safety assessment, such as base shear, drift.

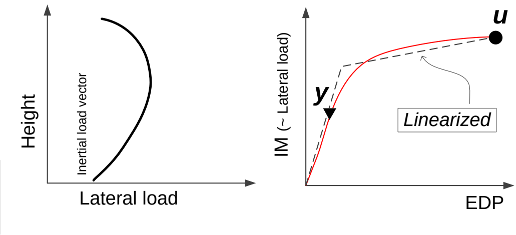
- We repeat this process *m* times for different intensity levels.
- There are four types of analysis that can be performed.

Method		S/D Analysis	*m*	*n*
Push Over Analysis	POA	Static	na	na
Multi Strip Analysis	MSA	Dynamic	3	*n*
Incremental Dynamic Analysis	IDA	Dynamic	Variable	*n*
Endurance Time Analysis	ETA	Dynamic	1	*n*

  where *m* be the number of ground motion intensity levels (or strips), and *n* the number of ground motions for a given *m*.

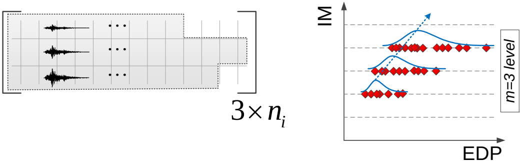
- In all cases we plot IM vs EDP (and not the other way around!)

PushOver Analysis



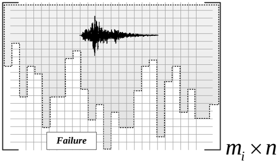
- Applies incrementally load or displacement
- Extensively used in building to capture failure mode in lieu of the more expensive transient nonlinear analysis.
- Assumed to be capable of mobilizing principal nonlinear modes of structural behavior up to collapse.

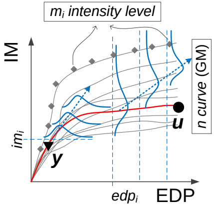
Multiple-Strip Analysis



- Hinges on a deterministic number of ground motion intensity levels m (or strips)
- Typically $m = 3$ corresponding to the exceedance probabilities of 10% in 50-year, 5% in 50-year, and 2% in 50-year.
- To each strip correspond n ground motions.
- Two possibilities:
    - Selection of n different ground motions scaled at m different levels.
    - Selection of n_i ground motions for each of the intensity levels with no scaling.
- Following the analysis, and for each m the usual IM versus EDP results are first plotted.
- Then for each IM histograms are generated and the most suitable probability distribution function (normal or log-normal) is selected.

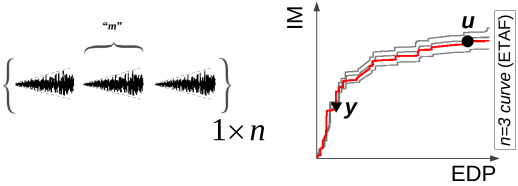
Incremental Dynamic Analysis





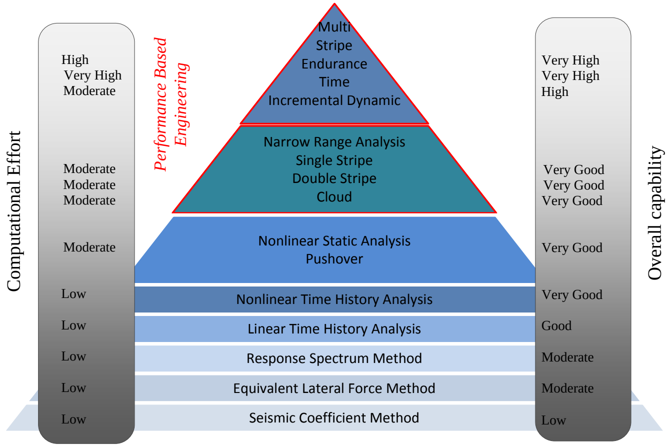
- Considers n ground motions which will all be incrementally scaled m times until failure.
- *a priori* m is unknown and each ground motion n will result in a corresponding failure at a different intensity level m_i .
- Following the analysis, the IDA curve connects the resulting m demand parameters for each of the n ground motions.
- Each one of those curve will be asymptotic to the corresponding failure.
- Capture of the overall response by a single measurable quantity at a given EDP (EDP = edp_i) can be determined through the corresponding probability distribution function.
- Similarly probability distribution function for a given IM (IM = im_i) can also be determined.
- Those curves can be used for the determination of the fragility plots, and probability of failure.

Endurance Time Analysis



- The preceding two methods started with actual recorded ground motion and required up to $m \times n$ analysis, computationally expensive and may force the analysis to make greatly simplified assumption in their model. Such assumptions may lead to erroneous conclusions.
- ETA method starts with a synthetic ground motion and modify it to be characterized with an increasing amplitude.
- Substitute to the *m* intensity levels previously determined and *n* endurance time acceleration function (ETAF) are used.
- Outcome of the analysis, is the average of the *n* analyses in terms of IM versus EDP. The resulting curve is analogous to the one of the POA or 50% fractile of IDA.

Summary


Computational Effort
High
Very High
Moderate
Moderate
Moderate
Moderate
Moderate
Low
Low
Low
Low
Low
Performance Based Engineering
Multi
Stripe
Endurance
Time
Incremental Dynamic
Narrow Range Analysis
Single Stripe
Double Stripe
Cloud
Nonlinear Static Analysis
Pushover
Nonlinear Time History Analysis
Linear Time History Analysis
Response Spectrum Method
Equivalent Lateral Force Method
Seismic Coefficient Method
Very High
Very High
High
Very Good
Very Good
Very Good
Very Good
Very Good
Good
Moderate
Moderate
Low
Overall capability

Non Linear Structural Analysis

Nonlinear Transient Analysis

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Spring 2019

Table of Contents I

- When the frequency of the applied load (excitation) of a structure is less than about a third of its lowest natural frequency of vibration, then we can neglect inertia effects and treat the problem as a quasi-static one, otherwise a dynamic analysis must be performed.
- For a very flexible structure, even a *slowly applied load* may necessitate a dynamic analysis.
- If the structure is subjected to an impact load, than one must be primarily concerned with (stress) wave propagation. In such a problem, we often have high frequencies and the duration of the dynamic analysis is about the time it takes for the wave to travel across the structure.
- If inertia forces are present, then we are confronted with a dynamic problem and can analyse it through any one of the following solution procedures:
  1. Response Spectrum (only linear elastic systems)
  2. Time history analysis through modal analysis (again linear elastic), or direct time integration.

- Prof. Wilson is reported to have said:

  Ray Clough and I regret we created the approximate response spectrum method for seismic analysis of structures in 1962.... At that time many members of the profession were using the sum of the absolute values of the modal values to estimate the maximum member forces. Ray suggested we use the SRSS method to combine the modal values. However, I am the one who put the approximate method in many dynamic analysis programs which allowed engineers to produce meaningless positive numbers of little or no value... After working with the RSM for over 50 years, I recommend it not be used for seismic analysis.

- Methods of structural dynamics are essentially independent of finite element analysis as these methods presume that we already have the stiffness, mass, and damping matrices. Those matrices may be obtained from a single degree of freedom system, from an idealization/simplification of a frame structure, or from a very complex finite element mesh. The time history analysis procedure remains the same.

- In a general three-dimensional continuum, the equations of motion of an elementary volume Ω without damping is $\mathbf{L}^T\boldsymbol{\sigma} + \mathbf{b} = \mathbf{m}\ddot{\mathbf{u}}$ where $\mathbf{m}$ is the mass density matrix equal to $\rho\mathbf{I}$, and $\mathbf{b}$ is the vector of body forces. The Differential operator $\mathbf{L}$ is $\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix}$
- For linear elastic material $\boldsymbol{\sigma} = \mathbf{D}_e\boldsymbol{\varepsilon}$ and for incremental nonlinear analysis, the constitutive equations can be written as $\dot{\boldsymbol{\sigma}} = \mathbf{D}_i\dot{\boldsymbol{\varepsilon}}$ where $\mathbf{D}_i$ is the tangent stiffness matrix.
- $\mathbf{L}^T\boldsymbol{\sigma} + \mathbf{b} = \mathbf{m}\ddot{\mathbf{u}}$ describes the body motion in a strong sense, a weak formulation is obtained by the principle of minimum complementary virtual work (or Weighted Residual/Galerkin) $\displaystyle\int_\Omega \delta\mathbf{u}^T[\mathbf{L}^T\boldsymbol{\sigma} + \mathbf{b} - \mathbf{m}\ddot{\mathbf{u}}]d\Omega = 0$

- Applying Gauss divergence theorem ($\int\int_A \phi \text{div}\, \mathbf{q} dA = \oint_s \phi \mathbf{q}^T \mathbf{n} ds - \int\int_A (\nabla \phi)^T \mathbf{q} dA$) and recalling that $\mathbf{L}\mathbf{u} = \varepsilon$, we obtain $\int_\Omega [\delta \mathbf{u}^T(\mathbf{m}\ddot{\mathbf{u}} - \mathbf{b}) + \delta \varepsilon^T \sigma] d\Omega - \int_\Gamma \delta \mathbf{u}^T \mathbf{t} d\Gamma = 0$ so far no assumption has been made with regard to material behavior.
- Next we will seek the spatial discretization of the virtual work equation.

$$\begin{array}{rclrclrcl}
\mathbf{u} & = & \mathbf{N}\overline{\mathbf{u}}; & \delta\mathbf{u}^T & = & \delta\overline{\mathbf{u}}^T\mathbf{N}^T; & \ddot{\mathbf{u}} & = & \mathbf{N}\ddot{\overline{\mathbf{u}}} \\
\mathbf{B} & = & \mathbf{L}\mathbf{N}; & \varepsilon & = & \mathbf{B}\overline{\mathbf{u}}; & \delta\varepsilon^T & = & \delta\overline{\mathbf{u}}^T\mathbf{B}^T \\
\dot{\varepsilon} & = & \mathbf{B}\dot{\overline{\mathbf{u}}} & & & & & &
\end{array}$$

- For linear problems $\sigma_{t,n} = \mathbf{D}_e \mathbf{B}\overline{\mathbf{u}}_{t,n}$, and with proper substitution, this would yield

$$\underbrace{\underbrace{\int_\Omega \mathbf{N}^T\mathbf{m}\mathbf{N}d\Omega}_{\mathbf{M}_{tt}}\ddot{\overline{\mathbf{u}}}_{t,n}}_{\mathbf{P}^{inertia}_{t,n}} + \underbrace{\underbrace{\int_\Omega \mathbf{B}^T\mathbf{D}_e\mathbf{B}d\Omega}_{\mathbf{K}}\overline{\mathbf{u}}_{t,n}}_{\mathbf{P}^{int}_{t,n}} - \underbrace{\left(\int_\Omega \mathbf{N}^T\mathbf{P}^{ext}_{t,n}d\Omega + \int_\Gamma \mathbf{N}^T\mathbf{t}_{t,n}d\Gamma\right)}_{\mathbf{P}^{ext}_{t,n}} = 0$$

or $\mathbf{M}\ddot{\overline{\mathbf{u}}}_{t,n} + \mathbf{K}\overline{\mathbf{u}}_{t,n} = \mathbf{P}^{ext}_{t,n}$ Which represents the semi-discrete linear equation of motion in the implicit time integration.

- Note similarity between the mass matrix and the geometric one,
  $\left[\mathbf{k}_g^{(e)}\right] = \left[P^{(e)} \int_L \{\mathbf{N}_{,x}\} \lfloor \mathbf{N}_{,x} \rfloor dx \right\}$
- Note the absence of the damping coefficient (which is a non-rational numerical "trick").
- If we assume viscous damping (and replacing $\overline{\mathbf{u}}$ by $\mathbf{u}$) we obtain

$$\mathbf{M}_{tt} \cdot \ddot{\mathbf{u}}_t + \mathbf{C}_{tt} \cdot \dot{\mathbf{u}}_t + \mathbf{P}_t^{int} = \mathbf{P}_t^{ext}$$

  where $\mathbf{M}_{tt}$ and $\mathbf{C}_{tt}$ are the mass and viscous damping matrices for the idealization of the structure; $\ddot{\mathbf{u}}_t$ is the nodal acceleration vector, $\dot{\mathbf{u}}_t$ is the nodal velocity vector, $\mathbf{P}_t^{int}$ is the static restoring or internal nodal force vector resulting from the nodal displacement vector $\mathbf{u}_t$, and $\mathbf{P}_t^{ext}$ is the vector of applied nodal forces due to a seismic loading.
- Numerical methods for solving this differential equation are divided into two major categories; explicit and implicit methods. We will limit coverage to implicit schemes and in particular: 1) Newmark β method, 2) the Hilber-Hughes-Taylor (HHT) method.

- There are two possible representation of the mass matrix: lumped and consistent.
- Lumped mass: it is assumed that all the masses are concentrated at the end nodes. Though not exactly correct, the advantage of this model is that we will have a diagonal matrix which can be easily inverted.

$$\mathbf{m}_e = \rho \cdot A \cdot L_e \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_r \cdot L_e^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_r \cdot L_e^2 \end{bmatrix}$$

Note: α_r zero will result in a singular mass matrix which is undesirable if we have to invert the mass matrix. An *ad hoc* solution to define α_r is to imagine that a uniform slender bar of length $L_e/2$ and mass $m/2$ is attached to each node and rotates with it. The associated mass moment of inertia would be $I_z = (m/2)(L_e/2)^2/3$, and consequently $\alpha_r = 1/24$. It should be noted that models based on lumped mass can run substantially faster than those based on consistent mass.

- Consistent mass uses a kinematically equivalent mass matrix where inertia forces are associated with all degrees of freedom.
- Given $\mathbf{m}_e = \int_0^{L_e} \rho \cdot A(x) \cdot \mathbf{N}_d(x)^T \cdot \mathbf{N}_d(x) dx$ and the shape functions of the beam column, it can be shown that the matrix is

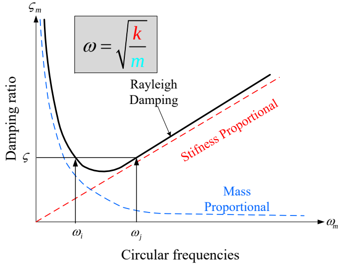
$$\mathbf{m}_e = \frac{\rho \cdot A \cdot L_e}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22 \cdot L_e & 0 & 54 & -13 \cdot L_e \\ 0 & 22 \cdot L_e & 4 \cdot L_e^2 & 0 & 13 \cdot L_e & -3 \cdot L_e^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13 \cdot L_e & 0 & 156 & -22 \cdot L_e \\ 0 & -13 \cdot L_e & -3 \cdot L_e^2 & 0 & -22 \cdot L_e & 4 \cdot L_e^2 \end{bmatrix}$$

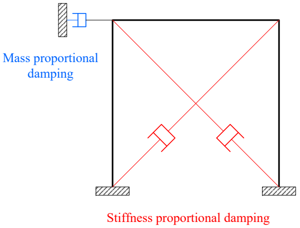
The mass matrix is then transformed into the global reference $\mathbf{M}_e = \boldsymbol{\Gamma}_e^T \cdot \mathbf{m}_e \cdot \boldsymbol{\Gamma}_e$

- All structures are damped, (2*nd* law of thermodynamic) otherwise their oscillations will never stop. Damping can be viewed as a frictional force which dissipates energy, and can take different form.
- Most commonly used form of damping is the so-called viscous or Rayleigh damping which, when inserted in the equation of motion, has the following form

$$\mathbf{M}_{tt} \cdot \ddot{\mathbf{u}}_{t,n} + \mathbf{C}_{tt} \cdot \dot{\mathbf{u}}_{t,n} + \mathbf{P}_{t,n}^{int} = \mathbf{P}_{t,n}^{ext}$$

  where, $\ddot{\mathbf{u}}_{t,n}$, $\dot{\mathbf{u}}_{t,n}$ and $\mathbf{u}_{t,n}$ are the nodal acceleration, velocity, and displacement vectors at the current time step, respectively; $\mathbf{P}_{t,n}^{int}$ is the static restoring or internal nodal force vector at the current time step.
- Damping is supposed to model the dissipation of energy. In a nonlinear analysis, this is accounted for by some constitutive models which include hysterisis damping, such as the Modified Kent and Park model for concrete.





- In linear elastic analysis, the most common form of damping is the so-called viscous damping (better known as Rayleigh damping). In this simplification, we assume the presence of a viscous damper (which by definition is sensitive to velocity) between the structure and an external fixed point (mass proportional), and another set of dampers inside the structure connecting all the degrees of freedom (stiffness proportional damper).
- Viscous damping: $\mathbf{C}_{tt} = a_m \cdot \mathbf{M}_{tt} + b_k \cdot \mathbf{K}_{tt}$ where, a_m and b_k are coefficients which pre-multiply the mass and stiffness terms respectively.

- Coefficients a_m and b_k are calculated based upon two circular frequencies (ω_1 and ω_2, radians/sec.) to be damped at ξ_1 and ξ_2 respectively. Where ω_m and ξ_m are the circular frequency and the damping ratio of the m^{th} mode.
- It can be easily shown that

$$\frac{1}{2}\begin{bmatrix} \frac{1}{\omega_i} & \omega_i \\ \frac{1}{\omega_j} & \omega_j \end{bmatrix}\left\{ \begin{array}{c} a_m \\ b_k \end{array} \right\} = \left\{ \begin{array}{c} \zeta_i \\ \zeta_j \end{array} \right\}$$

- If one assumes the same damping ratio ζ for both modes (reasonable practical assumption), then

$$a_m = \zeta\frac{2\omega_i \cdot \omega_j}{\omega_i + \omega_j}; \quad b_k = \zeta\frac{2}{\omega_i + \omega_j}$$

- Again, it should be emphasized that different damping coefficients should be used in linear and in nonlinear analysis (specially if the nonlinear constitutive model accounts for hysterisis damping. Furthermore, if Rayleigh damping is used in a nonlinear analysis, then coefficients a_m and b_k may have to be updated at each time increment to reflect the change in the tangential stiffness matrix $\mathbf{K}_t$.

- In practice we can obtain damping coefficients by exciting a structure with shakers albeit for only the elastic range.

- Euler method is a numerical procedure to solve initial values ordinary differential equations (as in structural dynamics). In other words, given a solution at time t_n, how do we get the solution at time t_{n+1}.
- Note that we referred to Newton's method for nonlinear analysis, and Euler for dynamic.
- In our case, we discretize space by the finite element, and discretize time by the finite difference.
- As with Newton's method, it all start with the Taylor's series.
- Forward Euler/Explicit

$$y(t_n + h) \equiv y_{n+1} = y(t_n) + h \left.\frac{dy}{dt}\right|_{t_n} + O(h^2) \Rightarrow y^{?}_{n+1} \simeq y^{\surd}_n + hf(y^{\surd}_n, t_n)$$

where $h = \Delta t$, and $f(y_n, t_n) = \left.\frac{dy}{dt}\right|_{t_n}$ This is also referred to as explicit since y_{n+1} is given explicitly in terms of known quantities such as y_n and $f(y_n, t_n)$ and there is no equation to solve.

Explicit methods are easy to implement but are conditionally sable (i.e. h should be smaller than a critical value). This is similar to the approximate step by step method used earlier for geometric nonlinear problems.

- Backward Euler/Implicit starts with the following backward Taylor series expansion

$$y(t_n) \equiv y_n = y(t_{n+1} - h) = y(t_{n+1}) - h \left. \frac{dy}{dt} \right|_{t_{n+1}} + O(h^2) \Rightarrow y_{n+1}^{?} \simeq y_n^{\surd} + hf(y_{n+1}^{?}, t_{n+1})$$

  It is an implicit method since $f(y_{n+1}, t_{n+1})$ is not known and a (usually) nonlinear equation must be solved at every time step (possibly by the Newton-Raphson method). Evidently, this is more computationally expensive than the explicit method, however the method is unconditionally stable.

- We note that the implicit method (at the cost of a Newton-Raphson solution) always provides an “exact” solution. In the context of structural dynamics, we can say that equilibrium is satisfied. This is not the case in the explicit method.

- Numerical example: Solve the following ordinary linear first order differential equation: $\frac{dy}{dt} = 1 + (t - y(t))^2; \qquad 2 \le t \le 3; \qquad y(0) = 1; n = 0.$

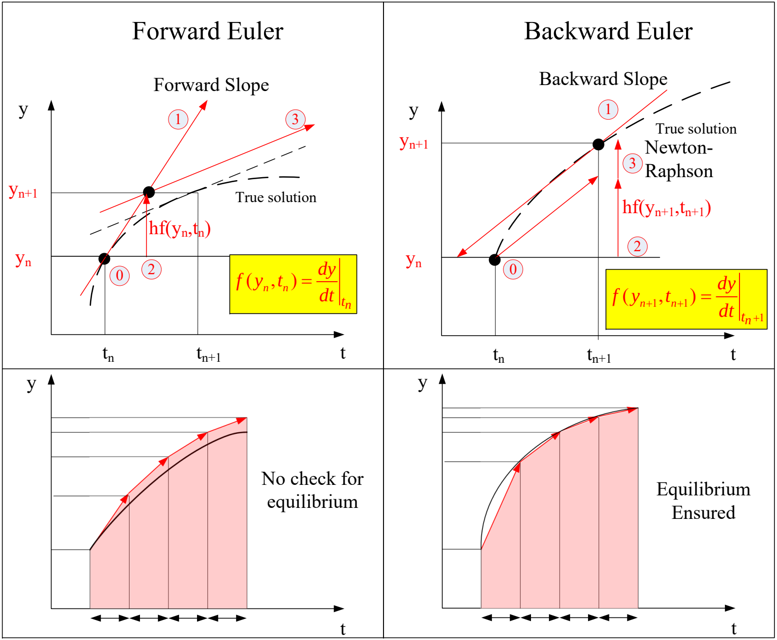
- Forward Euler with $h = 0.1$

$$y_{n+1} = y_n + hf(y_n, t_n) \Rightarrow y_1 = 1 + 0.1\left[1 + (2. - 1.)^2\right] = 1.2$$

- The Backward Euler will give

$$\begin{aligned} y_{n+1} &= y_n + hf(y_{n+1}, t_{n+1}) \\ \Rightarrow y_1 &= 1 + 0.1\left[1 + (2.1 - y_1)^2\right] \\ \Rightarrow 0 &= 0.1y_1^2 - 1.42y_1 + 1.541 \\ \Rightarrow y_1 &= 1.1839 \end{aligned}$$

  In this case we had a quadratic equation to solve, however in general we may have to use Newton's method to solve for y_n.
- In the context of nonlinear structural analysis, this would imply that we are checking equilibrium at $n = 1$, which is not the case in the explicit method.


Forward Euler
Forward Slope
y
y_{n+1}
y_n
1
3
True solution
$hf(y_n,t_n)$
0
2
$f(y_n,t_n) = \left.\frac{dy}{dt}\right|_{t_n}$
t_n
t_{n+1}
t
Backward Euler
Backward Slope
True solution
Newton-Raphson
$hf(y_{n+1},t_{n+1})$
$f(y_{n+1},t_{n+1}) = \left.\frac{dy}{dt}\right|_{t_{n+1}}$
No check for equilibrium
Equilibrium Ensured

- Newmark's method is a generalization of Euler's method for second order differential equations (equation of motion)
- Taylor's series (as usual) is our starting point

$$
\begin{aligned}
\begin{pmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{pmatrix}^{n} &= \begin{pmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{pmatrix}^{n-1} + \Delta t \begin{pmatrix} \dot{\mathbf{u}} \\ \ddot{\mathbf{u}} \end{pmatrix}^{n-1} \quad \text{Forward Euler} \\
\begin{pmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{pmatrix}^{n} &= \begin{pmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{pmatrix}^{n-1} + \Delta t \begin{pmatrix} \dot{\mathbf{u}} \\ \ddot{\mathbf{u}} \end{pmatrix}^{n} \quad \text{Backward Euler}
\end{aligned}
$$

- Newmark's method differs from Euler's method by replacing higher order derivatives with simpler expressions (and thus lower accuracy) for the sake of efficiency.
- Again, we first consider the Taylor series expansions of the nodal displacement and velocity vector terms about the values at the previous time $n-1$.

$$
\begin{aligned}
\mathbf{u}_{t,n} &\approx \mathbf{u}_{t,n-1} + \frac{\partial \mathbf{u}_{t,n-1}}{\partial t}\Delta t + \frac{\partial^2 \mathbf{u}_{t,n-1}}{\partial t^2}\frac{\Delta t^2}{2!} + \frac{\partial^3 \mathbf{u}_{t,n-1}}{\partial t^3}\frac{\Delta t^3}{3!} + \cdots \\
\dot{\mathbf{u}}_{t,n} &\approx \dot{\mathbf{u}}_{t,n-1} + \frac{\partial^2 \mathbf{u}_{t,n-1}}{\partial t^2}\Delta t + \frac{\partial^3 \mathbf{u}_{t,n-1}}{\partial t^3}\frac{\Delta t^2}{2!} + \cdots
\end{aligned}
$$

- Those two equations represent the approximate displacement and velocity vectors ($\mathbf{u}_{t,n}$ and $\dot{\mathbf{u}}_{t,n}$) except for high order terms of Taylor series. We represent the last terms of the above two equations as follow:

$$
\begin{aligned}
\frac{\partial^3 \mathbf{u}_{t,n-1}}{\partial t^3}\frac{\Delta t^3}{3!} &\approx \frac{\frac{\partial^2 \mathbf{u}_{t,n}}{\partial^2 t} - \frac{\partial^2 \mathbf{u}_{t,n-1}}{\partial^2 t}}{\Delta t}\frac{\Delta t^3}{3!} \approx \left(\ddot{\mathbf{u}}_{t,n} - \ddot{\mathbf{u}}_{t,n-1}\right)\frac{\Delta t^2}{3!} \\
&\approx \beta\left(\ddot{\mathbf{u}}_{t,n} - \ddot{\mathbf{u}}_{t,n-1}\right)\Delta t^2 \\
\frac{\partial^3 \mathbf{u}_{t,n-1}}{\partial t^3}\frac{\Delta t^2}{2!} &\approx \frac{\frac{\partial^2 \mathbf{u}_{t,n}}{\partial^2 t} - \frac{\partial^2 \mathbf{u}_{t,n-1}}{\partial^2 t}}{\Delta t}\frac{\Delta t^2}{2!} \approx \left(\ddot{\mathbf{u}}_{t,n} - \ddot{\mathbf{u}}_{t,n-1}\right)\frac{\Delta t}{2!} \\
&\approx \gamma\left(\ddot{\mathbf{u}}_{t,n} - \ddot{\mathbf{u}}_{t,n-1}\right)\Delta t
\end{aligned}
$$

where β and γ are parameters which depict numerical approximations.

- Substituting

$$
\begin{aligned}
\mathbf{u}_{t,n} &= \mathbf{u}_{t,n-1} + \Delta t \cdot \dot{\mathbf{u}}_{t,n-1} + \frac{\Delta t^2}{2} \cdot \ddot{\mathbf{u}}_{t,n-1} + \Delta t^2 \cdot \beta \cdot \left(\ddot{\mathbf{u}}_{t,n} - \ddot{\mathbf{u}}_{t,n-1}\right) \\
\dot{\mathbf{u}}_{t,n} &= \dot{\mathbf{u}}_{t,n-1} + \Delta t \cdot \ddot{\mathbf{u}}_{t,n-1} + \Delta t \cdot \gamma \cdot \left(\ddot{\mathbf{u}}_{t,n} - \ddot{\mathbf{u}}_{t,n-1}\right)
\end{aligned}
$$

- Hence, we obtain the Newmark β method, which consists of the following equations (forward difference):

$$\mathbf{P}^{ext}_{t,n} = \mathbf{M}_{tt} \cdot \ddot{\mathbf{u}}_{t,n} + \mathbf{C}_{tt} \cdot \dot{\mathbf{u}}_{t,n} + \mathbf{P}^{int}_{t,n} \quad (1)$$

$$\mathbf{u}_{t,n} = \mathbf{u}_{t,n-1} + \Delta t \cdot \dot{\mathbf{u}}_{t,n-1} + \frac{\Delta t^2}{2}\left[(1-2\beta)\ddot{\mathbf{u}}_{t,n-1} + 2\beta \cdot \ddot{\mathbf{u}}_{t,n}\right] \quad (2)$$

$$\dot{\mathbf{u}}_{t,n} = \dot{\mathbf{u}}_{t,n-1} + \Delta t\left[(1-\gamma)\ddot{\mathbf{u}}_{t,n-1} + \gamma \cdot \ddot{\mathbf{u}}_{t,n}\right] \quad (3)$$

where the first equation is the equation of equilibrium expressed at time n, and the other two are finite difference formulas describing the evolution of the approximation solution (Note we have three equations and three unknowns: $\mathbf{u}_{t,n}$, $\dot{\mathbf{u}}_{t,n}$ and $\ddot{\mathbf{u}}_{t,n}$. β and γ are parameters that determine the stability and accuracy characteristics. Stability conditions for the Newmark β method follows:

  - unconditionally stable if $\gamma \geq \frac{1}{2}$ and $\beta \geq \frac{\gamma}{2}$
  - conditionally stable if $\gamma \geq \frac{1}{2}$ and $\beta < \frac{\gamma}{2}$ with the following stability limit: $\frac{\Delta t}{T} \leq \frac{1}{2\pi}\frac{1}{\sqrt{\gamma - 2\beta}} = 0.551$

Method	Type	β	γ	Stability condition	Order of accuracy
Constant acceleration	Implicit	1/4	1/2	Unconditional	2
Linear acceleration	implicit	1/3!=1/6	1/2!=1/2	$\Delta t \leq 2\sqrt{3}/\omega$	2
Central difference	Explicit	0	1/2	$\Delta t \leq 2/\omega$	2

- Eq. 1, 2 and 3 can be rewritten as:

$$\begin{aligned}
\mathbf{P}^{ext}_{t,n} &= \mathbf{M}_{tt}\cdot\ddot{\mathbf{u}}_{t,n} + \mathbf{C}_{tt}\cdot\dot{\mathbf{u}}_{t,n} + \mathbf{P}^{int}_{t,n} & (4)\\
\mathbf{u}_{t,n} &= \tilde{\mathbf{u}}_{t,n} + \Delta t^2\cdot\beta\cdot\ddot{\mathbf{u}}_{t,n} & (5)\\
\dot{\mathbf{u}}_{t,n} &= \tilde{\dot{\mathbf{u}}}_{t,n} + \Delta t\cdot\gamma\cdot\ddot{\mathbf{u}}_{t,n} & (6)
\end{aligned}$$

  where the known quantities at time step $n-1$ have a $(\tilde{.})$ with

$$\begin{aligned}
\tilde{\mathbf{u}}_{t,n} &= \mathbf{u}_{t,n-1} + \Delta t\cdot\dot{\mathbf{u}}_{t,n-1} + \frac{\Delta t^2}{2}(1-2\beta)\ddot{\mathbf{u}}_{t,n-1}\\
\tilde{\dot{\mathbf{u}}}_{t,n} &= \dot{\mathbf{u}}_{t,n-1} + \Delta t(1-\gamma)\ddot{\mathbf{u}}_{t,n-1}
\end{aligned}$$

- Eq. 5 gives $\ddot{\mathbf{u}}_{t,n} = \frac{\mathbf{u}_{t,n}-\tilde{\mathbf{u}}_{t,n}}{\Delta t^2\cdot\beta}$;
- Substituting in Eq. 6 we can solve for $\dot{\mathbf{u}}_{t,n} = \tilde{\dot{\mathbf{u}}}_{t,n} + \frac{\gamma}{\Delta t\cdot\beta}(\mathbf{u}_{t,n} - \tilde{\mathbf{u}}_{t,n})$.

- Finally substituting in Eq. 4 we obtain:

$$\underbrace{\frac{1}{\Delta t^2 \cdot \beta}\mathbf{M}_{tt} \cdot \mathbf{u}_{t,n} + \frac{\gamma}{\Delta t \cdot \beta}\mathbf{C}_{tt} \cdot \mathbf{u}_{t,n} + \mathbf{P}^{int}_{t,n}}_{?}$$
$$= \underbrace{\mathbf{P}^{ext}_{t,n} + \frac{1}{\Delta t^2 \cdot \beta}\mathbf{M}_{tt} \cdot \tilde{\mathbf{u}}_{t,n} + \frac{\gamma}{\Delta t \cdot \beta}\mathbf{C}_{tt} \cdot \tilde{\mathbf{u}}_{t,n} - \mathbf{C}_{tt} \cdot \tilde{\dot{\mathbf{u}}}_{t,n}}_{\checkmark} \quad (7)$$

- If the trial solutions in given iteration step k are $\mathbf{u}^k_{t,n}$, and $\mathbf{P}^{int,k}_{t,n}$, then it does not satisfy the equations of motion. Hence, we can write for this particular step with residual force vector $\mathbf{P}^{R,k}_{t,n}$:

$$\mathbf{P}^{R,k}_{t,n} = \mathbf{P}^{ext}_{t,n} + \overline{\mathbf{M}}_{tt}\left(\tilde{\mathbf{u}}_{t,n} - \mathbf{u}^k_{t,n}\right) - \mathbf{C}_{tt} \cdot \tilde{\dot{\mathbf{u}}}_{t,n} - \mathbf{P}^{int,k}_{t,n}$$

where, $\overline{\mathbf{M}}_{tt} = \frac{\mathbf{M}_{tt} + \Delta t \cdot \gamma \cdot \mathbf{C}}{\Delta t^2 \cdot \beta}$

- Using initial stiffness iterative method, we can solve for $\Delta\mathbf{u}^k_{t,n}$ from $\mathbf{P}^{R,k}_{t,n} = \mathbf{K}_{eff} \cdot \Delta\mathbf{u}^k_{t,n}$where, $\mathbf{K}_{eff}$ is the effective stiffness matrix, and $\Delta\mathbf{u}^k_{t,n} = \mathbf{u}_{t,n} - \mathbf{u}^k_{t,n}$.

- In elastic section, $\mathbf{P}^{int}_{t,n} = \mathbf{K}_{tt} \cdot \mathbf{u}_{t,n}$; Substituting we can solve for $\mathbf{u}_{t,n}$:

$$\mathbf{K}_{eff} \cdot \mathbf{u}_{t,n} = \mathbf{P}^{ext}_{t,n} + \overline{\mathbf{M}}_{tt} \cdot \tilde{\mathbf{u}}_{t,n} - \mathbf{C}_{tt} \cdot \tilde{\dot{\mathbf{u}}}_{t,n}$$

where, $\mathbf{K}_{eff} = \overline{\mathbf{M}}_{tt} + \mathbf{K}_{tt}$ (lumped or consistent mass matrix).

- Finally, we solve for $\delta\mathbf{u}^{k}_{t,n}$ and the updated displacement vector $\mathbf{u}^{k+1}_{t,n}$ at the next iteration step $k+1$:

$$\delta\mathbf{u}^{k}_{t,n} = [\mathbf{K}_{eff}]^{-1} \cdot \mathbf{P}^{R,k}_{t,n}; \quad \mathbf{u}^{k+1}_{t,n} = \mathbf{u}^{k}_{t,n} + \delta\mathbf{u}^{k}_{t,n}$$

Note need to invert the mass matrix only for consistent matrix.

- Note analogy with nonlinear analysis where $\delta\mathbf{u}$ is equal to the tangent stiffness matrix times the residual force.

- A major drawback of Newmark β method is the tendency for high frequency noise to persist in the solution. On the other hand, when linear damping or artificial viscosity is added via the parameter γ, the accuracy is markedly degraded. The α method, improves numerical dissipation for high frequency without degrading the accuracy as much.
- Equation of motion in HHT method is written at current time step *n* (forward difference) as:

$$\underbrace{\mathbf{M}_{tt}\cdot\ddot{\mathbf{u}}_{t,n}}_{\mathbf{P}^{inertia}_{t,n}}+\mathbf{P}^{int}_{t,n}=\mathbf{P}^{ext}_{t,n}$$

Seeking an approximate solution of this equation by one-step difference, we write,

$$\mathbf{M}_{tt}\cdot\ddot{\mathbf{u}}_{t,n}+(1+\alpha)\mathbf{P}^{int}_{t,n}-\alpha\cdot\mathbf{P}^{int}_{t,n-1}=(1+\alpha)\mathbf{P}^{ext}_{t,n}-\alpha\mathbf{P}^{ext}_{t,n-1}$$

We note that the HHT method introduces $\alpha(\mathbf{P}^{int}_{t,n}-\mathbf{P}^{int}_{t,n-1})$ which is akin of stiffness proportional damping (indeed it is commonly said that the α method provides numerical damping). If the above equation is expanded, effect of damping introduced, and possible material nonlinearity introduced, we obtain:

$$(1+\alpha)\mathbf{P}^{ext}_{t,n}-\alpha\mathbf{P}^{ext}_{t,n-1}=\mathbf{M}_{tt}\cdot\ddot{\mathbf{u}}_{t,n}+(1+\alpha)\mathbf{C}_{tt}\cdot\dot{\mathbf{u}}_{t,n}-\alpha\cdot\mathbf{C}_{tt}\cdot\dot{\mathbf{u}}_{t,n-1}+(1+\alpha)\mathbf{P}^{int}_{t,n}-\alpha\cdot\mathbf{P}^{int}_{t,n-1}$$

If $-1/3 \leq \alpha \leq 0$, $\beta = (1-\alpha)^2/4$, and $\gamma = (1-2\alpha)/2$, then the α method is unconditionally stable and has a second-order accuracy.

- Assuming that we have obtained the response at the previous time step $n-1$, i.e. $\mathbf{u}_{t,n-1}$, $\dot{\mathbf{u}}_{t,n-1}$ and $\ddot{\mathbf{u}}_{t,n-1}$ which satisfy the equation of motion, we now seek to determine the solution at the current time step n by iteration.
- First of all, we need to determine effective external force and effective stiffness.

$$\underbrace{\frac{1}{\Delta t^2 \cdot \beta}\mathbf{M}_{tt} \cdot \mathbf{u}_{t,n} + \frac{\gamma}{\Delta t \cdot \beta}(1+\alpha)\mathbf{C}_{tt} \cdot \mathbf{u}_{t,n} + (1+\alpha)\mathbf{P}^{int}_{t,n}}_{?}$$
$$= \underbrace{(1+\alpha)\mathbf{P}^{ext}_{t,n} - \alpha \cdot \mathbf{P}^{ext}_{t,n-1} + \frac{1}{\Delta t^2 \cdot \beta}\mathbf{M}_{tt} \cdot \tilde{\mathbf{u}}_{t,n} + \frac{\gamma}{\Delta t \cdot \beta}(1+\alpha)\mathbf{C}_{tt} \cdot \tilde{\mathbf{u}}_{t,n}}_{\checkmark} \tag{8}$$
$$\underbrace{-(1+\alpha)\mathbf{C}_{tt} \cdot \tilde{\dot{\mathbf{u}}}_{t,n} + \alpha \cdot \mathbf{C}_{tt} \cdot \dot{\mathbf{u}}_{n-1} + \alpha \cdot \mathbf{P}^{int}_{t,n}}_{\checkmark}$$

- The trial solutions in iteration step k are $\mathbf{u}_{t,n}^{k}$, and $\mathbf{P}_{t,n}^{int,k}$, does not necessarily satisfy the equations of motion. Hence, we can write for this particular step:

$$\mathbf{P}_{t,n}^{R,k} = (1+\alpha)\mathbf{P}_{t,n}^{ext} - \alpha\cdot\mathbf{P}_{t,n-1}^{ext} + \overline{\mathbf{M}}_{tt}\left(\tilde{\mathbf{u}}_{t,n} - \mathbf{u}_{t,n}^{k}\right) - (1+\alpha)\mathbf{C}_{tt}\cdot\tilde{\dot{\mathbf{u}}}_{t,n} + \alpha\cdot\mathbf{C}_{tt}\cdot\dot{\mathbf{u}}_{t,n}$$
$$- (1+\alpha)\mathbf{P}_{t,n}^{int,k} + \alpha\cdot\mathbf{P}_{t,n-1}^{int}$$

where, $\overline{\mathbf{M}}_{tt} = \frac{\mathbf{M}_{tt}+\Delta t\cdot\gamma(1+\alpha)\mathbf{C}_{tt}}{\Delta t^2\cdot\beta}$ and $\mathbf{P}_{t,n}^{R,k}$ is the residual force vector.

- Using the initial stiffness iterative method, we can solve for $\Delta\mathbf{u}_{t,n}^{k}$ from $\mathbf{P}_{t,n}^{R,k} = \mathbf{K}_{eff}\cdot\Delta\mathbf{u}_{t,n}^{k}$ where, $\mathbf{K}_{eff}$ is the effective stiffness matrix, and $\Delta\mathbf{u}_{t,n}^{k} = \mathbf{u}_{t,n} - \mathbf{u}_{t,n}^{k}$
- In elastic section, we can express $\mathbf{P}_{t,n}^{int}$ to compute the effective stiffness matrix with initial stiffness matrix $\mathbf{K}_{tt}$ as: $\mathbf{P}_{t,n}^{int} = \mathbf{K}_{tt}\cdot\mathbf{u}_{t,n}$.
- Substituting we solve for $\mathbf{u}_{t,n}$:

$$\mathbf{K}_{eff}\cdot\mathbf{u}_{t,n} = (1+\alpha)\mathbf{P}_{t,n}^{ext}-\alpha\cdot\mathbf{P}_{t,n-1}^{ext}+\overline{\mathbf{M}}_{tt}\cdot\tilde{\mathbf{u}}_{t,n}-(1+\alpha)\cdot\mathbf{C}_{tt}\cdot\tilde{\dot{\mathbf{u}}}_{t,n}+\alpha\cdot\mathbf{C}_{tt}\cdot\dot{\mathbf{u}}_{t,n-1}+\alpha\cdot\mathbf{P}_{t,n-1}^{int}$$

where, $\mathbf{K}_{eff} = \overline{\mathbf{M}}_{tt} + (1+\alpha)\mathbf{K}_{tt}$

- Finally, we solve for $\delta\mathbf{u}_{t,n}^{k}$ and the updated displacement vector $\mathbf{u}_{t,n}^{k+1}$ at the next iteration step $k+1$:

$$\delta\mathbf{u}_{t,n}^{k} = [\mathbf{K}_{eff}]^{-1} \cdot \mathbf{P}_{t,n}^{R,k};\ \ \mathbf{u}_{t,n}^{k+1} = \mathbf{u}_{t,n}^{k} + \delta\mathbf{u}_{t,n}^{k}$$

- Final Remarks:
  - α introduces a damping that grows with the ratio of time increment to the period of vibration of a node.
  - Negative values of α cause damping
  - If $\alpha = 0$, we have no artificial damping (energy preseving) and is exactly the constant acceleration (trapezoidal rule) - Newmark's β method if $\beta = 1/4$ and $\gamma = 1/2$.
  - Minimum value is $\alpha = -1/3$ which provides the maximum artificial damping. This results in a damping ratio of about 6% when the time increment is 40% of the period of oscillation of the mode being studied and smaller if the oscillation period increases.
  - This artificial damping is not very substantial for realistic time increment and low frequencies, but is non-negligible for high frequencies.
  - A default value of -0.05 is recommended.

- We are accustomed to consider a signal in the time domain, i.e $f(t)$.
- Fourier series provides an alternate way of representing data: instead of representing the signal amplitude as a function of time, we represent the signal by how much information is contained at different frequencies.
- A Fourier series takes a signal and decomposes it into a sum of sines and cosines of different frequencies, $f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \sin(2\pi nt) + b_n \cos(2\pi nt)\right)$ where $f(t)$ is the signal in the time domain, a_n and b_n are unknown coefficients, n is an integer with units of Hertz (Hz)=$1/$s and corresponds to the frequency of the wave.
- Just as any function can be replaced by a corresponding Fourrier series, a signal originally expressed in the time domain, can be expressed in the frequency domain through a so-called Fast Fourrier Transform (FFT).

$$x(t) \xrightarrow{\texttt{FFT}} X(\omega) \Rightarrow X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi\omega t} \mathrm{d}t \tag{9}$$

while the inverse FFT takes us back from the frequency domain to the time domain through:

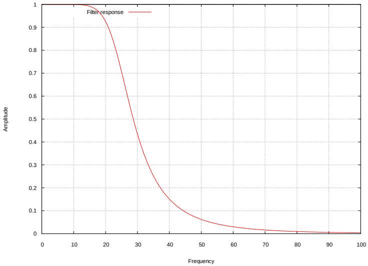
$$X(\omega) \xrightarrow{\mathtt{FFT}^{-1}} x(t) \Rightarrow x(t) = \int_{-\infty}^{\infty} X(\omega) e^{i2\pi\omega t} d\omega \tag{10}$$

- Reason we perform this operation:
  - Much "hidden" information contained in the signal can be best captured in the frequency domain (for instance, identify natural frequencies of a structural response to an excitation)
  - Filter response in the frequency domain, and then go back to the time domain.

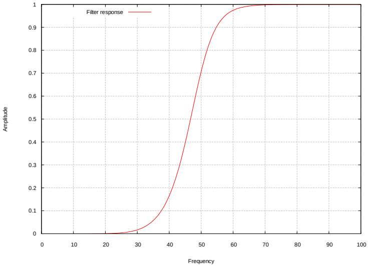
- Examples of so-called Butterworth filter:

$$|H(j\omega)|^2 = \begin{cases} \text{Low pass} & \frac{1}{1+\left(\frac{\omega}{\omega_L}\right)^{2n}} \\ \text{High pass} & \frac{1}{1+\left(\frac{\omega_U}{\omega}\right)^{2n}} \\ \text{Band pass} & \frac{1}{1+\left(\frac{\omega}{\omega_L}\right)^{2n}} \frac{1}{1+\left(\frac{omega_U}{\omega}\right)^{2n}} \\ \text{Band stop} & \frac{1}{1+\left(\frac{\omega_L}{\omega}\right)^{2n}} \frac{1}{1+\left(\frac{omega}{\omega_U}\right)^{2n}} \end{cases}$$

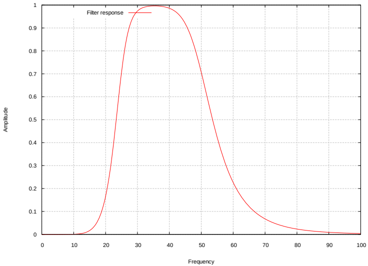
where ω, ω_L, ω_U and n are the frequency, the lower and upper filter frequency, and the order of the filter respectively. Following figure: Low Pass (25); High Pass (50); Band Pass (25-50); Band Stop (25-50) Filters, $N = 4$



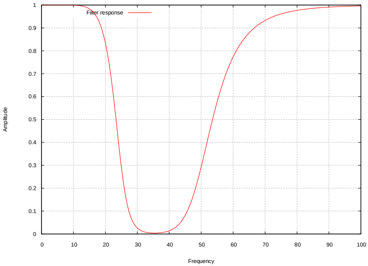
Low Pass 25;



High Pass 50;



Band Pass 25-50;

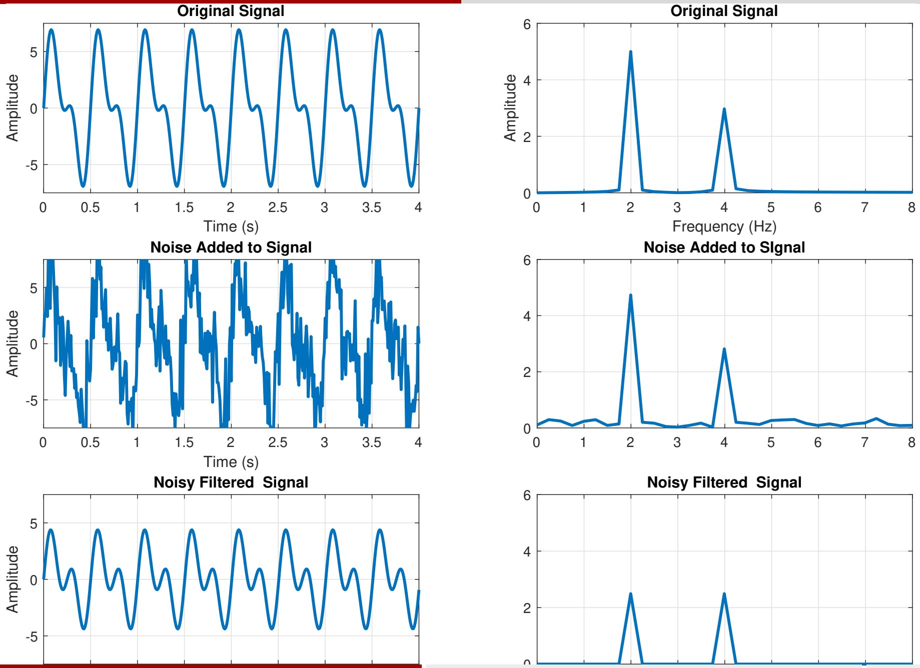


Band Stop 25-50;

```
%% First Example
clear; close all; clc
GS = 'c:/Program Files/gs/gs9.23/bin/gswin64.exe';
scrsz       = get(0,'ScreenSize'); % Screen size
fs=14;
x = rand(1,10);       % suppose 10 samples of a random signal
y = fft(x);           % Fourier transform of the signal
iy = ifft(y);         % inverse Fourier transform
x2 = real(iy);        % chop off tiny imaginary parts
norm(x x2)            % compare original with inverse of transformed
%=============================================================================
%% Create a signal of 4 seconds at a sampling rate of 0.01:
dt = 1/100;           % sampling rate
et = 4;               % end of the interval
t = 0:dt:et;          % sampling range
y = 3*sin(4*2*pi*t) + 5*sin(2*2*pi*t); % sample the signal
%
hf1 = figure('Position',[1 scrsz(4)/4.5 scrsz(3)/2.5 scrsz(4)/2.0]);
subplot(3,2,1);       % first of two plots
plot(t,y,'LineWidth',2); grid     % plot with grid
axis([0 et 8 8]);  % adjust scaling
xlabel('Time (s)'); % time expressed in seconds
ylabel('Amplitude');% amplitude as function of time
title('Original Signal');axis([0 4 7.5 7.5]);
%
%% Compute and plot Fourrier transform
Y = fft(y);           % compute Fourier transform
n = size(y,2)/2;      % 2nd half are complex conjugates
amp_spec = abs(Y)/n;% absolute value and normalize
%
subplot(3,2,2);
```

```
freq = (0:79)/(2*n*dt); % abscissa viewing wind
plot(freq,amp_spec(1:80),'LineWidth',2); grid % plot amplitude spectrum
xlabel('Frequency (Hz)');   % 1 Herz = number of cycles/second
ylabel('Amplitude');        % amplitude as function of frequency
title('Original Signal'); axis([0 8 0 6]);
%==========================================================================
%% Add Noise to  signal and compute the amplitude spectrum.
noise = randn(1,size(y,2)); % random noise
ey = y + 2*noise;           % samples with noise
eY = fft(ey);               % Fourier transform of noisy signal
n = size(ey,2)/2;           % use size for scaling
amp_spec = abs(eY)/n;       % compute amplitude spectrum
subplot(3,2,3);
plot(t,ey,'LineWidth',2); grid on        % plot noisy signal with grid
axis([0 et 8 8]);           % scale axes for viewing
xlabel('Time (s)');         % time expressed in seconds
ylabel('Amplitude');        % amplitude as function of time
title('Noise Added to Signal');axis([0 4 7.5 7.5]);
%
freq = (0:79)/(2*n*dt);     % abscissa viewing window
eY = fft(ey);               % compute Fourier transform
n = size(y,2)/2;            % 2nd half are complex conjugates
e_amp_spec = abs(eY)/n;     % absolute value and normalize
subplot(3,2,4);
plot(freq,e_amp_spec(1:80),'LineWidth',2); grid % plot amplitude spectrum
title('Noise Added to SIgnal'); axis([0 8 0 6]);
%==========================================================================
%% fiter noise
fY = fix(eY/500)*500;       % set numbers < 500 to zero
ifY = ifft(fY);             % inverse Fourier transform of fixed data
cy = real(ifY);             % remove imaginary parts
subplot(3,2,5)
```

```
plot(t,cy,'LineWidth',2); grid on        % plot corrected signal
axis([0 et 8 8]);              % adjust scale for viewing
xlabel('Time (s)');            % time expressed in seconds
ylabel('Amplitude');           % amplitude as function of time
title('Noisy Filtered  Signal');axis([0 4  7.5 7.5]);
%
cY = fft(cy);                  % compute Fourier transform
n = size(cy,2)/2;               % 2nd half are complex conjugates
e_amp_spec = abs(cY)/n;        % absolute value and normalize
subplot(3,2,6);
plot(freq,e_amp_spec(1:80),'LineWidth',2); grid % plot amplitude spectrum
title('Noisy Filtered  Signal');axis([0 8 0 6]);
set(gcf,'PaperPositionMode','auto');
FileName='fft example.eps';
print(FileName,' depsc');
eps2pdf(FileName,GS,0);
```


Original Signal
Amplitude
Time (s)
Original Signal
Amplitude
Frequency (Hz)
Noise Added to Signal
Amplitude
Time (s)
Noise Added to SIgnal
Noisy Filtered Signal
Amplitude
Noisy Filtered Signal

Non Linear Structural Analysis

Soil Structure Interaction

Victor E. Saouma
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University of Colorado, Boulder

Fall 2020

Table of Contents I

- We are accustomed to consider a signal in the time domain, i.e $f(t)$.
- Fourier series provides an alternate way of representing data: instead of representing the signal amplitude as a function of time, we represent the signal by how much information is contained at different frequencies.
- A Fourier series takes a signal and decomposes it into a sum of sines and cosines of different frequencies, $f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \sin(2\pi nt) + b_n \cos(2\pi nt)\right)$ where $f(t)$ is the signal in the time domain, a_n and b_n are unknown coefficients, n is an integer with units of Hertz (Hz)=$1/$s and corresponds to the frequency of the wave.
- Just as any function can be replaced by a corresponding Fourrier series, a signal originally expressed in the time domain, can be expressed in the frequency domain through a so-called Fast Fourrier Transform (FFT).

$$x(t) \xrightarrow{\text{FFT}} X(\omega) \Rightarrow X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi\omega t}dt \tag{1}$$

while the inverse FFT takes us back from the frequency domain to the time domain through:

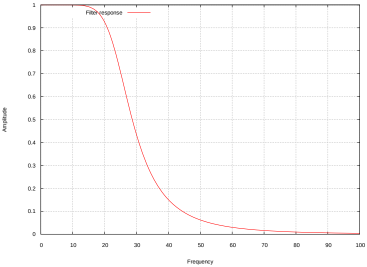
$$X(\omega) \xrightarrow{\mathtt{FFT}^{-1}} x(t) \Rightarrow x(t) = \int_{-\infty}^{\infty} X(\omega) e^{i2\pi\omega t} \mathrm{d}\omega \tag{2}$$

- Reason we perform this operation:
  - Much “hidden” information contained in the signal can be best captured in the frequency domain (for instance, identify natural frequencies of a structural response to an excitation)
  - Filter response in the frequency domain, and then go back to the time domain.

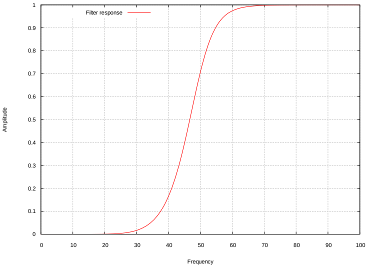
- Examples of so-called Butterworth filter:

$$|H(j\omega)|^2 = \begin{cases} \text{Low pass} & \frac{1}{1+\left(\frac{\omega}{\omega_L}\right)^{2n}} \\ \text{High pass} & \frac{1}{1+\left(\frac{\omega_U}{\omega}\right)^{2n}} \\ \text{Band pass} & \frac{1}{1+\left(\frac{\omega}{\omega_L}\right)^{2n}} \frac{1}{1+\left(\frac{omega_U}{\omega}\right)^{2n}} \\ \text{Band stop} & \frac{1}{1+\left(\frac{\omega_L}{\omega}\right)^{2n}} \frac{1}{1+\left(\frac{omega}{\omega_U}\right)^{2n}} \end{cases}$$

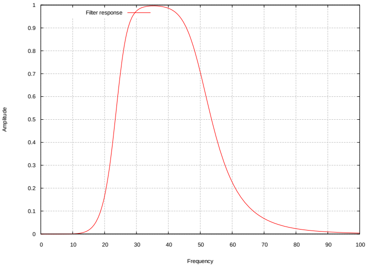
where ω, ω_L, ω_U and n are the frequency, the lower and upper filter frequency, and the order of the filter respectively. Following figure: Low Pass (25); High Pass (50); Band Pass (25-50); Band Stop (25-50) Filters, $N = 4$



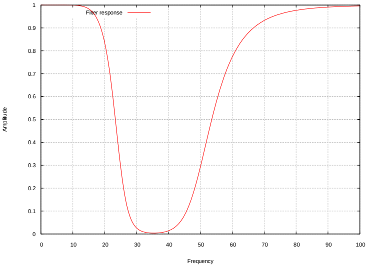
Low Pass 25;



High Pass 50;



Band Pass 25-50;

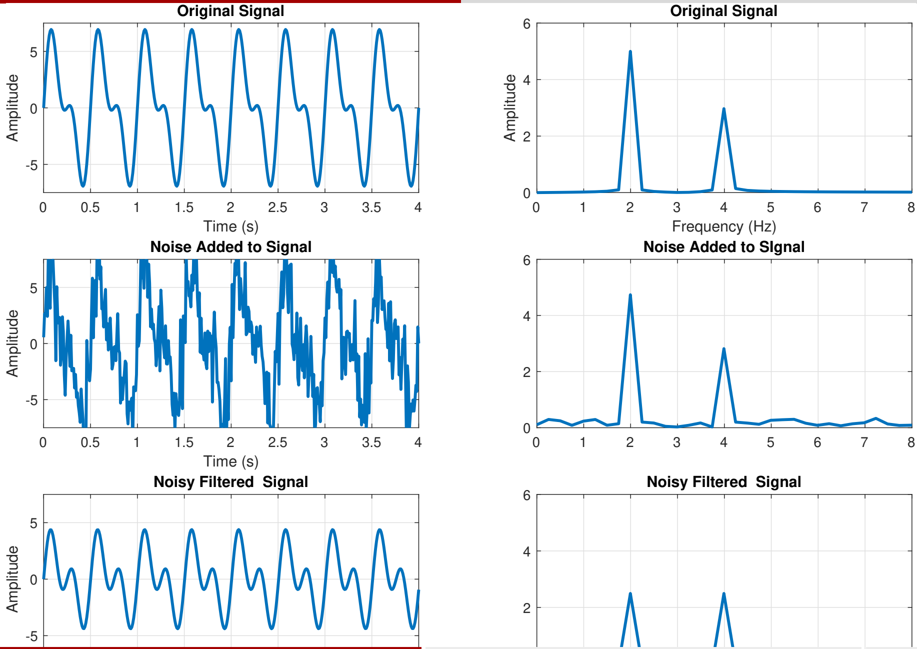


Band Stop 25-50;

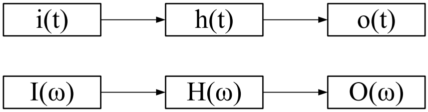
```matlab
%% First Example
clear; close all; clc
GS = 'c:/Program Files/gs/gs9.23/bin/gswin64.exe';
scrsz        = get(0,'ScreenSize'); % Screen size
fs=14;
x = rand(1,10);      % suppose 10 samples of a random signal
y = fft(x);          % Fourier transform of the signal
iy = ifft(y);        % inverse Fourier transform
x2 = real(iy);       % chop off tiny imaginary parts
norm(x-x2)           % compare original with inverse of transformed
%======================================================================
%% Create a signal of 4 seconds at a sampling rate of 0.01:
dt = 1/100;          % sampling rate
et = 4;              % end of the interval
t = 0:dt:et;         % sampling range
y = 3*sin(4*2*pi*t) + 5*sin(2*2*pi*t); % sample the signal
%
hf1 = figure('Position',[1 scrsz(4)/4.5 scrsz(3)/2.5 scrsz(4)/2.0]);
subplot(3,2,1);      % first of two plots
plot(t,y,'LineWidth',2); grid     % plot with grid
axis([0 et -8 8]);  % adjust scaling
xlabel('Time (s)'); % time expressed in seconds
ylabel('Amplitude');% amplitude as function of time
title('Original Signal');axis([0 4 -7.5 7.5]);
%----------------------------------------------------------------------
%% Compute and plot Fourrier transform
Y = fft(y);          % compute Fourier transform
n = size(y,2)/2;     % 2nd half are complex conjugates
amp_spec = abs(Y)/n;% absolute value and normalize
%
subplot(3,2,2);
```

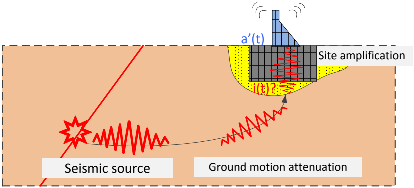
```
freq = (0:79)/(2*n*dt); % abscissa viewing wind
plot(freq,amp_spec(1:80),'LineWidth',2); grid % plot amplitude spectrum
xlabel('Frequency (Hz)');   % 1 Herz = number of cycles/second
ylabel('Amplitude');        % amplitude as function of frequency
title('Original Signal'); axis([0 8 0 6]);
%==================================================================
%% Add Noise to  signal and compute the amplitude spectrum.
noise = randn(1,size(y,2)); % random noise
ey = y + 2*noise;           % samples with noise
eY = fft(ey);               % Fourier transform of noisy signal
n = size(ey,2)/2;           % use size for scaling
amp_spec = abs(eY)/n;       % compute amplitude spectrum
subplot(3,2,3);
plot(t,ey,'LineWidth',2); grid on        % plot noisy signal with grid
axis([0 et -8 8]);          % scale axes for viewing
xlabel('Time (s)');         % time expressed in seconds
ylabel('Amplitude');        % amplitude as function of time
title('Noise Added to Signal');axis([0 4 -7.5 7.5]);
%------------------------------------------------------------------
freq = (0:79)/(2*n*dt);     % abscissa viewing window
eY = fft(ey);               % compute Fourier transform
n = size(y,2)/2;            % 2nd half are complex conjugates
e_amp_spec = abs(eY)/n;     % absolute value and normalize
subplot(3,2,4);
plot(freq,e_amp_spec(1:80),'LineWidth',2); grid % plot amplitude spectrum
title('Noise Added to SIgnal'); axis([0 8 0 6]);
%==================================================================
%% fiter noise
fY = fix(eY/500)*500;       % set numbers < 500 to zero
ifY = ifft(fY);             % inverse Fourier transform of fixed data
cy = real(ifY);             % remove imaginary parts
```

```
subplot(3,2,5)
plot(t,cy,'LineWidth',2); grid on        % plot corrected signal
axis([0 et -8 8]);             % adjust scale for viewing
xlabel('Time (s)');            % time expressed in seconds
ylabel('Amplitude');           % amplitude as function of time
title('Noisy Filtered  Signal');axis([0 4 -7.5 7.5]);
%---------------------------------------------------------------------
cY = fft(cy);                  % compute Fourier transform
n = size(cy,2)/2;               % 2nd half are complex conjugates
e_amp_spec = abs(cY)/n;        % absolute value and normalize
subplot(3,2,6);
plot(freq,e_amp_spec(1:80),'LineWidth',2); grid % plot amplitude spectrum
title('Noisy Filtered  Signal');axis([0 8 0 6]);
set(gcf,'PaperPositionMode','auto');
FileName='fft-example.eps';
print(FileName,'-depsc');
eps2pdf(FileName,GS,0);
```

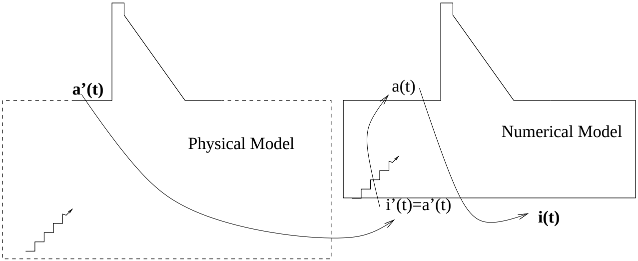

Original Signal
Amplitude
Time (s)
Original Signal
Amplitude
Frequency (Hz)
Noise Added to Signal
Amplitude
Time (s)
Noise Added to SIgnal
Noisy Filtered Signal
Amplitude
Noisy Filtered Signal

- In dynamic event, we can define an input record $i(t)$ which is amplified by $h(t)$ resulting in an output signal $o(t)$.
- Similarly, the operation can be defined in the frequency domain. This output to input relationship is of major importance in many disciplines.
- The transfer function is the Laplace transform of the output divided by the Laplace transform of the input.
- Hence, in 1D, we can determine the transfer function as follows:
  1. $i(t) \xrightarrow{\texttt{FFT}} I(\omega)$
  2. $o(t) \xrightarrow{\texttt{FFT}} O(\omega)$
  3. Transfer Function is $TF_{I-O} = O(\omega)/I(\omega)$





- Seismic events originate through tectonic slips and elastic waves (*p* and *s*) traveling through rock/soil foundation up to the surface. Hence, the seismographs (usually installed at the foot of the dam) record only the manifestation of the event.
- On the other hand, modelling the foundation is essential for proper and comprehensive analysis of the dam, and as such the seismic excitation will have to be applied at the base of the foundation.
- If we were to apply at the base the accelerogram recorded on the surface $I(t)$, the output signal $A(t)$ at the surface will be different than the one originally recorded (unless we have rigid foundation).
- Hence, the accelerogram recorded on the surface must be deconvoluted into a new one $I'(t)$, such that when the new signal is applied at the base of the foundation, the computed signal at the dam base matches the one recorded by the accelerogram.

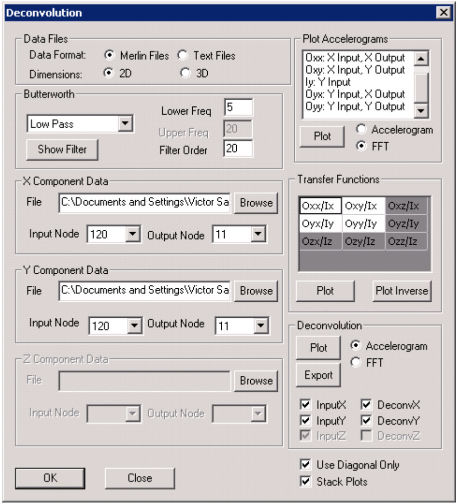


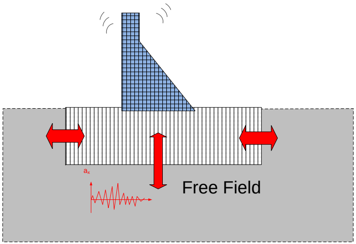
1. We record the earthquake induced acceleration on the surface $a'(t)$. and apply it as $i'(t)$ at the base of the foundation.
2. Perform a transient finite element analysis.
3. Determine the surface acceleration $a(t)$ (which is obviously different from $i(t)$.
4. Compute: $i'(t) \xrightarrow{\text{FFT}} I'(\omega) = A'(\omega)$ and $a(t) \xrightarrow{\text{FFT}} A(\omega)$
5. Compute transfer function from base to surface as $TF_{I'-A} = A(\omega)/I'(\omega)$.
6. Compute the inverse transfer function $TF^{-1}_{I'-A}$.

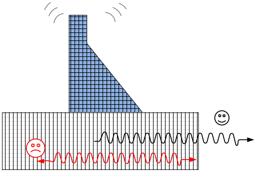
7. Determine the updated excitation record in the frequency domain $I(\omega) = TF^{-1}_{I'-A}A'(\omega) = \frac{I'(\omega)}{A(\omega)}A'(\omega)$

8. Determine the updated excitation in the time domain $i(t) \xrightarrow{\texttt{FFT}^{-1}} I(\omega)$

Process automated in our FE code Merlin.


Deconvolution
Data Files
Data Format: Merlin Files Text Files
Dimensions: 2D 3D
Butterworth
Low Pass
Show Filter
Lower Freq 5
Upper Freq 20
Filter Order 20
X Component Data
File C:\Documents and Settings\Victor Sa
Browse
Input Node 120 Output Node 11
Y Component Data
File C:\Documents and Settings\Victor Sa
Browse
Input Node 120 Output Node 11
Z Component Data
File
Browse
Input Node Output Node
OK
Close
Plot Accelerograms
Oxx: X Input, X Output
Oxy: X Input, Y Output
Iy: Y Input
Oyx: Y Input, X Output
Oyy: Y Input, Y Output
Plot
Accelerogram
FFT
Transfer Functions
Oxx/Ix Oxy/Ix Oxz/Ix
Oyx/Iy Oyy/Iy Oyz/Iy
Ozx/Iz Ozy/Iz Ozz/Iz
Plot
Plot Inverse
Deconvolution
Plot
Export
Accelerogram
FFT
InputX DeconvX
InputY DeconvY
InputZ DeconvZ
Use Diagonal Only
Stack Plots





Bull Earthquake Eng (2011) 9:1387–1402
DOI 10.1007/s10518-011-9261-7

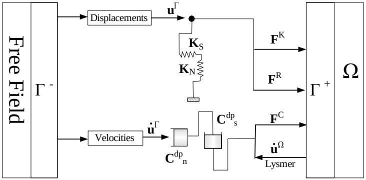
ORIGINAL RESEARCH PAPER

A simplified 3D model for soil-structure interaction with radiation damping and free field input

V. Saouma · F. Miura · G. Lebon · Y. Yagome

- Base of the Structure excited by a seismic wave that will travel through the model, and eventually hit the boundary.

- As with all waves, it will be reflected by the free surface whereas actually it propagates in the foundation to the free-field.
- Reflected wave may either amplify or decrease seismic excitation, in either case, it must be eliminated.
- Reflection can be eliminated either by a) “infinitely” large large mesh (expensive), b) “infinite” (boundary) element; or through Radiation Damping which will absorb the incident waves (P and S).
- Effect of free field on model must also be accounted for.

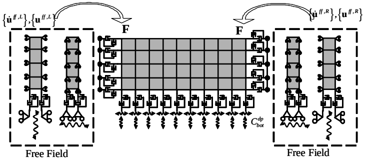


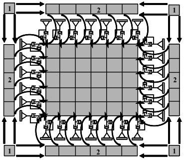
We identify four distinct parts:

1. The free field itself (F) without its contact surface Γ^-;
2. The contact surface of the free field Γ^-;
3. the contact surface of the model Γ^+;
4. the model Ω without its contact surface Γ^+.

$$\left[\mathrm{M}^\Omega \ddot{\mathrm{u}}^\Omega + \mathrm{C}^\Omega \dot{\mathrm{u}}^\Omega + \mathrm{K}^\Omega \mathrm{u}^\Omega\right] + \left[\mathrm{C}^{dp}_{\text{lft}} \dot{\mathrm{u}}^\Omega_{\text{lft}} + \mathrm{C}^{dp}_{\text{rgt}} \dot{\mathrm{u}}^\Omega_{\text{rgt}} + \mathrm{C}^{dp}_{\text{bot}} \dot{\mathrm{u}}^\Omega_B\right]$$
$$= \mathrm{t}^\Omega_{\text{bot}} + \left[\mathrm{F}^C_{\text{lft}} + \mathrm{F}^K_{\text{lft}} + \mathrm{F}^R_{\text{lft}}\right] + \left[\mathrm{F}^C_{\text{rgt}} + \mathrm{F}^K_{\text{rgt}} + \mathrm{F}^R_{\text{rgt}}\right]$$

Where F^C, F^K, and F^R are the vectors of nodal equivalent forces caused by the free field velocities, stiffness and damping respectively.

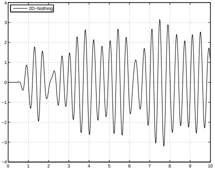


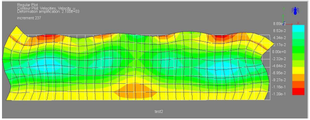


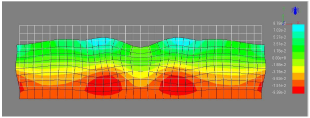
1. Discretize the free field with an arbitrary mesh. Place dashpots at the base of the mesh.
2. Constrain the vertical displacements of all the nodes (thus allowing only shear deformation), apply an horizontal excitation and analyze.
3. If the seismic record includes a vertical component, repeat the analysis by constraining all the horizontal displacements (thus allowing only axial deformation), apply the vertical component of the excitation and analyze.
4. Determine the nodal equivalent forces $\mathtt{F}^C$, $\mathtt{F}^K$ and $\mathtt{F}^R$.
5. Apply these as external (time dependent) boundary forces to the bounded domain and analyze.

Major advantage of this method, is that there is no need to modify existing finite element programs

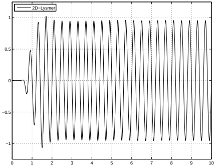
- Excite base with a harmonic excitation with period of 0.4 sec, a full wave length develops over 200 m which is the height of the model.
- Free Boundaries, bad

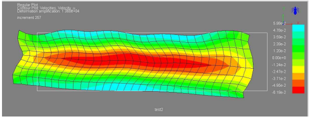


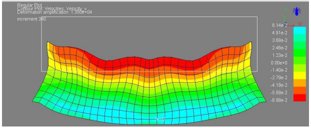




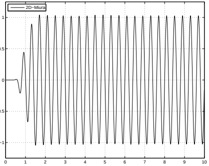
- 2D Lysmer

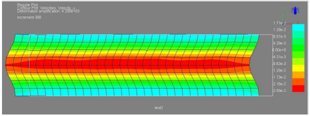


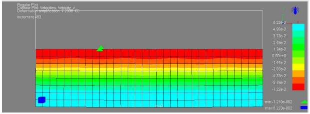




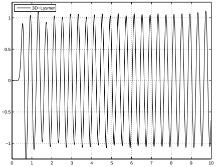
- 2D Miura-Saouma model

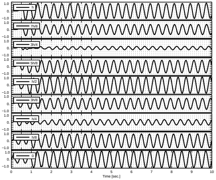


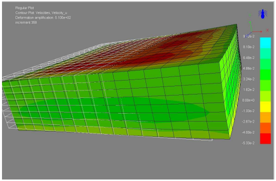




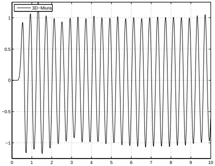
- 3D Lysmer

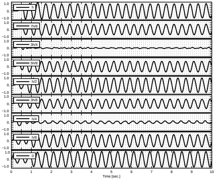


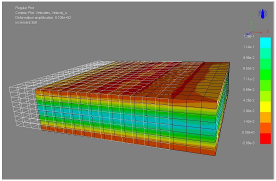


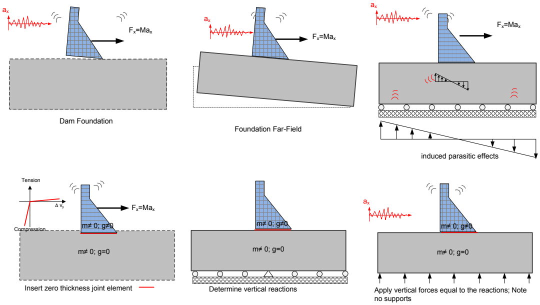


- 3D Miura-Saouma


3D−Miura


h
7h/8
3h/4
5h/8
h/2
3h/8
h/4
h/8
0
Time [sec.]





- Horizontal foundation modeling requires special attention as fixed supports would also reflect elastic waves resulting in "rocking" and can not be used.
- Foundation must simply be "supported" by vertical dashpots.

- To account for gravity loads, first a static analysis is performed with vertical supports, then supports are removed, and reactions replaced by nodal forces for the dynamic analysis along with dashpots.
- Process automated in Merlin

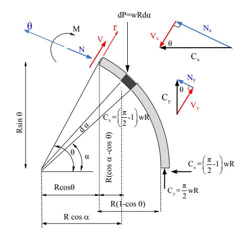
The Four Books of

Structural Analysis

August 9, 2022

Victor E. Saouma

University of Colorado, Boulder
2022



In Preparation;
Expected completion date:
Dec. 2023

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NOTICES

1. Intentionally, this book can not be printed. It is best read on a computer to easily follow the multiple hyperlinks and bookmarks.
2. It is particularly important that you start with the Preface, as this is an atypical book.
3. This book is free, feel free to share it.

Dedication

To my grandfather

whom I never met.

And to all future Structural Engineers.

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Preface

Genesis

This book, like so many others, had its genesis in notes of three courses taught over the span of over thirty years. But not only notes, but also a multitude of documents collected over the years in anticipation of this book. This resulted in a big puzzle where all the pieces had to smoothly fit together.
Hence, at the dusk of my academic career, and with a shade of vanity, I thought that I could share my 35+ years of teaching Structural Analysis with intrepid readers through a *magnum opus*.

Coverage

Broadly speaking the book is divided into four parts:

Book I is an extensive history of structural analysis. It does not pretend to be exhaustive, but was probably the most intersting part for me to write. Unconstrained, I have selected key events at first, and then when Galileo Galilei came, it had to follow a more disciplined path.

Book II is what one would expect students to be exposed in a first course in structural analysis (following Statics and Mechanics of Deformable Bodies (a.k.a. Materials). I have greatly expanded the coverage of some topics insufficiently covered in most books such as Cables, Arches, 3D structures. This book ends with a chapter containing numerous examples of preliminary design as it is important for the structural analyst to also have a sense of design.

Book III is what many institutions refer to as *Matrix Structural Analysis*. It is entirely devoted to the finite element method of framework members at first, but then rapidly expand into continuum elements. Along the way extensive coverage is given to variational methods as the foundation of the finite element method.

Book IV is based on a new course I had introduced, and which has only few counterparts in academia. It is devoted to the nonlinear analysis of framed structures, but also addresses plasticity, stability dynamics, and last but not least Performance Based Earthquake Engineering.

Hence, the pertinacious reader will be reward with an encyclopedic knowledge of structural engineering.

Yet another book?

The casual reader would wonder why is there yet another book on Structural Analysis? I have found that most textbooks on structural analysis) are really variation on a theme, all practically identical (and some have had as many of 15 editions and counting).
Many of them provide a rudimentary coverage of the underlying theory, and most importantly limit the examples to simple structures.
Finally, throughout the book I have attempted to correlate the various procedures of structural analysis with the principles of applied mechanics and mathematics on which they are based.

Audience

Students: This book is appropriate for three consecutive courses: Structural Analysis, Intermediate Structural Analysis, and Nonlinear Structural Analysis, combining in a single volume what has traditionally has required in at least two books. It further benefits from consistent notation throughout the coverage and includes illustrative examples prepared intentionally be challenging to the student.

However, only "mature audience' should consult it. By that, I mean those students who do not necessarily look for a simple and verbose coverage of the basics[1], of students motivated enough to explore sub-topics traditionally not covered, and students who aim to be structural engineers.

This book will also be of great values to those students who would like to see a *unified* (notation, philosophy) coverage of structural engineering with smooth transitions from fundamentals to intermediary and into advanced.

Engineers: This book is also addressed to structural engineers, wise enough to take a pause from computer programs, and explore the beauty of analytical solutions that can be of much greater value than thought of. Indeed too often many of them run to the computer before any attempt to obtain an exact or approximate analytical solution which could thne be validated by a program.

Historians: The first of the four books exhaustively covers the history of structural analysis. Aside from the great classical books that addressed this them, this is by far the most exhaustive coverage that can be found in a structural analysis book.

Style

A book is characterized by its content and its form. The form (or style) is utterly and blatantly personal, it reflects the teaching style, the focus of interest, ultimately, it reflects the delivery system of the author. As such, I have at times peppered this book with personal comment, and the depth and breadth of the coverage reflect my personal take on the topic.

For over 35 years, I have been a big fan of LaTeX(and felt pity for those who insisted in writing technical documents in a tool originally meant for lawyers: Word). As such, with a decent command of LaTeX; countless *packages* developed by others, and few macros I wrote myself I always tried to make sure that any manuscript I author is not only rigorous, complete, but also "looks nice"[2]

So, I have paid great attention to the layout, have personally drawn all the figures[3] and wrote the MATLAB®programs found in the appendix.

Oh "my English" is far from perfect, evidently is is not my native language, tried my best, so be kind and try not to be too critical.

[1]hence, for most, this book should never be assigned as the primary textbook in a course

[2]I have been fortunate to collaborate for nine years with the Tokyo Electric Power Company (TEPCO) on the nonlinear seismic analysis of tall arch dams. After about six years, I thought that we had accomplished all the work. *No! no! Prof. Saouma, in Japan, a program has not only to work properly but it must look beautiful.* This simple comment, along with my many visits/stays in Switzerland (where a great value is placed on sobriety), and a certain taste for architecture, influenced me.

[3]Starting with Xfig on Unix, ending with Visio on Windows

Why is it free?

On the supply side, there are two main reasons books are written. One is it may provide financial reward to its author, the second it may bring self-satisfaction and then possibly fame. In both cases, there is an anticipation that the publisher will provide text-editing, page layouts, and marketing that is unachievable by the author.
In our disciplines, I would venture to say that very few were awarded sufficient royalties to pay for a transatlantic flight. Fame on the other hand (in theory) should not be of concern to true scholars.
As to the demand side, students/readers have seen the price of books sky-rocket, even though nowadays there are clever marketing strategies whereas a reader may rent a book (or even specific chapters) for a limited time for a fixed fee (akin of renting a movie from Netflix).
As to formatting/marketing!. Any author sufficiently familiar with LaTeXcan quasi-professionally format any scientific book. No need to have a professional accomplish this task (unless one is stuck with Word that is). Marketing is also nowadays made so much easier, suffice it to publish a book through Amazon and it will be instantaneously be within reach of millions.
On the other hand if a book is well written (as this one pretends to be), and is free, then it will be naturally disseminated.
Finally, as a University Professor, our responsibility is to acquire and *share* knowledge. We are semi-decently paid by our institution, and the crumbs given to us by publisher are not worth a Faustian bargain.
Accordingly, this book can be freely downloaded and freely shared.

Books Consulted

In writing my notes and this book, I have consulted numerous books that have lend me some their coverage or examples. The following are the primary (but not only) ones.

- *Indeterminate Structural Analysis*
  Kinney, 1957
- *Elementary Structural Analysis*
  Norris and Wilbur, 1960
- *Theory of Matrix Structural Analysis*
  Przemieniecki, 1968
- *Basic Structural Analysis*
  Gerstle, 1974
- *Programming the matrix analysis of skeletal structures*
  Bhatt, 1986
- *Mechanics of Structures, Variational and Computational Methods*
  Pilkey and Wunderlich, 1994

Finally, I have tried in as much as possible to give proper credit within the book. If some were missing, it was certainly not intentional, and apologies are hereby offered.

A major challenge in teaching Structural Analysis is motivation. Hence, one should always keep in mind that structural analysis is not an end by itself, but only an indispensable tool to design or structural safety assessment (or design).

Victor E. Saouma
Boulder, CO 2023

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